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Article

$(d, 1)$-Total Labelling of Generalized Petersen Graphs $P(n, k)$

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Abstract: In this paper, we investigate the $(d, 1)$-total labelling of generalized Petersen graphs $P(n, k)$ for $d \geq 3$. We find that the $(d, 1)$-total number of $P(n, k)$ with $d \geq 3$ is $d + 3$ for even n and odd k or even n and $k = \frac{n}{2}$, and $d + 4$ for all other cases.

Keywords: (d,1)-Total labelling, (d,1)-Total number, Generalized Petersen graph
Mathematics Subject Classification: 05C15, 05C78

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1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A $(d, 1)$-total labelling of a graph G is an assignment of integers to $V(G) \cup E(G)$ such that:

(i) any two adjacent vertices of G receive distinct integers,

(ii) any two adjacent edges of G receive distinct integers, and

(iii) a vertex and its incident edge receive the integers that differ by at least d in absolute value.

The span of a $(d, 1)$-total labelling is the maximum difference between two labels. The minimum span of labels required for such a $(d, 1)$-total labelling of G is called the $(d, 1)$-total number and is denoted by $\lambda_d^{\mathrm{T}}(G)$. It reduces to the traditional total coloring when d is taken as 1.

The notion of $(d, 1)$-total labelling of a graph G was first introduced by Havet and Yu [1, 2]. They conjectured that $\lambda_d^{\mathrm{T}}(G) \leq \Delta(G) + 2d - 1$ for every graph G [1, 2], where $\Delta(G)$ is the maximum degree of G. The validity of conjecture has been verified for complete graphs [3, 4], complete bipartite graphs [5], planar graphs [6–11] and graphs with a given maximum average degree [12]. The exact values of $\lambda_d^{\mathrm{T}}(G)$ were determined for some graphs, such as $(d, 1)$-total labelings of flower snarks and quasi flower snarks [13] and the (2,1)-total number of joins of paths and cycles [14].

A well-known class of graphs is the generalized Petersen graphs $P(n, k)$. By definnition [15], $P(n, k)$ is a graph on $2n$ ($n \geq 3$) vertices with $V(P(n, k)) = \{u_i, v_i : 0 \leq i \leq n-1\}$ and $E(P(n, k)) = \{u_i u_{i+k}, u_i v_i, v_i v_{i+1} : 0 \leq i \leq n-1\}$, where subscripts are to be read modulo n and $1 \leq k \leq n-1$. For example, $P(5, 1)$ and $P(5, 2)$ are shown in Figure 1.

A number of articles has been devoted to the study of the labeling and coloring of $P(n, k)$, in particular, to the study of L(2,1)-labeling and total coloring [16–21]. However, the $(d, 1)$-total labeling

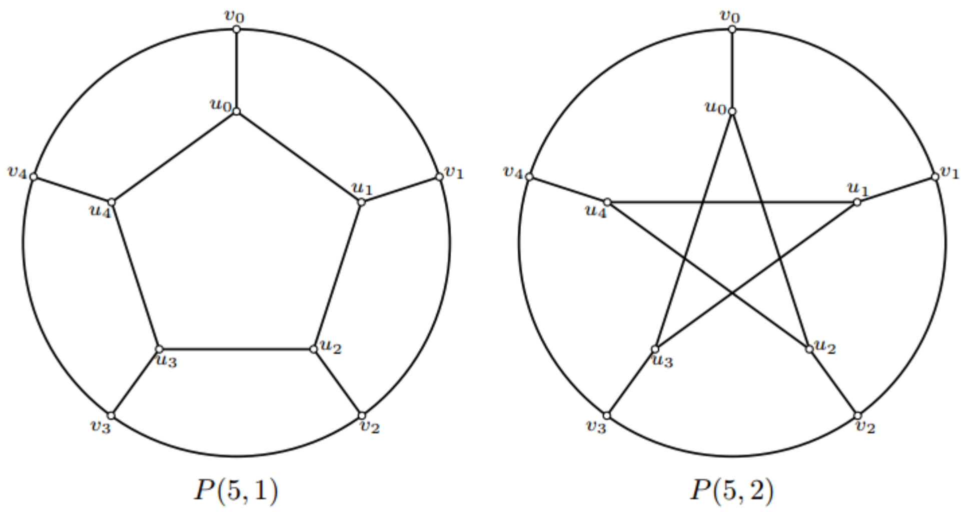


Figure 1. Generalized Petersen Graphs $P(5,1)$ and $P(5,2)$

of generalized Petersen graphs still remains open. To fill the gap, in this paper, we study the $(d,1)$-total numbers of generalized Petersen graphs. We aim to determine the $(d,1)$-total numbers of $P(n,k)$ for $d \geq 3$. This paper is organized as follows. We first prove that the $(d,1)$-total number of $P(n,k)$ for $k = \frac{n}{2}$ is $d+3$ in Section 2, and we then discuss the $(d,1)$-total number of $P(n,k)$ for $k \neq \frac{n}{2}$ in Section 3.

2. (d,1)-Total Labelling of $P(n,k)$ for $k = \frac{n}{2}$

For a graph G with $\Delta(G) \leq d$, there is always $\lambda_d^{\mathrm{T}}(G) \geq d + \Delta(G)$ [4]. Since $\Delta(P(n,k)) = 3$, we have Lemma 1.

Lemma 1. $\lambda_d^T(P(n,k)) \geq d+3$ *for* $d \geq 3$.

To obtain the exact value of $\lambda_d^{\mathrm{T}}(P(n,k))$, we prove Lemma 2.

Lemma 2. $\lambda_d^T(P(n,k)) \leq d+3$ *if* $k = \frac{n}{2}$.

Proof. We use f to represent the assignment of integers to $V(G) \cup E(G)$. In the case of $k = \frac{n}{2}$, we can define the assignment as follows,

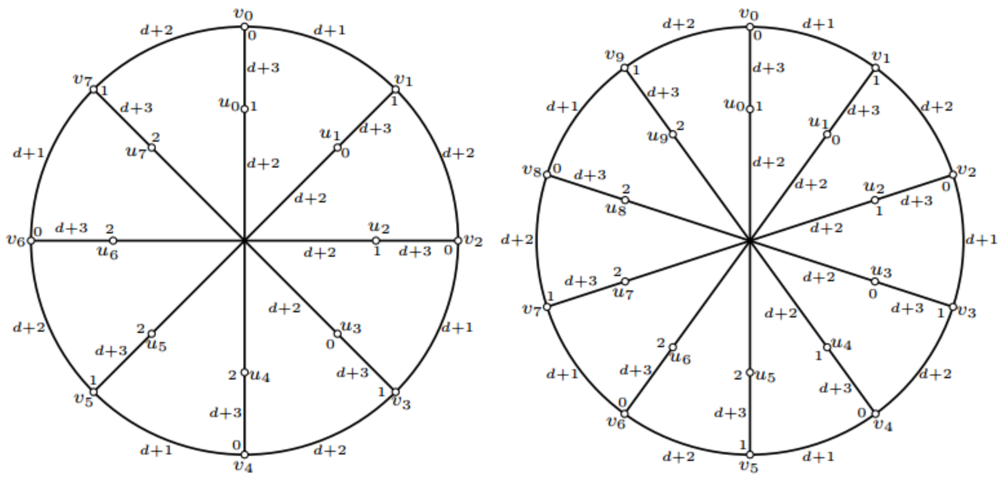
$$f(u_i) = \begin{cases} \frac{1+(-1)^i}{2}, & 0 \leq i \leq \frac{n}{2}-1, \\ 2, & \frac{n}{2} \leq i \leq n-1. \end{cases}$$
$$f(u_iu_{i+k}) = d+2,\ \ 0 \leq i \leq \tfrac{n}{2}-1.$$
$$f(u_iv_i) = d+3,\ \ 0 \leq i \leq n-1.$$
$$f(v_i) = \tfrac{1-(-1)^i}{2},\ \ 0 \leq i \leq n-1.$$
$$f(v_iv_{i+1}) = d+1+\tfrac{1-(-1)^i}{2},\ \ 0 \leq i \leq n-1.$$

Clearly, the assignment f, defined above, gives a $(d,1)$-total labelling. Indeed, it is easy to verify that f fulfills all the three conditions, i.e.,

(i) any two adjacent vertices receive distinct integers,

(ii) any two adjacent edges receive distinct integers, and

(iii) a vertex and its incident edge receive the integers that differ by at least d.



$P(8,4)$ $P(10,5)$

Figure 2. (d,1)-Total Labelings of $P(8,4)$ and $P(10,5)$

We then have $\lambda_d^{\mathrm{T}}(P(n,k)) \leq d+3$ for $k = \frac{n}{2}$. This completes the proof of Lemma 2.

Figure 2 shows (d,1)-total labelings of $P(8,4)$ and $P(10,5)$.

□

Lemma 2 determines the upper bound of $\lambda_d^{\mathrm{T}}(P(n,k))$ for $k = \frac{n}{2}$ while Lemma 1 gives the lower bound of $\lambda_d^{\mathrm{T}}(P(n,k))$. Hence, we have Theorem 1.

Theorem 1. *$\lambda_d^T(P(n,k)) = d+3$ for $d \geq 3$ if $k = \frac{n}{2}$.*

3. (d,1)-Total Labelling of $P(n,k)$ for $k \neq \frac{n}{2}$

Since $P(n,k) \cong P(n,n-k)$, we need only to consider the labeling of $P(n,k)$ for $1 \leq k < \frac{n}{2}$. Therefore, we assume $1 \leq k < \frac{n}{2}$ in the following discussion.

We first consider $P(n,k)$ for even n and odd k. In this case, $P(n,k)$ is 3-regular bipartite graphs, and by borrowing the result obtained in Ref. [4], which indicates that $\lambda_d^{\mathrm{T}}(G) = d+r$ if G is an r-regular bipartite graph, we immediately obtain the $(d,1)$-total number of $P(n,k)$ for even n and odd k. We state it as Theorem 2.

Theorem 2. *$\lambda_d^T(P(n,k)) = d+3$ if n is even and k is odd.*

We now consider $P(n,k)$ for odd n or even n and even k ($k \neq \frac{n}{2}$). In this case, $P(n,k)$ is 3-regular nonbipartite graph, and by borrowing the result obtained in Ref. [13], which indicates $\lambda_d^{\mathrm{T}}(G) \geq d+r+1$ if G is an r-regular nonbipartite graph and $d \geq r \geq 3$, we have Lemma 3.

Lemma 3. *$\lambda_d^T(P(n,k)) \geq d+4$ for $d \geq 3$ if n is odd or both n and k are even ($k \neq \frac{n}{2}$).*

To determine the exact value of $\lambda_d^{\mathrm{T}}(P(n,k))$, we prove the following lemma.

Lemma 4. *$\lambda_d^T(P(n,k)) \leq d+4$ if n is odd or both n and k are even.*

Proof. Let $g = \gcd(n,k)$ and $l = n/g$. We refer to cycle $u_i u_{i+k} u_{i+2k} \cdots u_{i+(l-1)k} u_i$ $(0 \leq i \leq g-1)$ as an inner cycle, cycle $v_0 v_1 \cdots v_{n-1} v_0$ as an outer cycle, and $u_i v_i$ $(0 \leq i \leq n-1)$ as a spoke. Obviously, $P(n,k)$ contains g non-intersecting inner cycles with length l. With these notions, we can divide our proof of Lemma 4 into the following two cases.

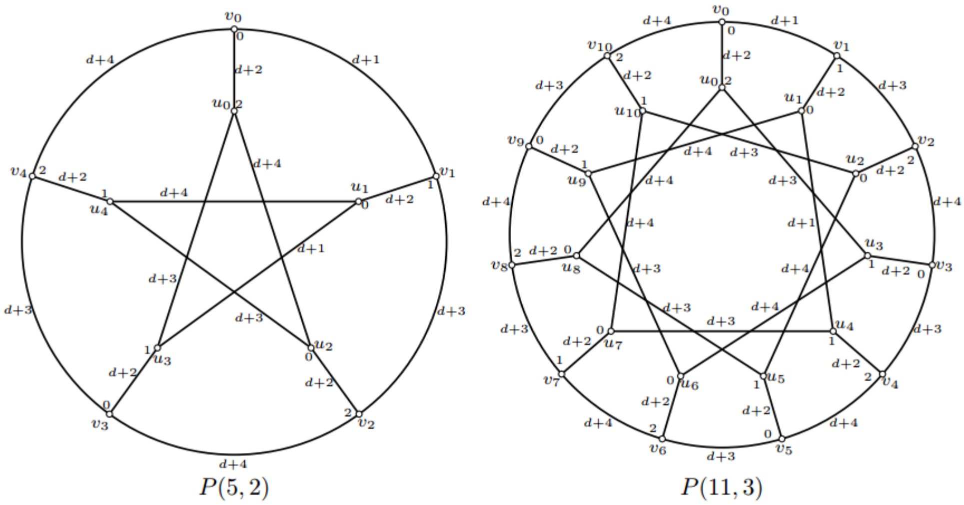


Figure 3. (d,1)-Total Labelings of $P(5,2)$ and $P(11,3)$

Case 1. $g = 1$. In this case, l and n are odd. It follows that both the inner cycle and the outer cycle are odd cycles. We define the assignment f as follows,

$$f(u_{ik}) = \begin{cases} 2, & i = 0, \\ \frac{1-(-1)^{i+j}}{2}, & 1 \le i, j \le n-1 \text{ and } j \text{ is defined by } jk \text{ (mod n)} = 1. \end{cases}$$
$$f(u_{1+ik}u_{1+(i+1)k}) = \begin{cases} d+1, & i = 0, \\ d+3+\frac{1+(-1)^{i}}{2}, & 1 \le i \le n-1. \end{cases}$$
$$f(u_iv_i) = d+2,\ 0 \le i \le n-1.$$
$$f(v_i) = \begin{cases} \frac{1-(-1)^{i}}{2}, & 0 \le i \le 1, \\ 2, & 2 \le i \le n-1 \text{ and } i \text{ (mod 2)} = 0, \\ 1-f(u_i), & 2 \le i \le n-2 \text{ and } i \text{ (mod 2)} = 1. \end{cases}$$
$$f(v_iv_{i+1}) = \begin{cases} d+1, & i = 0, \\ d+3+\frac{1+(-1)^{i}}{2}, & 1 \le i \le n-1. \end{cases}$$

Figure 3 shows (d,1)-total labelings of $P(5,2)$ and $P(11,3)$.

Case 2. $g \ge 2$. We further divide this case into the following two subcases.

Case 2.1. l is even. In this case, n must be even. It follows that both the outer cycle and the inner cycles are even cycles. We define the assignment f as follows,

$$f(u_{ik+j}) = \frac{1-(-1)^{i}}{2},\ \ 0 \le i \le l-1,\ 0 \le j \le g-1.$$
$$f(u_{ik+j}u_{(i+1)k+j}) = d+3+\frac{1-(-1)^{i}}{2},\ \ 0 \le i \le l-1,\ 0 \le j \le g-1.$$
$$f(u_iv_i) = d+2,\ \ 0 \le i \le n-1.$$
$$f(v_i) = \begin{cases} f(u_{n-1}), & i = 0, \\ \min(\{0,1,2\} - \{f(v_{i-1}), f(u_i)\}), & 1 \le i \le n-1. \end{cases}$$
$$f(v_iv_{i+1}) = d+3+\frac{1-(-1)^{i}}{2}, 0 \le i \le n-1.$$

Figure 4 shows (d,1)-total labelings of $P(8,2)$ and $P(16,6)$.

Case 2.2. l is odd. In this case, the inner cycles are odd cycles. We define the assignment f as

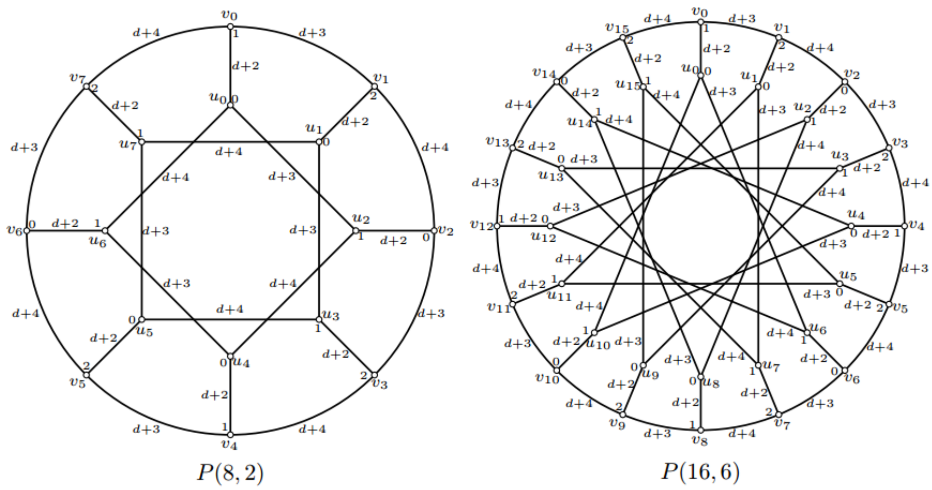


Figure 4. (d,1)-Total Labelings of $P(8,2)$ and $P(16,6)$

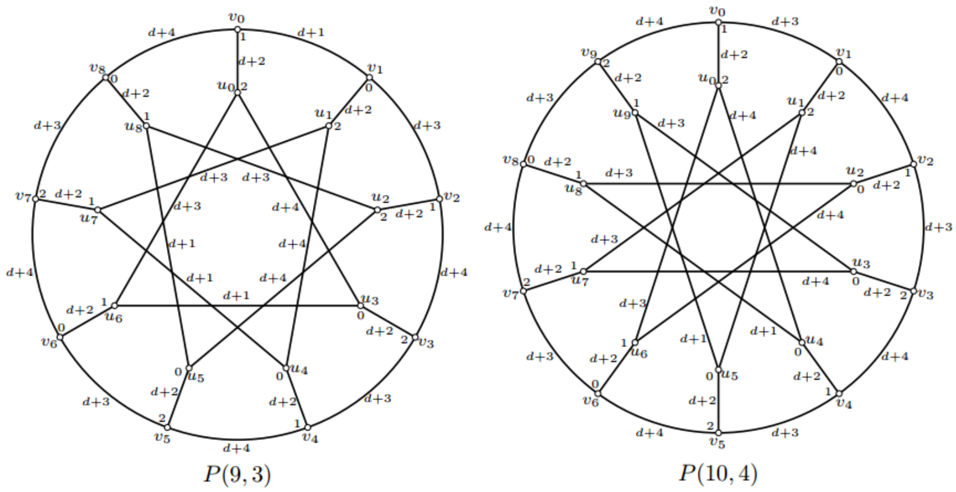


Figure 5. (d,1)-Total Labelings of $P(9,3)$ and $P(10,4)$

follows,

$$f(u_{ik+j}) = \begin{cases} 2, & i=0,\ 0 \le j \le g-1, \\ \frac{1+(-1)^i}{2}, & 1 \le i \le l-1,\ 0 \le j \le g-1. \end{cases}$$

$$f(u_{(1+i)k+j}u_{(1+i+1)k+j}) = \begin{cases} d+1, & i=0,\ 0 \le j \le g-1, \\ d+3+\frac{1+(-1)^i}{2}, & 1 \le i \le l-1,\ 0 \le j \le g-1. \end{cases}$$

$$f(u_iv_i) = d+2,\ 0 \le i \le n-1.$$

$$f(v_i) = \begin{cases} f(u_{n-1}), & i=0, \\ \min(\{0,1,2\} - \{f(v_{i-1}), f(u_i)\}), & 1 \le i \le n-1. \end{cases}$$

$$f(v_iv_{i+1}) = \begin{cases} d+2+(-1)^n, & i=0, \\ d+3+\frac{1-(-1)^{i+n}}{2}, & 1 \le i \le n-1. \end{cases}$$

Figure 5 shows (d,1)-total labelings of $P(9,3)$ and $P(10,4)$.

It is obvious that in the above labeling scheme, any two adjacent edges receive distinct integers, and each vertex and its incident edges receive the integers that differ by at least d. Besides, it is also obvious that any two adjacent vertices of inner cycles receive distinct integers. To complete our proof, we need further to verify that each vertex of outer cycle and its adjacent vertices receive distinct integers. We examine the labeling in each case. In case 1, according to the assignment of $f(u_{ik})$ and $f(v_i)$, we can easily find that $f(v_i) \neq f(v_{i+1})$ for $0 \leq i \leq n-1$ and $f(v_i) \neq f(u_i)$ for $i = 0$ and $2 \leq i \leq n-1$. Furthermore, since jk (mod n)=1 and ik (mod n)=1 for $f(u_1)$, we have $f(u_1) = 0 \neq f(v_1)$. In case 2, since $f(v_i) = \min(\{0,1,2\} - \{f(v_{i-1}), f(u_i)\})$ for $1 \leq i \leq n-1$, we then have $f(v_i) \neq f(v_{i-1})$ and $f(v_i) \neq f(u_i)$ for $1 \leq i \leq n-1$. Furthermore, since $f(v_0) = f(u_{n-1})$, we have $f(v_{n-1}) \neq f(v_0)$ and $f(v_0) = 1 \neq f(u_0)$. That is, our labeling scheme presented above fulfills all the three conditions (i) (ii) and (iii), being a $(d,1)$-total labeling. This concludes the proof of Lemma 4. □

By Lemmas 3 and 4, we have Theorem 3.

Theorem 3. $\lambda_d^T(P(n,k)) = d+4$ *for* $d \geq 3$ *if* n *is odd or both* n *and* k *are even* ($k \neq \frac{n}{2}$).

In summary, we have presented a $(d,1)$-total labelling of generalized Petersen graphs $P(n,k)$ for $d \geq 3$, and given the exact values of $\lambda_d^{\mathrm{T}}(P(n,k))$. By combing Theorems 1, 2 and 3, we have the following conclusion.

The $(d,1)$-total number of $P(n,k)$ with $d \geq 3$ is $d+3$ for even n and odd k or even n and $k = \frac{n}{2}$, and $d+4$ for all other cases.

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Conflict of Interest

The authors declare no conflict of interest.

References

1. Havet, F. and Yu, M. L., 2002. (d,1)-total labelling of graphs (Technical Report No. 4650). *INRIA*.
2. Havet, F., 2003. (d,1)-Total labelling of graphs. In *Workshop on Graphs and Algorithms, Dijon, France*.
3. Chia, M. L., Kuo, D., Yan, J. H. and Yang, S. R., 2013. (p,q)-Total labeling of complete graphs. *Journal of Combinatorial Optimization, 25*(4), pp.543-561.
4. Havet, F. and Yu, M. L., 2008. (p,1)-Total labelling of graphs. *Discrete Mathematics, 308*, pp.496-513.
5. Lih, K., Liu, D. F. and Wang, W. F., 2008. On (d,1)-Total numbers of graphs. *Discrete Mathematics, 309*, pp.3767-3773.
6. Bazzaro, F., Montassier, M. and Raspaud, A., 2007. (d,1)-Total labelling of planar graphs with large girth and high maximum degree. *Discrete Mathematics, 307*(16), pp.2141–2151.
7. Chen, D. and Wang, W. F., 2007. (2,1)-Total labelling of outerplanar graphs. *Discrete Applied Mathematics, 155*, pp.2585–2593.
8. Sun, L. and Li, H. Y., 2011. (2,1)-Total labeling of planar graphs with large girth and low maximum degree. *ARS Combinatoria, 100*, pp.65–72.
9. Sun, L. and Cai, H., 2015. On (p,1)-Total labelling of special 1-planar graphs. *ARS Combinatoria, 123*, pp.87–96.
10. Sun, L. and Wu, J. L., 2017. On (p,1)-Total labelling of planar graphs. *Journal of Combinatorial Optimization, 33*(1), pp.317–325.

11. Zhang, X., Yu, Y. and Liu, G. Z., 2011. On (p,1)-Total labelling of 1-planar graphs. *Central European Journal of Mathematics,* 9, pp.1424–1434.
12. Montassier, M. and Raspaud, A., 2006. (d,1)-Total labelling of graphs with a given maximum average degree. *Journal of Graph Theory,* 51, pp.93–109.
13. Tong, C. L., Lin, X. H., Yang, Y. S. and Hou, Z. W., 2010. (d,1)-Total labellings of regular nonbipartite graphs and an application to flower snarks. *ARS Combinatoria,* 96, pp.33–40.
14. Wang, W. F., Huang, J., Sun, H. N. and Huang, D. J., 2012. (2,1)-Total number of joins of paths and cycles. *Taiwanese Journal of Mathematics,* 16, pp.605–619.
15. Watkins, M. E., 1969. A theorem on Tait colorings with an application to generalized Petersen graphs. *Journal of Combinatorial Theory,* 6, pp.152–164.
16. Adams, S. S., Cass, J. and Troxell, D. S., 2006. An extension of the channel-assignment problem: L(2,1)-labelings of generalized Petersen graphs. *IEEE Transactions on Circuits and Systems I: Regular Papers, 53*(5), pp.1101–1107.
17. Adams, S. S., Cass, J., Tesch, M., Troxell, D. S. and Wheeland, C., 2007. The minimum span of L(2,1)-labelings of certain generalized Petersen graphs. *Discrete Applied Mathematics, 155*, pp.1314–1325.
18. Adams, S. S., Booth, P., Jaffe, H., Troxell, D. S. and Zinnen, S. L., 2012. Exact λ-numbers of generalized Petersen graphs of certain higher-orders and on Möbius strips. *Discrete Applied Mathematics, 160*, pp.436–447.
19. Georges, J. P. and Mauro, D. W., 2002. On generalized Petersen graphs labeled with a condition at distance two. *Discrete Mathematics, 259*, pp.311–318.
20. Huang, Y. Z., Chiang, C. Y., Huang, L. H. and Yeh, H. G., 2012. On L(2,1)-labeling of generalized Petersen graphs. *Journal of Combinatorial Optimization, 24*(3), pp.266–279.
21. Tong, C. L., Lin, X. H. and Yang, Y. S., 2019. Equitable total coloring of generalized Petersen graphs. *ARS Combinatoria, 143*, pp.321-336