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Article

Algorithmic Aspects of Vertex-edge Domination in Some Graphs

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Abstract: Let $G = (V, E)$ be a simple graph. A vertex $v \in V(G)$ *ve*-dominates every edge uv incident to v, as well as every edge adjacent to these incident edges. A set $D \subseteq V(G)$ is a vertex-edge dominating set if every edge of $E(G)$ is *ve*-dominated by a vertex of D. The MINIMUM VERTEX-EDGE DOMINATION problem is to find a vertex-edge dominating set of minimum cardinality. A linear time algorithm to find the minimum vertex-edge dominating set for proper interval graphs is proposed. The vertex-edge domination problem is proved to be APX-complete for bounded-free graphs and NP-Complete for Chordal bipartite and Undirected Path graphs.

Keywords: Vertex-edge dominating set, Chordal Bipartite graph, Undirected Path graph, Proper interval graph, NP-complete, APX-complete
Mathematics Subject Classification: 05C69, 05C85

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1. Introduction

Let $G = (V, E)$ be a simple connected graph of order n and size m. By an *open neighborhood* of a vertex v of G, we mean the set $N_G(v) = \{u \in V(G) \colon uv \in E(G)\}$ and the *closed neighborhood*, $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex v, denoted by $d_G(v)$, is the cardinality of an open neighborhood. By a end vertex we mean a vertex of degree one, while a support vertex is a vertex adjacent to a end vertex. We denote by P_n, a *path* of order n. A closed path is a cycle. For a set $S \subseteq V$, the *subgraph* of G induced by S is defined as $G[S] = (S, E_S)$, where $E_S = \{xy : xy \in E(G), x, y \in S\}$. A set of vertices S is a *clique* in G if $G[S]$ is a maximal complete subgraph of G.

A graph G is a *chordal graph* if every cycle in G of length at least 4 has a chord. Let $\mathcal{F}$ be a nonempty family of sets. A graph $G = (V, E)$ is called an intersection graph for a finite family $\mathcal{F}$ of a nonempty set if there is a one-to-one correspondence between $\mathcal{F}$ and V such that two sets in $\mathcal{F}$ have nonempty intersection if and only if their corresponding vertices in V are adjacent. We call $\mathcal{F}$ an intersection model of G. For an intersection model $\mathcal{F}$, we use $G(\mathcal{F})$ to denote the intersection graph for $\mathcal{F}$. If $\mathcal{F}$ is a family of intervals on a real line, then $G(\mathcal{F})$ is called an *interval graph* for $\mathcal{F}$ and $\mathcal{F}$ is called an interval model of G. If $\mathcal{F}$ is a family of intervals on a real line such that no interval in $\mathcal{F}$ properly contains an other interval in $\mathcal{F}$, then $G(\mathcal{F})$ is called a *proper interval graph* for $\mathcal{F}$ and $\mathcal{F}$ is called a proper interval model of G.

A vertex $v \in V(G)$ is a *simplicial vertex* of G if $N_G[v]$ is a clique of G. An ordering $\alpha = (v_1, v_2, \ldots, v_n)$ is a *perfect elimination ordering* (PEO) of G if v_i is a simplicial vertex of $G_i = G[v_i, v_{i+1}, \ldots, v_n]$ for all i, $1 \leq i \leq n$. A PEO $\alpha = (v_1, v_2, \ldots, v_n)$ of a chordal graph is a *bi-compatible* *elimination ordering* (BCO) if $\alpha^{-1} = (v_n, v_{n-1}, \ldots, v_1)$ is also a PEO of G. This implies that v_i is simplicial in $G[v_1, v_2, \ldots, v_i]$ as well as in $G[v_i, v_{i+1}, \ldots, v_n]$. A graph G is chordal if and only if it has a PEO and proper interval graphs are characterized in terms of BCO, see [1].

A set S of vertices is a *dominating set*, abbreviated DS, of G if every vertex not in S is adjacent to some vertex in S. The *domination number* of a graph G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. The MINIMUM DOMINATION problem is to find a dominating set of minimum cardinality. For more details on domination and its variants, see [2,3]. The decision version of domination problem is defined as follows:

Domination Problem

INSTANCE: A graph $G = (V, E)$ and a positive integer $k \le |V|$.
QUESTION: Does G have a dominating set of cardinality at most k?

A vertex $v \in V(G)$ vertex-edge dominates every edge uv incident to v, as well as every edge adjacent to these incident edges. A set $S \subseteq V$ is a *vertex-edge dominating set* (or simply, a *ve-dominating set*) if for every edge $e \in E$, there exists a vertex $v \in S$ such that v *ve*-dominates e. The *vertex-edge domination number* of a graph G, denoted by $\gamma_{ve}(G)$, is the minimum cardinality of a *ve*-dominating set of G. The *Minimum Vertex-edge domination problem* is to find a vertex-edge dominating set of minimum cardinality. The concept of vertex-edge domination was introduced by Peters [4] and studied further in [5–8]. The decision version of vertex-edge domination problem is defined as follows:

Vertex-edge Domination Problem

INSTANCE: A graph $G = (V, E)$ and a positive integer $k \le |V|$.
QUESTION : Does G have a vertex-edge dominating set of cardinality at most k?

Lewis [7] showed that the decision version of vertex-edge domination problem is NP-complete for bipartite graphs, chordal graphs and planar graphs and it is linearly solvable for trees.

An edge $e \in E(G)$ edge-vertex dominates a vertex $v \in V(G)$ if e is incident with v or e is incident with a vertex adjacent to v. A subset $D \subseteq E(G)$ is an edge-vertex dominating set, abbreviated EVDS, of a graph G if every vertex of G is edge-vertex dominated by an edge of D. The edge-vertex domination number of G, denoted by $\gamma_{ev}(G)$, is the minimum cardinality of an edge-vertex dominating set of G. An edge-vertex dominating set of G of minimum cardinality is called a $\gamma_{ev}(G)$-set. Edge-vertex domination in graphs was introduced in [4] and studied further in [7, 9, 10].

Approximation, Hardness and APX-complete results of Restrained domination and secure domination are given in [11, 12]. In section 2, we give the complexity difference between minimum domination problem and minimum vertex-edge domination problem. In section 3, it has been proved that minimum vertex-edge domination problem is NP-complete for Chordal bipartite and Undirected Path graphs. In the fourth section, a linear time algorithm is presented to find a minimum vertex-edge dominating set of proper interval graphs. In the final section, the minimum vertex-edge dominating set is proved APX-complete for graphs with maximum degree 7. From this APX-completeness result, we conclude that there is an approximation algorithm for the minimum vertex-edge domination problem for graphs with degree at most 7 with an approximation factor at least 1.0025694.

2. Preliminary Result

Every dominating set is a vertex-edge dominating set and we have the following:

Proposition 1. *[4] For any graph G, $\gamma_{ve}(G) \le \gamma(G)$.*

We now present some complexity difference between domination problem and vertex-edge domination problem. We define a graph class, called $GP3$-graphs, for which the decision version of domination problem is NP-Complete, but the MINIMUM VERTEX-EDGE DOMINATION problem is easily solvable.

Definition 1. *A graph $G = (V, E)$ is a $GP3$-graph if it can be obtained from a general connected graph $H = (V', E')$ where $V' = \{v_1, v_2, \ldots, v_{n'}\}$, by adding a path P_3, say $x_iy_iz_i$ to every vertex v_i of H.*

Let G be a $GP3$-graph of order $n = 4n'$ as constructed above. Let $V_i = \{v_i, x_i, y_i, z_i\}$ for $i = 1, 2, \ldots, n'$. If D is a vertex-edge dominating set of G, then the set D contains at least one vertex from each set V_i, to ve-dominate the edges incident to vertices of V_i. Thus, $\gamma_{ve}(G) \geq n'$. However, the set $\{x_i : 1 \leq i \leq n'\}$ is a VED-set of G and so $\gamma_{ve}(G) \leq n'$. Consequently, $\gamma_{ve}(G) = n' = \frac{n}{4}$. Thus we have the following observation.

Observation 1. *If G is $GP3$-graph, then $\gamma_{ve}(G) = \frac{|V(G)|}{4}$.*

Lemma 1. *If G is a $GP3$-graph constructed from a graph H, then H has a dominating set of cardinality k if and only if G has a dominating set of cardinality $n' + k$.*

Proof. Let D be a dominating set of H and $|D| = k$. Then $D \cup \{y_i : 1 \leq i \leq k\}$ is a dominating set of G of cardinality $n' + k$.

Conversely, Suppose that D' is a dominating set of G with cardinality $n' + k$. In order to dominate the vertices z_i, the dominating set D' contains the vertices y_i. It is easy to see that $D \setminus \{y_i : 1 \leq i \leq n'\}$ is a dominating set of H of cardinality at most k. □

Since the decision version of the domination problem is known to be NP-complete for general graphs [13], the following theorem follows directly from Lemma 1.

Theorem 1. *The decision version of the domination problem is NP-complete for $GP3$-graphs.*

3. NP-completeness Result

In [7], lewis proved the NP-Completeness for the class of chordal graphs, the same is found to be true for subclass of chordal graphs, namely chordal bipartite graphs and undirected path graphs. To this we construct a new graph G' from a connected graph G as follows : Let $G = (V, E)$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. We construct a graph $G' = (V', E')$ from G such that $V' = V(G') = V \cup \{a_i, b_i, c_i, d_i, e_i, f_i : 1 \leq i \leq n\}$ and $E' = E(G') = E \cup \{v_ia_i, a_ib_i, b_ic_i, c_id_i, c_ie_i, e_if_i : 1 \leq i \leq n\}$.

One can refer [14] for the definitions of Chordal Bipartite and Undirected Path graph. The following lemma is easy to prove.

Lemma 2. *Let graph G' be constructed from a graph G.*

- *If G is Chordal Bipartite, then G' is also Chordal Bipartite*
- *If G is Undirected Path graph, then G' is also Undirected Path graph.*

Now, we present a polynomial reduction of Vertex-edge domination problem from the domination problem.

Lemma 3. *If G has a dominating set of cardinality k if and only if G' has a vertex-edge dominating set of cardinality $k + n$.*

Proof. Let D be a minimum dominating set of G. The vertices in D ve-dominate the edges of G and the edges v_ia_i. Thus $D \cup \{c_i : 1 \leq i \leq n\}$ is a vertex-edge dominating set of G' of cardinality $k + n$.

Conversely, Let D' be a minimum vertex-edge dominating set of G'. To ve-dominate the edge e_if_i, the vertex c_i should belongs to D'. The vertex $c_i \in D'$ also ve-dominates the edges $a_ib_i, b_ic_i, c_id_i, d_ie_i$.

To ve-dominate the edge $v_i a_i$ either v_i or a_i or b_i or a neighbor of v_i other than a_i should belong to D'. If $x_i \in D'$ where $x_i = a_i$ or b_i, then we can define $D'' = (D' \cup \{v_i\}) \setminus \{x_i\}$ and D'' is still a ve-dominating set of G'. Thus without loss of generality, the set D' contains the vertices from $V(G)$ and $\{c_i : 1 \leq i \leq n\}$. Let $D^* = D' \cap V(G)$.

Claim: D^* is a dominating set of G.

Suppose there exist $v_i \in V(G) \setminus D^*$ such that $uv_i \notin E$ for any $u \in D^*$, the edge $v_i a_i$ is not ve-dominated by any vertex of D', a contradiction. Thus $|D^*| = |D'| - n = k$. □

The decision version of domination problem is NP-complete for Chordal Biparite [15] and Undirected Path graphs [16]. From Lemma 2 and Lemma 3, we have the following theorem;

Theorem 2. *The Vertex-edge domination problem is NP-complete for Chordal Bipartite and Undirected Path graphs.*

4. Algorithm

Lewis [7], proved that the vertex-edge domination problem is NP-Complete for Chordal graphs. Here, we present a linear time algorithm for a subclass of chordal graph, namely Proper Interval graph. Let G be a connected proper interval graph with a BCO $\sigma = (v_1, v_2, \ldots, v_n)$. Algorithm VED-PROPER INTERVAL GRAPHS takes G as a input and returns a minimum vertex-edge dominating set of G. Algorithm VED-PROPER INTERVAL GRAPHS maintains an array D for selecting the vertices in to the set VED. If $D[v] = 1$, then all the edges incident to vertex v is ve-dominated; otherwise $D[v] = 0$.

VED-PROPER INTERVAL GRAPHS

Input: A connected proper interval graph G with BCO $\sigma = (v_1, v_2, \ldots, v_n)$ of G and an array with $D[v_i] = 0$ for all v_i where $1 \leq i \leq n$.

Output: A minimum vertex-edge dominating set of G.

Algorithm:

$VED = \emptyset$;
for $i = 1$ *to* $n - 1$ **do**
  if $D[v_i] \neq 1$ **then**
    Let $N_{G_i}(v_i) = \{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$, where $i_1 < i_2 < \ldots < i_r$;
    if $|N_{G_{i+1}}(v_{i+1})| \geq 1$ **then**
      $VED = VED \cup \{v_{i+1_r}\}$
      $D[v] = 1$ for all $v \in N_{G_{i+1}}[v_{i+1_r}]$;
    else
      $VED = VED \cup \{v_{i+1}\}$
    end if
  end if
end for
return VED.

Theorem 3. *For* $0 \leq i \leq n - 1$, *the set* VED_i *is contained in some minimum vertex-edge dominating set of* G.

Proof. We prove the result by induction on the number of iteration i of the algorithm. The base case $i = 0$ is true as $VED_0 = \emptyset$ is contained in a minimum vertex-edge dominating set. Assume that the induction hypothesis is true for all positive integers less than or equal to $i - 1$. Equivalently, the set VED_{i-1} is contained in some minimum vertex-edge dominating set, say D' of G. Notice that at the

i^{th} iteration of algorithm, the vertex v_i is being processed. If $D[v_i] = 1$, then the algorithm does not select any new vertex in to the set VED_i. So $VED_i = VED_{i-1}$ and hence it is contained in D'. Now assume that $D[v_i] = 0$.

Case 1: $|N_{G_{i+1}}(v_{i+1})| = r \geq 1$

Let $N_{G_{i+1}}(v_{i+1}) = \{v_{i+1_1}, v_{i+1_2} \ldots, v_{i+1_r}\}$ where $i + 1_1 \leq i + 1_2 \leq \ldots \leq i + 1_r$. In this case, we have $VED_i = VED_{i-1} \cup \{v_{i+1_r}\}$. If $l = i + 1_r$, then VED_i is contained in a minimum vertex-edge dominating set of G. Suppose $l > i + 1_r$, the vertex v_l is not adjacent to v_{i+1}. The vertex v_l is not adjacent to v_i, since v_i is simplicial in $G_i[v_i, v_{i+1}, \ldots, v_n]$. The vertex v_l does not *ve*-dominate the edge $v_i v_{i+1}$. Hence $l < i + 1_r$. The set $D'' = (D' \setminus \{v_l\}) \cup \{v_{i+1_r}\}$ is a minimum vertex-edge dominating set of G containing the vertex v_{i+1_r}.

Case 2: $|N_{G_{i+1}}(v_{i+1})| = 0$

In this case we have $VED_i = VED_{i-1} \cup \{v_{i+1}\}$. Since v_{i+1} has no neighbors in G_{i+1}, to *ve*-dominate the edge $v_i v_{i+1}$ the vertex v_{i+1} should belong to minimum vertex-edge dominating set of G. Thus $D' \cup \{v_{i+1}\}$ is a minimum vertex-edge dominating set of G containing the vertex v_{i+1}. □

Next we show that how the algorithm VED-PROPER INTERVAL GRAPHS runs in Linear time.

The BCO $\sigma = (v_1, v_2, \ldots, v_n)$ of a proper interval graph can be computed in Linear time [17]. Each iteration of the algorithm VED-PROPER INTERVAL GRAPHS checks the degree of the vertex v_{i+1} in the graph G_{i+1}. Thus the total time taken is $O(n + m)$.

Theorem 4. *For a given connected proper interval graph G with n vertices and m edges, the algorithm VED-PROPER INTERVAL GRAPHS takes $O(n + m)$ time to compute a minimum vertex-edge dominating set of G.*

5. APX-completeness of Bound Degree Graphs and Approximation Result

In this section we show that Minimum vertex-edge dominating set is *APX*-complete for graphs with maximum degree 7. First we provide an approximation ratio of Minimum vertex-edge dominating set in terms of maximum degree.

Proposition 2. *[4] For any graph G with maximum degree $\Delta(G)$, $\gamma_{ev}(G) \leq \Delta(G)\gamma_{ve}(G)$.*

Proposition 3. *[7] For any graph G of order n, without isolates, and maximum degree $\Delta(G)$,*

$$\left\lceil \frac{n}{2\Delta(G) - 2} \right\rceil \leq \gamma_{ev}(G).$$

From Proposition 2 and 3, one can easily prove the following proposition.

Proposition 4. *For any graph G of order n, without isolates, and maximum degree $\Delta(G)$,*

$$\left\lceil \frac{n}{2\Delta(G)(\Delta(G) - 1)} \right\rceil \leq \gamma_{ve}(G).$$

Hence for any graph $G = (V, E)$, $D' = V(G)$ is an vertex-edge dominating set such that $|D'| \leq 2\Delta(G)(\Delta(G) - 1)\gamma_{ve}(G)$. Thus we have the following theorem;

Theorem 5. *The MINIMUM VERTEX-EDGE DOMINATION problem in any graph G with maximum degree $\Delta(G)$ can be approximated with an approximation ratio of $2\Delta(G)(\Delta(G) - 1)$.*

To prove the APX-completeness of MINIMUM VERTEX-EDGE DOMINATION problem, it is enough to prove that there is a L-reduction from Minimum domination problem. We recall the notation of L-reduction [18]: Given two NP optimization problems π_1 and π_2 and a polynomial time transformation f from instances of π_1 to instances of π_2, we say that f is an L-reduction if there are positive constants α and β such that for every instance x of π_1 the following holds.

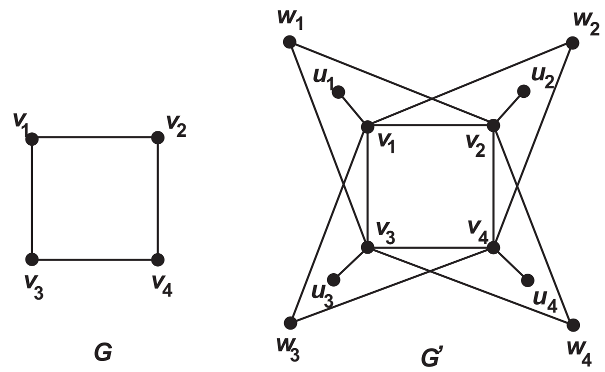

Figure 1. Graph Transformation for Theorem 7

1. $opt_{\pi_2}(f(x)) \leq \alpha.opt_{\pi_1}(f(x))$.
2. for every feasible solution y of $f(x)$ with objective value $m_{\pi_2}(f(x), y) = c_2$ we can in polynomial time find a solution y' of x with $m_{\pi_1}(x, y') = c_1$ such that $|opt_{\pi_1}(x) - c_1| \leq \beta|opt_{\pi_2}(f(x)) - c_2|$.

We define the problems considered in this section as given below:

MIN DOM SET-B

Instance: A graph $G = (V, E)$ with degree at most B.
Solution: A dominating set of G, a subset $V' \subset V$ such that each vertex $u \in V \setminus V'$ has at least one neighbor in V'.
Measure: Cardinality of dominating set $|V'|$.

MIN VE-DOM SET-B

Instance: A graph $G = (V, E)$ with degree at most B.
Solution: A vertex-edge dominating set of G, a subset $V' \subset V$ such that each edge $e \in E(G)$ gets *ve*-dominated by the vertices of V' .
Measure: Cardinality of vertex-edge dominating set $|V'|$.

Theorem 6. *[19] MIN DOM SET-3 is APX-complete.*

Theorem 7. *MIN VE-DOM SET-7 is APX-complete.*

Proof. The MINIMUM DOMINATION problem is APX-complete for graphs with maximum degree 3. It is enough to establish an L-Reduction f from the instances of the MINIMUM DOMINATION SET problem for graphs with maximum degree 3 to the instances of the MINIMUM VE-DOMINATION PROBLEM for graphs with maximum degree 7. Given a graph $G = (V, E)$, with degree at most 3, construct a graph $G' = (V', E')$ as follows. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$, then $V' = V(G') = V \cup \{u_1, u_2, \ldots, u_n\} \cup \{w_1, w_2, \ldots, w_n\}$ and $E' = E(G') = E \cup \{u_i v_i : v_i \in V(G)\} \cup \{w_i v_j : v_j \in N_G(v_i)\}$. See Figure 1 for transformation from G to G'.
Claim: G has a dominating set of cardinality at most k if and only if G' has a *ve*-dominating set of cardinality k.

Proof of Claim. Let D be a minimum dominating set of G, where $|D| \leq k$. It is easy to see that the set D is a vertex-edge dominating set for G'. Thus, $\gamma_{ve}(G') \leq \gamma(G)$. Let D' be a minimum vertex-edge dominating set of G', where $|D'| \leq k$. If $x_i \in D'$ where $x_i = w_i$ or u_i, then we can define,

$D'' = (D' \cup \{v_i\}) \setminus \{x_i\}$ and D'' is still a ve-dominating set of G'. Thus, without loss of generality we can assume D' contains vertices from $V(G)$. Assume that $v_i \in V(G)$ is not adjacent to any vertex of D'. The edge u_iv_i is not ve-dominated by any vertex in D', a contradiction to D' is a ve-dominating set. Hence D' is a dominating set of G. Therefore, $\gamma(G) \leq |D'| = \gamma_{ve}(G')$. Thus we have $\gamma_{ve}(G') = \gamma(G)$. This proves our claim. □

Let D^* and S^* be a minimum dominating set of G and a minimum vertex-edge dominating set of G', respectively. By claim, we have $|S^*| = |D^*|$. Again $||D^*| - \gamma(G)| \leq ||S^*| - \gamma_{ve}(G')|$. Therefore, the reduction is an L-reduction with $\alpha = 1$ and $\beta = 1$. The proof of theorem is complete. □

We now provide the lower bounds for MIN VE-DOM SET-7 problem by using the reduction in Theorem 7 and by using following Theorem 8.

Theorem 8. *[20] It is NP-hard to decide whether an instance of the MIN DOM SET-3 problem with n vertices has a dominating set of cardinality greater than $2.01549586n$, or less than $2.0103305n$. Equivalently, it is NP-hard to approximate the MIN DOM SET-3 problem 1.0025641.*

By Theorem 8, we have $2.0103305n \leq \gamma(G) \leq 2.01549586n$, where n is the number of vertices in G. By Theorem 7, we have $\gamma_{ve}(G') = \gamma(G)$. Thus $2.0103305n \leq \gamma_{ve}(G') = \gamma(G) \leq 2.01549586n$. Since $n' = |V(G')| = 3n$, $(2.0103305)\frac{n'}{3} \leq \gamma_{ve}(G') \leq (2.01549586)\frac{n'}{3}$, implies $0.67011016n' \leq \gamma_{ve}(G') \leq 0.67183195n'$. So we have the following theorem;

Theorem 9. *It is NP-hard to decide whether an instance of the MIN VE-DOM SET-7 problem with n' vertices has an vertex-edge dominating set of cardinality greater than $0.67183195n'$ or less than $0.67011016n'$. Equivalently, it is NP-hard to approximate the MIN VE-DOM SET-4 problem within* $\frac{0.67183195n'}{0.67011016n'} \sim 1.0025694$.

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Conflict of Interest

The authors declare no conflict of interest.

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