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Article

5-Regular Subgraphs in Hypercubes

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Abstract: One of the fundamental properties of the hypercube Q_n is that it is bipancyclic as Q_n has a cycle of length l for every even integer l with $4 \leq l \leq 2^n$. We consider the following problem of generalizing this property: For a given integer k with $3 \leq k \leq n$, determine all integers l for which there exists an l-vertex, k-regular subgraph of Q_n that is both k-connected and bipancyclic. The solution to this problem is known for $k = 3$ and $k = 4$. In this paper, we solve the problem for $k = 5$. We prove that Q_n contains a 5-regular subgraph on l vertices that is both 5-connected and bipancyclic if and only if $l \in \{32, 48\}$ or l is an even integer satisfying $52 \leq l \leq 2^n$. For general k, we establish that every k-regular subgraph of Q_n has $2^k, 2^k + 2^{k-1}$ or at least $2^k + 2^{k-1} + 2^{k-3}$ vertices.

Keywords: Hypercube, k-connected, Regular, Bipancyclic, Subgraphs
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1. Introduction

A graph $G = (V, E)$ on n vertices is an *n-vertex graph.* A $2n$-vertex graph is *bipancyclic* if it has a cycle of length l for all even integers l satisfying $4 \leq l \leq 2n$. The *Cartesian product* of two graphs G and H is the graph $G \square H$ with vertex set $V(G) \times V(H)$ in which two vertices (x, y) and (u, v) are adjacent if and only if either $x = u$ and y is adjacent to v in H, or $y = v$ and x is adjacent to u in G. Throughout the paper n denote a positive integer.

The n-dimensional hypercube Q_n is the Cartesian product of n copies of the complete graph K_2. It is an n-regular, n-connected, bipartite, and bipancyclic graph on 2^n vertices with diameter n. Because of such rich properties, hypercubes are one of the most widely used interconnection network topologies [1]. The connectivity of a network is an important parameter to evaluate the reliability and fault tolerance of a network [2]. Bipancyclicity is a fundamental property of the hypercube networks as it allows the embedding of cycles of various lengths effectively into hypercubes. Cycle networks are used to design simple algorithms with low communication cost and it has applications in image processing and signal processing [1, 3].

We consider the following problem of generalizing the property of bipancyclicity of hypercubes to the existence of l-vertex, k-regular subgraphs for various values of l that are also k-connected and bipancyclic. This will give subgraphs of Q_n with less number of vertices which retain the important properties of Q_n such as regularity, high connectivity, and bipancyclicity.

Problem 1. *For a given integer k with $3 \leq k \leq n$, determine all integers l for which there exists an l-vertex, k-regular subgraph of Q_n that is both k-connected and bipancyclic.*

This problem is also related to the problem of embedding regular graphs into hypercubes. Cybenko et al. [4] proved that the problem of deciding whether or not a given graph is embeddable into a hypercube is NP-complete, in fact, the problem is NP-complete even for trees [5].

Since the hypercube Q_n is a bipartite graph, every regular subgraph of it has even number of vertices. For $k = 3$, Ramras [6] established that every even integer from 8 to 2^n except 10 can be the number of vertices of a 3-regular subgraph of Q_n. Borse and Shaikh [7] improved this result by showing that such a 3-regular subgraph can be bipancyclic also. They solved the above problem for $k = 3$ and $k = 4$ in [7] and [8], receptively. They established that, for $k \in \{3, 4\}$, Q_n has a k-regular, k-connected, bipancyclic subgraph on l vertices if and only if $l = 2^k$ or l is an even integer with $2^k + 2^{k-1} \leq l \leq 2^n$. The problem remains open for $k \geq 5$.

Besides hypercubes, the special case $k = 3$ of the above problem is settled for the class of the Cartesian product of cycles in [9] and for the class of the Cartesian product of paths in [10]. Also, Borse et al. [11] proved the existence of a factorization of the Cartesian product of r cycles, each of length a power of 2, into isomorphic k-regular, k-connected and bipancyclic subgraphs with the number of vertices a power of 2, for $2 \leq k < 2r$. Moreover, the number of vertices of a smallest k-regular subgraph of an r-regular graph G is related to the conditional k-edge-connectivity of G [12].

In this paper, we settle Problem 1 for the case $k = 5$. The following is the main result of the paper.

Theorem 1 (Main Theorem)**.** *For $n \geq 5$, there exists a 5-regular, 5-connected and bipancyclic subgraph of the hypercube Q_n on l vertices if and only if $l \in \{32, 48\}$ or l is an even integer such that $52 \leq l \leq 2^n$.*

For general k, Borse and Shaikh [8] obtained the following result about the non-existence of k-regular subgraphs of Q_n on a certain number of vertices.

Theorem 2 ([8])**.** *For a given integer k with $1 \leq k \leq n$, every subgraph of Q_n with minimum degree at least k either is isomorphic to Q_k or has at least $2^k + 2^{k-1}$ vertices.*

In this paper, we improve this theorem as follows.

Theorem 3. *For a given integer k with $2 \leq k \leq n$, if H is a subgraph of Q_n with minimum degree at least k, then one of the following holds:*

(i) H is isomorphic to Q_k.

(ii) H is a spanning subgraph of the subgraph of Q_n induced by $V(C \square Q_{k-2})$ for some cycle C of length six.

(iii) H has at least $2^k + 2^{k-1} + 2^{k-3}$ vertices.

Thus, if $2 \leq k \leq n$ and $1 \leq l < 2^k + 2^{k-1} + 2^{k-3}$ with $l \notin \{2^k, 2^k + 2^{k-1}\}$, then Q_n does not have a k-regular subgraph on l vertices and hence no k-regular, l-vertex graph is embeddable into Q_n.

We provide preliminary results in Section 2 and prove Theorem 3 in Section 3. The proof of Main Theorem 1 is divided into the next three sections. A construction of 5-regular subgraphs of Q_n is given in Section 4. The connectivity and bipancyclicity properties of these subgraphs are dealt in Section 5 and Section 6, respectively.

2. Preliminaries

We can write Q_n as $Q_{n-k} \square Q_k$ for $0 \leq k \leq n$, where Q_0 is the complete graph K_1. A *k-cycle* means a cycle of length k. We need the following lemmas.

Lemma 1 ([3])**.** *Let G_i be an m_i-regular and m_i-connected graph for $i = 1, 2$. Then the graph $G_1 \square G_2$ is $(m_1 + m_2)$-regular and $(m_1 + m_2)$-connected.*

Lemma 2 ([13]). *If P and Q are non-trivial paths and one of them has an even number of vertices, then the graph $P\Box Q$ is bipancyclic.*

Hence $C\Box K_2$ is bipancyclic if C is a non-trivial path or a cycle of length at least three.

Lemma 3 ([8]). *For $n \geq 3$, the hypercube Q_n has a Hamiltonian cycle C with a chord e such that there is a 4-cycle in Q_n containing e and three edges of C.*

Lemma 4 ([8]). *Let l be an even integer such that $6 \leq l \leq 2^n - 2$. Then there exists an l-cycle C in Q_n containing six vertices x, y, z, u, v, w and there are two vertices g, h in $V(Q_n) - V(C)$ such that*

(i) g is adjacent to x, y, z;

(ii) h is adjacent to u, v, w;

(iii) xu, uy and yv are edges of C.

We obtain a similar result as follows.

Lemma 5. *Let l be an even integer such that $8 \leq l \leq 2^n - 2$. Then there exists an l-cycle C in Q_n and a vertex u in $V(Q_n) - V(C)$ having four pairwise non-adjacent neighbours in C.*

Proof. Suppose $n = 4$. Clearly, $C = < v_1, v_2, \ldots, v_8, v_1 >$ is a required 8-cycle in Q_4 as shown by bold lines in Figure 1. Replacing the edge v_1v_2 of C by a path of length 3 avoiding u gives a 10-cycle C' and replacing the edge v_3v_4 of C' by a path of length 3 or 5 that is edge-disjoint with C' produces required 12-cycle and 14-cycle. Thus the result holds for $n = 4$.

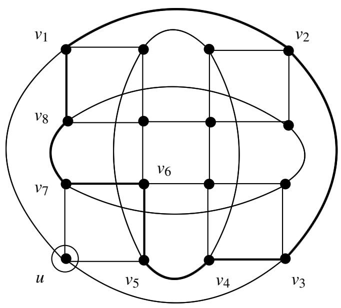
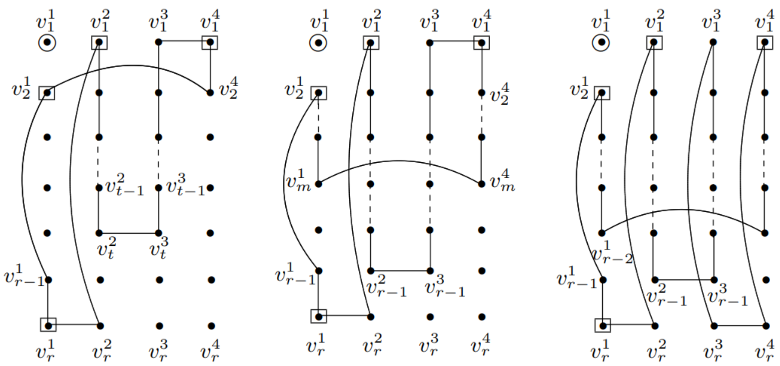


Figure 1. 8-Cycle C in Q_4

Suppose $n \geq 5$. Write Q_n as $Q_{n-2}\Box Q_2$. Then Q_n is obtained from four copies Q^1_{n-2}, Q^2_{n-2}, Q^3_{n-2}, Q^4_{n-2} of Q_{n-2} such that Q^i_{n-2} is joined to Q^{i+1}_{n-2} for $i = 1, 2, 3$ by a matching between their corresponding vertices. Vertices of Q^1_{n-2} are joined to the corresponding vertices of Q^4_{n-2}.



a) $8 \leq l \leq 2^{n-1} + 4$ b) $2^{n-1} + 6 \leq l \leq 2^n - 4$ c) $l = 2^n - 2$

Figure 2. The l-cycle Z

Since $n-2 \geq 3$, by Lemma 3, each Q^i_{n-2} contains a Hamiltonian cycle C^i with a chord e_i such that there is a 4-cycle in Q^i_{n-2} containing e_i and three edges from C^i. For simplicity let $r = 2^{n-2}$. Label the set of vertices of Q^i_{n-2} by $\{v^i_p \mid p = 1, 2, \ldots, r\}$ so that $C^i = < v^i_1, v^i_2, \ldots, v^i_r, v^i_1 >$ and $e = v^i_2 v^i_{r-1}$. We now construct a cycle Z of length l in Q_n, as required, from the cycles C^1, C^2, C^3, C^4 and the chord e_1 of C^1. If $8 \leq l \leq 2^{n-1} + 4$, then $l = 2t + 6$, where $1 \leq t = l/2 - 3 \leq 2^{n-2} - 1 = r - 1$. In this case, take Z as the cycle shown in Figure 2(a). If $2^{n-1} + 6 \leq l \leq 2^n - 4$, then $l = 2m + 2^{n-1}$ with $3 \leq m = l/2 - 2^{n-2} \leq 2^{n-1} - 2^{n-2} - 2 = r - 2$. In this case, choose Z to be the cycle shown in Figure 2(b). Finally, for $l = 2^n - 2$ take Z as the cycle given in Figure 2(c). In each case, Z is a cycle of length l in Q_n, and further, the vertex v^1_1 is not on Z but it has four pairwise non-adjacent neighbours $v^1_2, v^1_r, v^2_1, v^4_1$ in Z. This completes the proof.

□

3. Proof of Theorem 3

In this section, we prove Theorem 3. For a graph G, let $G[S]$ denote the induced subgraph of G on a vertex subset $S \subseteq V(G)$. The minimum degree of G is denoted by $\delta(G)$. If G is isomorphic to a graph H, then we write $G \cong H$. Since $Q_n = Q_{n-1} \square K_2$, we can split Q_n into two copies Q^0_{n-1} and Q^1_{n-1} of Q_{n-1}. If H is a subgraph of Q^0_{n-1}, then there is a subgraph H' of Q^1_{n-1} isomorphic to H such that the vertex set of H' is the set of neighbours of H in Q^1_{n-1}. We say that H' is the subgraph of Q^1_{n-1} corresponding to H.

We need the following result.

Lemma 6 ([8]). *For a given integer k with $1 \leq k < n$, if H is a subraph of Q_n isomorphic to Q_k, then every vertex in $V(Q_n) - V(H)$ has at most one neighbour in H.*

Proof of Theorem 3. By Theorem 2, (i) holds or $|V(H)| \geq 2^k + 2^{k-1}$. Suppose (i) does not hold. Then $|V(H)| \geq 2^k + 2^{k-1}$. We prove, by induction on k, that (ii) holds if $|V(H)| = 2^k + 2^{k-1}$, otherwise (iii) holds. Suppose $k = 2$. Then $|V(H)| = 6$ or $|V(H)| \geq 7$. If $|V(H)| = 6$, then it follows that H is a chordless 6-cycle or a 6-cycle with a chord and so (ii) holds. If $|V(H)| \geq 7 > 2^2 + 2^{2-1} + 2^{2-3}$, then (iii) follows. Thus the result is true for $k = 2$.

Suppose $k \geq 3$. Assume that the result holds for the subgraphs of Q_n of minimum degree at least $k-1$. Let e be an edge of H. Without loss of generality, we may assume that the end vertices of e differ in the first coordinate only. Write $Q_n = Q^0_{n-1} \cup Q^1_{n-1} \cup D$, where D is the set of all edges in Q_n whose end vertices differ in the first coordinate only. Then $e \in D$. Hence H intersects both Q^0_{n-1} and Q^1_{n-1}. Let $H_i = H \cap Q^i_{n-1}$ for $i = 0, 1$. Then $\delta(H_i) \geq k - 1$. We may assume that $|V(H_0)| \leq |V(H_1)|$.

By induction hypothesis, $H_0 \cong Q_{k-1}$ or H_0 is a spanning subgraph of $Q_n[V(C \square Q_{k-3})]$ for some 6-cycle C or $|V(H_0)| \geq 2^{k-1} + 2^{k-2} + 2^{k-4}$. Hence $|V(H_0)| = 2^{k-1}$ or $|V(H_0)| = 3(2^{k-2})$ or $|V(H_0)| \geq 2^{k-1} + 2^{k-2} + 2^{k-4}$. We consider these three cases separately.

Case (i). Suppose $|V(H_0)| \geq 2^{k-1} + 2^{k-2} + 2^{k-4}$.

Consider $|V(H)| = |V(H_1)| + |V(H_0)| \geq 2|V(H_0)| = 2^k + 2^{k-1} + 2^{k-3}$ as $|V(H_1)| \geq |V(H_0)|$. Therefore (iii) holds.

Case (ii). Suppose $|V(H_0)| = 2^{k-1}$.

In this case, H_0 is isomorphic to Q_{k-1}. As $\delta(H) \geq k$, each vertex of H_0 has a neighbour in H_1. Let W_1 be the subgraph of Q^1_{n-1} corresponding to H_0. Then $W_1 \cong Q_{k-1}$ and $V(W_1) \subseteq V(H_1)$. Let W_2 be the subgraph of H_1 induced by $V(H_1) - V(W_1)$. Observe that no vertex of W_2 has a neighbour in H_0 and by Lemma 6, it has at most one neighbour in W_1. Therefore $\delta(W_2) \geq k - 1$. By Theorem 2, $W_2 \cong Q_{k-1}$ or $|V(W_2)| \geq 2^{k-1} + 2^{k-2}$. In the later case, (iii) holds as $|V(H))| = |V(H_0)| + |V(W_1)| + |V(W_2)| \geq 2^{k-1} + 2^{k-1} + 2^{k-1} + 2^{k-2} > 2^k + 2^{k-1} + 2^{k-3}$. Suppose $W_2 \cong Q_{k-1}$. Then the subgraph of Q_n induced by the vertices of H_0, W_1, W_2 is isomorphic to $P_3 \square Q_{k-1} = (P_3 \square K_2) \square Q_{k-2}$, where P_3 is a path on three vertices. Since $P_3 \square K_2$ is a 6-cycle with a chord, (ii) holds.

Case (iii). H_0 is a spanning subgraph of $Q_n[V(C\square Q_{k-3})]$ for some 6-cycle C.

We consider two subcases depending on whether the cycle C has a chord or not.

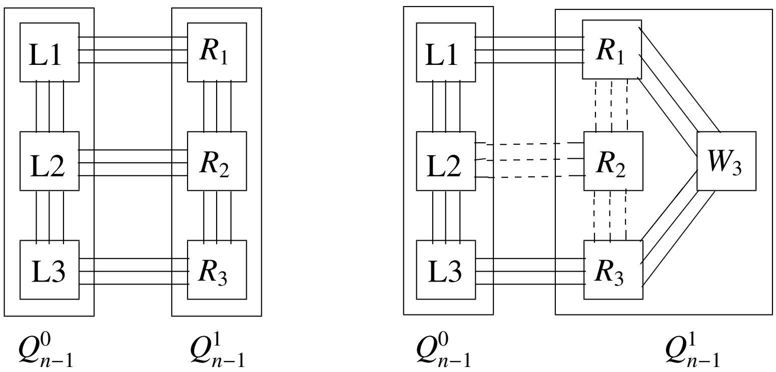
Subcase (i). Suppose C is chordless.

Then H_0 is $(k-1)$-regular and in fact, $H_0 \cong C\square Q_{k-3}$. Let W_1 be the subgraph of Q^1_{n-1} corresponding to H_0. Then $V(W_1) \subseteq V(H_1)$. If $V(W_1) = V(H_1)$, then $W_1 = H_1$. As $H \cong H_0\square K_2 \cong (C\square Q_{k-3})\square K_2 \cong C\square Q_{k-2}$, (ii) follows. Suppose $V(H_1) - V(W_1) \neq \emptyset$. Let W_2 be the subgraph of H_1 induced by $V(H_1) - V(W_1)$. As Q_n is triangle-free, it follows from Lemma 6 that every vertex of W_2 has at most three neighbours in W_1. This shows that $\delta(W_2) \geq k-3$. By Theorem 2, $|V(W_2)| \geq 2^{k-3}$. Thus $|V(H)| = |V(H_0)| + |V(W_1)| + |V(W_2)| \geq 3(2^{k-2}) + 3(2^{k-2}) + 2^{k-3} = 2^k + 2^{k-1} + 2^{k-3}$. Therefore (iii) holds in this case.

Subcase (ii). Suppose C has a chord.

Then C is a spanning subgraph of $P_3\square K_2$ and hence H_0 is a spanning subgraph of $P_3\square Q_{k-2}$. Let $1, 2, 3$ be the vertices of the path P_3 in order and let L_i the copy of Q_{k-2} in H_0 corresponding to the vertex i for $i \in \{1, 2, 3\}$. Let R_i be the subgraph of Q^1_{n-1} corresponding to L_i for $i \in \{1, 2, 3\}$. Then $R_i \cong Q_{k-2}$. Since the degree of every member of $V(L_1) \cup V(L_3)$ is $k-1$ in H_0, we have $V(R_1) \subseteq V(H_1)$ and $V(R_3) \subseteq V(H_1)$. If $V(H_1) = V(R_1) \cup V(R_2) \cup V(R_3)$, then (ii) holds (see Figure 3(a)). Suppose $V(H_1) - V(R_1) \cup V(R_2) \cup V(R_3)$ is non-empty and let W_3 be the subgraph of H_1 induced by this set. Then, by Lemma 6, $\delta(W_3) \geq k-2$. Therefore, by Theorem 2, $W_3 \cong Q_{k-2}$ or $|V(W_3)| \geq 2^{k-2} + 2^{k-3}$. In the later case, (iii) holds.

Suppose $W_3 \cong Q_{k-2}$. It follows from Lemma 6 that every vertex of W_3 has a neighbour in R_i for $i = 1, 3$ but no neighbour in R_2. Suppose $V(R_2) \cap V(H_1) = \emptyset$. Then $V(H_1) = V(R_1) \cup V(R_3) \cup V(W_3)$ (see Figure 3(b)). It follows that $H = Z\square Q_{k-2}$, where Z is a 6-cycle whose six vertices correspond to $L_1, L_2, L_3, R_3, W_3, R_1$ in order, and so, (ii) holds. Suppose $V(R_2) \cap V(H_1) \neq \emptyset$. Then the graph $R_2 \cap H_1$ has minimum degree at least $k-3$ and hence, it has at least 2^{k-3} vertices by Theorem 2. Thus $|V(H)| = |V(H_0)| + |V(R_1)| + |V(R_3)| + |V(W_3)| + |V(R_2 \cap H_1)| \geq 6(2^{k-2}) + 2^{k-3}$. Therefore (iii) holds. This completes the proof. □



a) $V(H_1) = V(R_1) \cup V(R_2) \cup V(R_3)$ b) $V(R_2) \cap V(H_1) = \emptyset$

Figure 3. The Subgraph H of Q_n

Corollary 1. *Every* 5*-regular subgraph of* Q_n *has* 32, 48 *or at least* 52 *vertices.*

4. Construction of 5-regular Subgraphs of Q_n

In this section, we give a construction of an l-vertex, 5-regular subgraph of the hypercube Q_n. Suppose Q_n has a 5-regular subgraph on l vertices. Then l is an even integer and by Corollary 1, we have $l \in \{32, 48\}$ or $52 \leq l \leq 2^n$. We prove that for every such l there exists a 5-regular subgraph of Q_n with $n \geq 6$ that is both 5-connected and bipancyclic. The case when l is a multiple of 4 follows trivially from the following result.

Theorem 4 ([8]). *If $n \geq 4$ and l is an even integer such that $l = 16$ or $24 \leq l \leq 2^n$, then there exists a l-vertex, 4-regular subgraph of Q_n that is both 4-connected and bipancyclic.*

Lemma 7. *If $n \geq 6$ and l is a multiple of 4 such that $l \in \{32, 48\}$ or $52 \leq l \leq 2^n$, then there exists a 5-regular, 5-connected and bipancyclic subgraph of Q_n on l vertices.*

Proof. We can write $l = 4m$, where $m = 8$ or m is an integer such that $12 \leq m \leq 2^{n-2}$. Express Q_n as $Q_{n-1}\Box K_2$. Since $n-1 \geq 4$, Q_{n-1} has a 4-regular, 4-connected and bipancyclic subgraph H on $2m$ vertices by Theorem 4. Therefore, by Lemma 1, the graph $H\Box K_2$ is a 5-regular and 5-connected subgraph of Q_n on $4m = l$ vertices. As H is bipancyclic, it has a Hamiltonian cycle and so has a Hamiltonian path. Hence, by Lemma 2, $H\Box K_2$ is bipancyclic. □

Suppose l is an even integer such that $52 \leq l \leq 2^n$ but not a multiple of 4. Then $n \geq 6$ and $54 \leq l \leq 2^n - 2$. We have the following four cases.

(*i*) $l = 16k + 2$ with $4 \leq k \leq 2^{n-4} - 1$

(*ii*) $l = 16k + 6$ with $3 \leq k \leq 2^{n-4} - 1$.

(*iii*) $l = 16k + 10$ with $3 \leq k \leq 2^{n-4} - 1$.

(*iv*) $l = 16k + 14$ with $3 \leq k \leq 2^{n-4} - 1$.

Case (i). $l = 16k + 2$ with $4 \leq k \leq 2^{n-4} - 1$.

In this case, $n \geq 7$ and $66 \leq l \leq 2^n - 14$. Write $Q_n = Q_{n-3}\Box Q_3$. By Lemma 5, Q_{n-3} contains a cycle C of length $2k$ and there is a vertex $g \in V(Q_{n-3}) - V(C)$ with four pairwise non-adjacent neighbours x, y, z, w in C. Let $V(Q_3) = \{a_0, a_1, \ldots, a_7\}$ so that a_0, a_1, a_2, a_3, a_0 and a_4, a_5, a_6, a_7, a_4 are 4-cycles and a_0a_7, a_1a_6, a_2a_5 and a_3a_4 are edges of Q_3. Then $C\Box Q_3$ is a 5-regular subgraph of Q_n with $16k$ vertices containing a copy C^i of C corresponding to the vertex a_i of Q_3. Let g_i, x_i, y_i, z_i, w_i be the vertices of Q^i_{n-3} corresponding to the vertices g, x, y, z, w, respectively.

Let L_1 be the subgraph of Q_n with $V(L_1) = \{x_i, y_i, z_i, w_i, g_i : i = 0, 7\}$ and $E(L_1) = \{g_ix_i, g_iy_i, g_iz_i, g_iw_i : i = 0, 7\} \cup \{g_0g_7\}$. We define a graph H_1 from $C\Box Q_3$ and L_1 as follows.

$$H_1 = (C\Box Q_3) \cup L_1 - \{x_0x_7, y_0y_7, z_0z_7, w_0w_7\} \quad \text{(see Figure 4)}.$$

Clearly, H_1 is a 5-regular subgraph of Q_n on $16k + 2 = l$ vertices.

Now, to construct 5-regular subgraphs in Cases (ii), (iii) and (iv), we choose a cycle C in Q_{n-3} of length $2k$ by Lemma 4. Then there are six vertices x, y, z, u, v, w on C and two vertices g, h in $V(Q_{n-3}) - V(C)$ such that g is adjacent to x, y, z, and h is adjacent to u, v, w, and xu, uy, yv are edges of C. Let

$$H = C\Box Q_3.$$

Then H is a 5-regular subgraph of Q_n on $16k$ vertices. As in Case (i), let C^i be a copy of C corresponding to the vertex a_i of Q_3 and let $g_i, h_i, x_i, y_i, z_i, u_i, v_i, w_i$ be the vertices of Q^i_{n-3} corresponding to g, h, x, y, z, u, v, w, respectively.

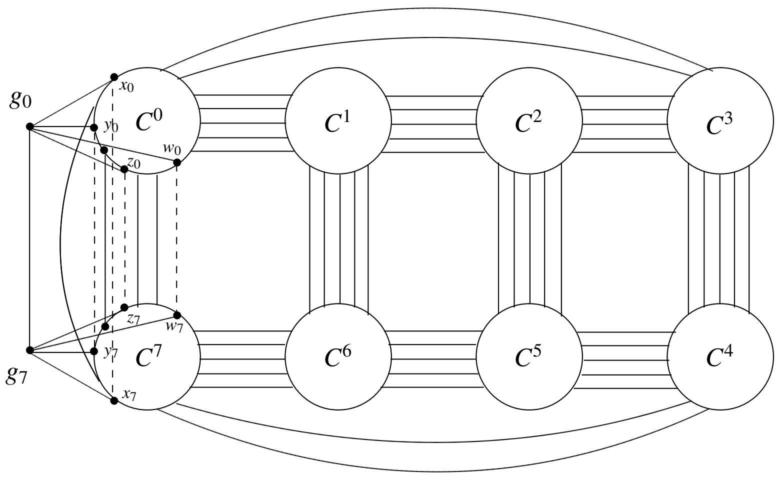


Figure 4. The Subgraph H_1 on $16k + 2$ Vertices

Case (ii). Suppose $l = 16k + 6$ with $3 \leq k \leq 2^{n-4} - 1$.

Clearly, $54 \leq l \leq 2^n - 10$. Let L_2 be the subgraph of Q_n with vertex set $\{g_i, x_i, y_i, z_i \colon i = 0, 1, 2, 5, 6, 7\}$ and edge set

$$\{g_i x_i, g_i y_i, g_i z_i \colon i = 0, 1, 2, 5, 6, 7\} \cup \{g_0 g_1, g_1 g_2, g_2 g_5, g_5 g_6, g_6 g_7, g_7 g_0\}.$$

Define a subgraph H_2 of Q_n from the graphs H and L_2 as follows.

$$H_2 = (H \cup L_2) - \{x_0 x_1, y_0 y_1, z_0 z_1, x_2 x_5, y_2 y_5, z_2 z_5, x_6 x_7, y_6 y_7, z_6 z_7\}.$$

The graph H_2 is depicted in Figure 5. It is easy to see that H_2 is a 5-regular subgraph of Q_n on l vertices.

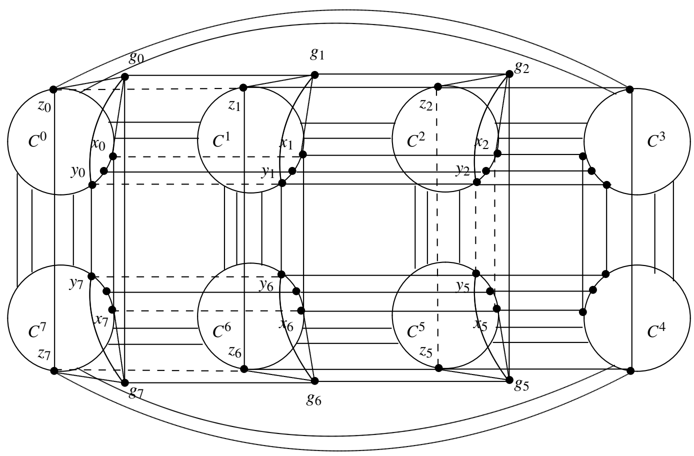


Figure 5. The Subgraph H_2 on $16k + 6$ Vertices

Case (iii). $l = 16k + 10$ with $3 \leq k \leq 2^{n-4} - 1$.

Consider the three edge sets: $F_1 = \{g_0 g_1, g_1 g_2, g_2 g_5, g_5 g_6, g_6 g_7, g_7 g_0\}$, $F_2 = \{h_2 h_3, h_3 h_4, h_4 h_5, h_5 h_2\}$, and

$$F_3 = \{x_0 x_7, y_0 y_7, z_0 z_7, x_1 x_2, y_1 y_2, z_1 z_2, x_5 x_6, y_5 y_6, z_5 z_6, u_2 u_3, v_2 v_3, w_2 w_3, u_4 u_5, v_4 v_5, w_4 w_5\}.$$

Let L_3 be the subgraph of Q_n with vertex set $\{g_i, x_i, y_i, z_i \colon i = 0, 1, 2, 5, 6, 7\} \cup \{h_j, u_j, v_j, w_j \colon j = 2, 3, 4, 5\}$ and edge set

$$\{g_i x_i, g_i y_i, g_i z_i \colon i = 0, 1, 2, 5, 6, 7\} \cup \{h_j u_j, h_j v_j, h_j w_j \colon j = 2, 3, 4, 5\} \cup F_1 \cup F_2.$$

We now define a subgraph H_3 of Q_n which is shown in Figure 6 as

$$H_3 = (H \cup L_3) - F_3$$

Clearly, H_3 is a 5-regular subgraph of Q_n on $16k + 10$ vertices.

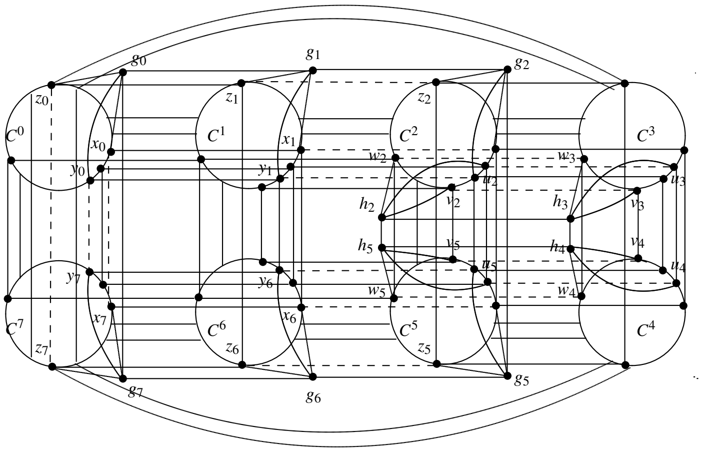


Figure 6. The subgraph H_3 on $16k + 10$ vertices

Case (iv). $l = 16k + 14$ with $3 \leq k \leq 2^{n-4} - 1$.

We define the four edge sets M_1, M_2, M_3, M_4 as follows.

$$M_1 = \{g_0g_1, g_1g_2, g_2g_3, g_3g_4, g_4g_5, g_5g_6, g_6g_7, g_7g_0\};$$

$$M_2 = \{h_0h_1, h_1h_2, h_2h_5, h_5h_6, h_6h_7, h_7h_0\};$$

$$M_3 = \{u_0u_7, v_0v_7, w_0w_7, u_1u_6, v_1v_6, w_1w_6, u_2u_5, v_2v_5, w_2w_5\}$$

and

$$M_4 = \{x_0x_1, y_0y_1, z_0z_1, x_2x_3, y_2y_3, z_2z_3, x_4x_5, y_4y_5, z_4z_5, x_6x_7, y_6y_7, z_6z_7\}.$$

Let L_4 be the subgraph of Q_n having vertex set $\{g_i, x_i, y_i, z_i \colon i = 0, 1, \ldots, 7\} \cup \{h_j, u_j, v_j, w_j \colon j = 0, 1, 2, 5, 6, 7\}$ and edge set $\{g_ix_i, g_iy_i, g_iz_i \colon i = 0, 1, \ldots, 7\} \cup \{h_ju_j, h_jv_j, h_jw_j \colon j = 0, 1, 2, 5, 6, 7\} \cup M_1 \cup M_2$. We define a subgraph H_4 of Q_n as follows.

$$H_4 = (H \cup L_4) - (M_3 \cup M_4).$$

The graph H_4 is shown in Figure 7. Clearly, it is a 5-regular subgraph of Q_n on $16k + 14$ vertices.

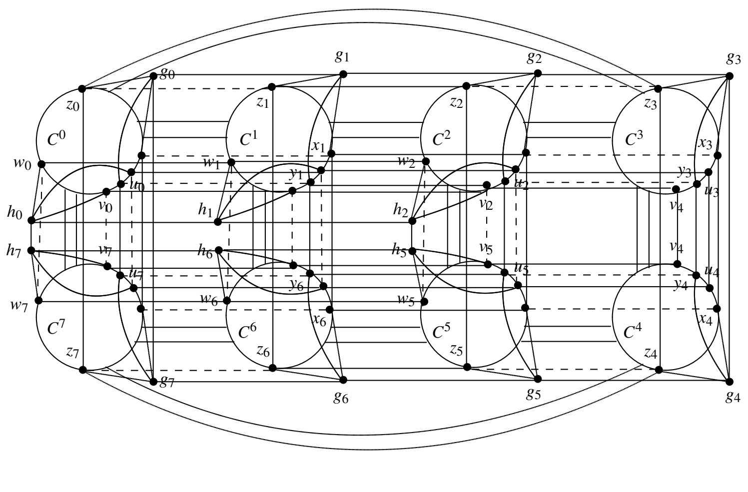


Figure 7. The subgraph H_4 on $16k + 14$ vertices

Thus we have constructed 5-regular subgraphs H_1, H_2, H_3, and H_4 of the hypercube Q_n on l vertices in each of the four cases. In the next two sections, we prove that these subgraphs are 5-connected and bipancyclic.

5. Connectivity

In this section, we prove that all the four subgraphs H_1, H_2, H_3, and H_4 of Q_n that are constructed in Section 4 and shown in Figures 4, 5, 6, and 7, respectively are 5-connected.

If F is the matching in H_i consisting of edges having one end vertex in the lower side (L) and the other end vertex in the upper side (R) in the figure of H_i, then F is an edge-cut of H_i. The graph $H_i - F$ has two components and further, the components are 4-connected and isomorphic to each other. We prove that these components together with the matching F give a 5-connected graph. We first prove the following observations for general graphs.

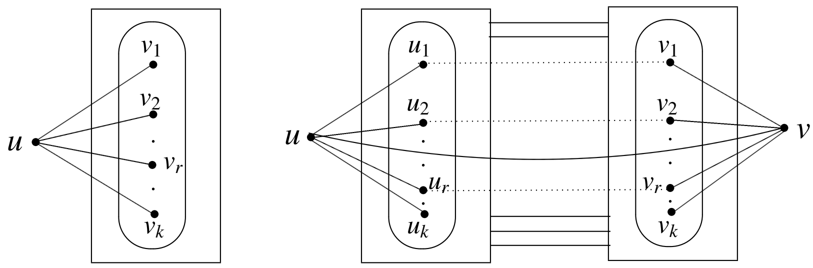
Lemma 8. *Let G be a simple k-connected graph and let $v_1, v_2, \ldots, v_k$ be distinct vertices of G. Let $\hat{G}$ be a new graph obtained from G by adding a new vertex u and k edges uv_i for $i = 1, 2, \ldots, k$. Then $\hat{G}$ is k-connected.*

Proof. The graph $\hat{G}$ is shown in Figure 8(a). Suppose $S \subseteq V(\hat{G})$ with $|S| < k$. If $u \notin S$, then $S \subseteq V(G)$ and so $G - S$ is connected. Therefore $\hat{G} - S$ is connected as the vertex u has a neighbour in $G - S$. Suppose $u \in S$. Then $S - \{u\}$ contains at most $k - 2$ vertices of G and hence $G - (S - \{u\}) = \hat{G} - S$ is connected. Thus $\hat{G}$ is k-connected. □

Lemma 9. *Let G be a simple k-connected graph with at least $2k$ vertices and an independent set $\{u_1, u_2, \ldots, u_r\}$, where $1 \leq r \leq k$. Suppose G' is a simple graph obtained from G by adding a new vertex u and k new edges each having one end vertex u including the r edges $uu_1, uu_2, \ldots, uu_r$. Let H be the graph obtained from the graph $G' \square K_2$ by deleting the matching consisting of r edges between the two copies of G' corresponding to the vertices $u_1, u_2, \ldots, u_r$. Then the graph H is $(k+1)$-connected.*

Proof. The graph H is shown in Figure 8(b). The graph $G' \square K_2$ is obtained from G' and a copy G'' of G' by adding edges between their corresponding vertices. Let v and v_i be the vertices of G'' corresponding to u and u_i for $1 \leq i \leq r$, respectively and let $M = \{u_1v_1, u_2v_2, \ldots, u_rv_r\}$. Then $H = (G' \square K_2) - M$. Since G' has at least $2k + 1$ vertices, there are at least $k + 1$ edges in H between G' and G''.

Suppose $S \subseteq V(H)$ with $|S| \leq k$. It is sufficient to prove that $H - S$ is connected. Since G is k-connected, by Lemma 8, the graph G' is k-connected. Suppose S intersects both $V(G')$ and $V(G'')$. Then both $G' - S$ and $G'' - S$ are connected and they are joined to each other by an edge. Hence $H - S$ is connected. Suppose $S \subseteq V(G')$. The degree of u_i in G is at least k and so it is at least $k + 1$ in G' for any $i \in \{1, 2, \ldots, r\}$. If $G' - S$ has a component with vertex set a subset of the independent set $\{u_1, u_2, \ldots, u_r\} - S$, then that component has just one vertex and so $|S| = k + 1$, a contradiction. Thus every component of $G' - S$ has a neighbour in the connected graph G'' in $H - S$. This shows that $H - S$ is connected. □



(a): G' (b): H

Figure 8. The Graphs of Lemma 8 and Lemma 9

Let H_1, H_2, H_3, and H_4 be the 5-regular subgraphs of Q_n constructed in Section 4. We now prove that these graphs are 5-connected.

Lemma 10. *The graph H_1 is 5-connected.*

Proof. The subgraphs of H_1 induced by $V(C^0) \cup V(C^1) \cup V(C^2) \cup V(C^3)$ and by $V(C^4) \cup V(C^5) \cup V(C^6) \cup V(C^7)$ are isomorphic to $C^0 \square Z$ for some 4-cycle Z. Hence, by Lemma 1, these two subgraphs are 4-connected. Now, by Lemma 9, H_1 is 5-connected. □

Lemma 11. *The graph H_2 is 5-connected.*

Proof. The subgraphs of H_2 induced by $V(C^0) \cup V(C^3)$ and by $V(C^1) \cup V(C^2)$ are isomorphic to $C^0 \square K_2$ and so, by Lemma 1, they are 3-connected. Hence, by Lemma 9, the upper half subgraph R of H_2 that is induced by $V(C^0) \cup V(C^1) \cup V(C^2) \cup V(C^3) \cup \{g_0, g_1\}$ is 4-connected. If $L = H_2[V(C^4) \cup V(C^5) \cup V(C^6) \cup V(C^7) \cup \{g_6, g_7\}]$, then L is isomorphic to R. Thus, by Lemma 9, H_2 is 5-connected. □

Lemma 12. *The graph H_3 is 5-connected.*

Proof. By Lemmas 1 and 9, the subgraphs $H_3[V(C^0) \cup V(C^3)]$ and $H_3[V(C^1) \cup V(C^2) \cup \{g_1, g_2\}]$ of H_3 are 3-connected. By similar arguments of the proof of Lemma 11, we see that the upper half subgraph R of H_3 induced by $V(C^0) \cup V(C^1) \cup V(C^2) \cup V(C^3) \cup \{g_1, g_2, h_2, h_3\}$ is 4-connected. If $L = H_3[V(C^4) \cup V(C^5) \cup V(C^6) \cup V(C^7) \cup \{g_5, g_6, h_4, h_5\}]$, then L is isomorphic to R. Now, the result follows from Lemma 9. □

Lemma 13. *The graph H_4 is 5-connected.*

Proof. Let $V_1 = V(C^0) \cup V(C^1) \cup \{g_0, g_1\}$ and let $V_2 = V(C^2) \cup V(C^3) \cup \{g_2, g_3\}$. By Lemma 9, the subgraphs $H_4[V_1]$ and $H_4[V_2]$ of H_4 are 3-connected. Therefore the graph $H_4[V_1 \cup V_2]$ is 4-connected. Hence the graph $R = H_4[V_1 \cup V_2 \cup \{h_0, h_1, h_2\}]$ is also 4-connected. The subgraph L of H_4 induced by $V(H_4) - V(R)$ is isomorphic to R and so, it is 4-connected.

We now prove that H_4 is 5-connected. Suppose $S \subseteq V(H_4)$ with $|S| \leq 4$. It is sufficient to prove that $H_4 - S$ is connected. If S intersects both $V(R)$ and $V(L)$, then $R - S$ and $L - S$ are connected and they are joined to each other by an edge leaving $H_4 - S$ connected. Suppose $S \subseteq V(R)$. The set of vertices of R that do not have any neighbour in $V(L)$ is $A = \{h_1, g_1, g_2, u_0, v_0, w_0, u_1, v_1, w_1, u_2, v_2, w_2\}$. Assume that $R - S$ has a component D such that $V(D) \subseteq A$. No member of A is isolated in $R - S$ as each has degree five in R. Hence $\delta(D) \geq 1$. Observe that the subgraph of H_4 induced by A is the forest as shown in Figure 9. Therefore D is a tree containing at least two pendant vertices. So $|S| = 4$ and every pendant vertex of D is adjacent to all the four members of S. This gives $K_{2,3}$ as a subgraph of H_4 and so of Q_n, a contradiction. Hence every component of $R - S$ has a neighbour in the connected graph L. Therefore $H_4 - S$ is connected. Similarly, $H_4 - S$ is connected if $S \subseteq V(L)$. □

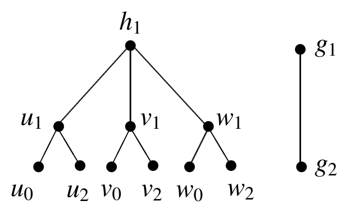


Figure 9. Subgraph of H_4 Induced by A

6. Bipancyclicity

In this section, we prove that the 5-regular graphs H_1, H_2, H_3, and H_4 constructed in Section 4 are all bipancyclic.

A ladder on $n \geq 4$ vertices has two edges at its two ends. If we identify one of these two edges with an edge of a k-cycle, then it follows that the resulting graph has cycles of every even length from k to $k + n - 2$. The following lemma is based on this fact.

Lemma 14. *Let l, m, n be even integers and C be an m-cycle conatining an edge xy and L be a ladder on n vertices containing an end edge $x'y'$. Then the graph $C \cup L \cup \{xx', yy'\}$ has cycles of all even lengths from $m + 2$ to $m + n$ (see Figure 10).*

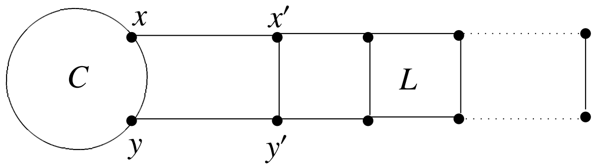


Figure 10. The Graph $C \cup L \cup \{xx', yy'\}$

Lemma 15. *The graph H_1 is bipancyclic.*

Proof. Recall that H_1 contains the eight copies $C^0, C^1, \ldots, C^7$ of a $2k$-cycle C. Let E_{ij} be the set of edges of H_1 with one end vertex in the cycle C^i and the other in the cycle C^j. Consider the subgraph $W = C^0 \cup C^1 \cup C^2 \cup C^3 \cup C^4 \cup C^5 \cup C^6 \cup C^7 \cup E_{03} \cup E_{32} \cup E_{25} \cup E_{54} \cup E_{47} \cup E_{76} \cup E_{61} \cup E_{10}$ of H_1. Then W is isomorphic to $C \square Z$, where Z is an 8-cycle. By Lemma 2, W is bipancyclic. Therefore W and so H_1 contains a cycle of every even length from 4 to $16k = |V(W)|$. Let $C^i =< v_1^i, v_2^i, \ldots v_{2k}^i, v_1^i >$ for $i = 0, 1, \ldots 7$ such that v_1^i is the label to the neighbor of g_i. Then the cycle shown in Figure 11 is a Hamiltonian cycle of H_1. Thus the graph H_1 is bipancyclic. □

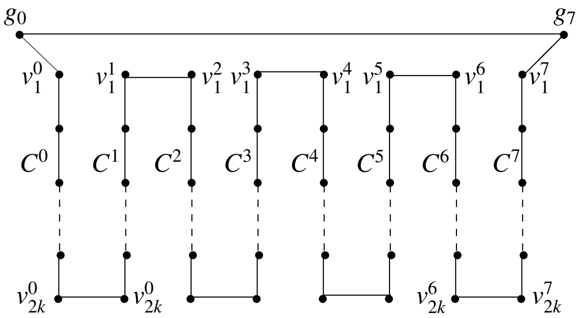


Figure 11. Hamiltonian Cycle in H_1

Recall that we have written Q_n as $Q_{n-3} \square Q_3$ where the vertices of Q_3 are labeled by $a_0, a_1, \ldots, a_7$, so that $a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_0$ is a Hamiltonian cycle of Q_3 with chords a_0a_3, a_1a_6, a_2a_5 and a_4a_7. Furthermore, C is a $2k$-cycle in Q_{n-3} and C^i is its copy in Q_n corresponding to vertex a_i of Q_3 in the construction of the graphs H_2, H_3 and H_4. Let E_{ij} be the matching between cycles C^i and C^j in Q_n corresponding to the edge a_ia_j of Q_3. For each i, label the vertices of C^i by $v_1^i, v_2^i, \ldots, v_{2k}^i$ so that $C^i =< v_1^i, v_2^i, v_3^i, v_4^i, \ldots, v_p^i, \ldots, v_q^i, \ldots, v_{2k}^i, v_1^i >$ and v_1^i, v_3^i, v_p^i are the neighbours of the vertex g_i, and v_2^i, v_4^i, v_q^i are the neighbours of the vertex h_i for some p, q with $4 < p, q \leq 2k$ and $p \neq q$. Hence the neighbours x_i, y_i, z_i of g_i on C^i are relabeled as v_1^i, v_3^i, v_p^i, respectively. Similarly, the vertices u_i, v_i, w_i are relabeled as v_2^i, v_4^i, v_q^i, respectively.

Lemma 16. *The graph H_2 is bipancyclic.*

Proof. The graph H_2 has $16k + 6$ vertices. Consider the subgraph W of H_2, where $W = C^0 \cup C^1 \cup C^2 \cup C^3 \cup C^4 \cup C^5 \cup C^6 \cup C^7 \cup E_{03} \cup E_{32} \cup E_{21} \cup E_{16} \cup E_{65} \cup E_{54} \cup E_{47} \cup E_{70}$. Then W_2 is isomorphic to $C^0 \square Z$, where Z is an 8-cycle. By Lemma 2, it contains cycles of every even length from 4 to $16k$. A cycle Z of length $16k + 6$ in H_2 is shown in Figure 12. Furthermore, $Z' = (Z - \{g_1, g_6\}) \cup \{v_1^1 v_1^6, g_0g_7\}$ is a cycle of length $16k + 4$ while $Z'' = (Z' - \{g_0, g_7\}) \cup \{v_1^0 v_1^7\}$ is a cycle of length $16k + 2$ in H_2. Thus H_2 is bipancyclic. □

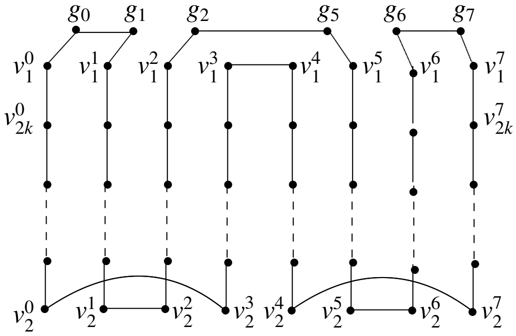
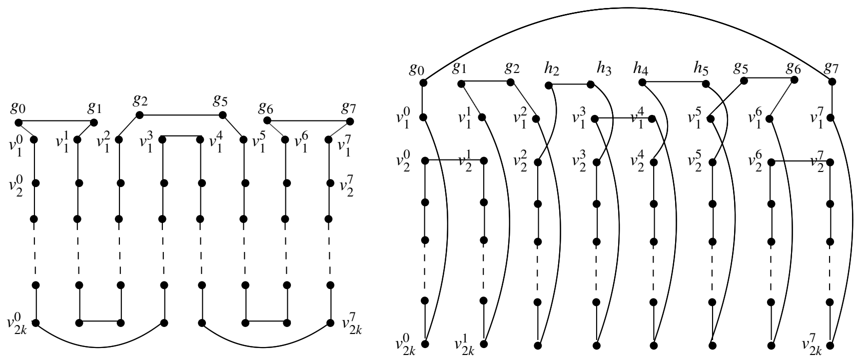


Figure 12. $(16k + 6)$-cycle in H_2



a) $l = 16k + 6$ b) $l = 16k + 10$

Figure 13. l-cycles in H_3 for $l \in \{16k + 6, 16k + 10\}$

Lemma 17. *The graph H_3 is bipancyclic.*

Proof. The graph H_3 is on $16k + 10$ vertices. Consider the subgraph W of H_3 on $12k$ vertices defined as

$$W = C^0 \cup C^1 \cup C^3 \cup C^4 \cup C^6 \cup C^7 \cup E_{03} \cup E_{34} \cup E_{47} \cup E_{67} \cup E_{16} \cup E_{01}.$$

Then W is isomorphic to $C^0 \square Z$, where Z is a 6-cycle. Hence W contain cycles of all even lengths from 4 to $12k$. Therefore these cycles are also contained in H_3. Let L be the ladder on $4k$ vertices defined as

$$L = (C^2 - v_1^2 v_2^2) \cup (C^5 - v_1^5 v_2^5) \cup \{v_i^2 v_i^5 : i = 1, 2, \ldots, 2k\}.$$

Note that W has a Hamiltonian cycle $Z_1 =< v_2^1, v_3^1, \ldots, v_{2k}^1, v_1^1, v_1^0, v_{2k}^0, v_{2k-1}^0, \ldots, v_2^0, v_2^3, \ldots, v_{2k}^3, v_1^3, v_1^4, v_{2k}^4, \ldots, v_2^4, v_2^7, \ldots, v_{2k}^7, v_1^7, v_1^6, v_{2k}^6, \ldots, v_2^6, v_2^1 >$. Then the subgraph $Z_1 \cup L \cup \{v_2^2 v_2^1, v_2^5 v_2^6\}$ of H_3 has $16k$ vertices. By Lemma 14, it has cycles of every even length from $12k + 2$ to $16k$. Let Z_2 and Z_3 be the cycles of length $16k + 6$ and $16k + 10$ as shown in Figure 13(a) and 13(b), respectively. Then $Z_2' = (Z_2 - \{g_0, g_1\}) \cup \{v_1^0 v_1^1\}$ is a cycle of length $(16k + 4)$ whereas $(Z_2' - \{g_6, g_7\}) \cup \{v_1^6 v_1^7\}$ is a $(16k + 2)$-cycle in H_3. Finally, $(Z_3 - \{h_3, h_4\}) \cup \{v_2^3 v_2^4, h_2 h_5\}$ is a cycle of length $16k + 8$ in H_3. Thus H_3 is bipancyclic. □

Lemma 18. *The graph H_4 is bipancyclic.*

Proof. Recall that H_4 has $16k + 14$ vertices. We prove that H_4 contain cycles of every even length from 4 to $16k + 14$. The subgraph of H_4,

$$W = C^0 \cup C^3 \cup C^4 \cup C^7 \cup E_{03} \cup E_{34} \cup E_{47}$$

is bipancyclic as it is isomorphic to $C^0 \square P_4$, where P_4 is a path on four vertices. This implies that H_4 contain cycles of every even length from 4 to $8k$. Consider the Hamiltonian cycle Z_1 of W, where $Z_1 =< v_2^0, v_3^0, \cdots, v_{2k}^0, v_1^0, v_1^3, v_{2k}^3, v_{2k-1}^3, \cdots, v_{k+2}^3, v_{k+2}^4, v_{k+3}^4, \cdots, v_{2k}^4, v_1^4, v_1^7, v_{2k}^7, v_{2k-1}^7, \cdots, v_2^7, v_2^4, v_3^4, \cdots, v_{k+1}^4, v_{k+1}^3, v_k^3, \cdots, v_2^3, v_2^0 >$. We get two paths P and Q each of length $4k$ from $C^1 \cup C^6$ and $C^2 \cup C^5$, respectively as follows.

$$P =< v_2^1, v_3^1, \ldots, v_{2k}^1, v_1^1, v_1^6, v_{2k}^6, v_{2k-1}^6, \ldots, v_2^6 >,$$

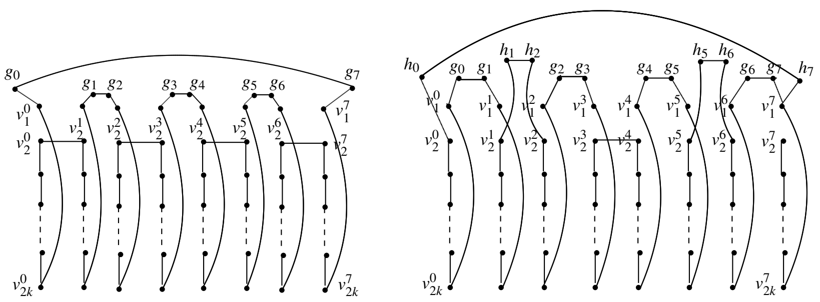
$$Q =< v_2^2, v_3^2, \ldots, v_{2k}^2, v_1^2, v_1^5, v_{2k}^5, v_{2k-1}^5, \ldots, v_2^5 > .$$

Then

$$L_1 = P \cup Q \cup (\{v_i^1 v_i^2, v_i^5 v_i^6 : i = 1, 2, \ldots, 2k\})$$

is a ladder on $8k$ vertices. Hence, by Lemma 14, $Z_1 \cup L_1 \cup \{v_2^0 v_2^1, v_2^2 v_2^3\}$ is a subgraph of H_4 on $16k$ vertices contains cycle of length p for every even integer p with $8k + 2 \leq p \leq 16k$.

Let Z_2 and Z_3 be the cycles in H_4 of length $16k + 8$ and $16k + 14$ as shown in Figures 14(a) and 14(b), respectively. If $Z_2' = (Z_2 - \{g_5, g_6\}) \cup \{v_1^5 v_1^6\}$; $Z_2'' = (Z_2' - \{g_3, g_4\}) \cup \{v_1^3, v_1^4\}$ and $Z_2''' = (Z_2'' - \{g_1, g_2\}) \cup \{v_1^1, v_1^2\}$, then Z_2', Z_2'' and Z_2''' are cycles of lengths $16k + 6$, $16k + 4$ and $16k + 2$, respectively. Finally, $Z_3' = (Z_3 - \{h_5, h_6\}) \cup \{v_2^5 v_2^6\}$ and $Z_3'' = (Z_3' - \{h_1, h_2\}) \cup \{v_2^1 v_2^2\}$ are cycles of lengths $16k + 12$ and $16k + 10$, respectively. Hence H_4 is bipancyclic. □



(a) $l = 16k + 8$ (b) $l = 16k + 14$

Figure 14. l-cycles in H_4 for $l \in \{16k + 8, 16k + 14\}$

Remark 1. *The problem of the existence of a k-regular subgraph of the hypercube Q_n of a given order is completely solved for $k = 3,\ 4,\ 5$. To solve the problem for the general value of k one needs to identify the even integers in between 2^k and 2^n that cannot be the order of any k-regular subgraph of Q_n. By Theorem 3, we have two intervals $(2^k,\ 2^k + 2^{k-1})$ and $(2^k + 2^{k-1},\ 2^k + 2^{k-1} + 2^{k-3})$ with the property that no even integer belonging to any of these intervals is the order of a k-regular subgraph of Q_n. There is no further gap for $k \in \{3, 4, 5\}$. However, there seems a further gap for $k \geq 6$. The next interval of the gaps can be found by characterizing the k-regular subgraph of Q_n on $2^k + 2^{k-1} + 2^{k-3}$ vertices. Thus, Problem 1 remains open for $k \geq 6$.*

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Conflict of Interest

The authors declare no conflict of interest

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