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Article

Combinatorial Bounds of the Regularity of Elimination Ideals

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Abstract: Elimination ideals are monomial ideals associated to simple graphs, not necessarily square–free, was introduced by Anwar and Khalid. These ideals are Borel type. In this paper, we obtain sharp combinatorial upper bounds of the Castelnuovo–Mumford regularity of elimination ideals corresponding to certain family of graphs.

Keywords: primary decomposition, Castelnuovo–Mumford regularity, stable ideal, strongly stable ideals
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1. Introduction

Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring in n–variables over an infinite field k. We say that a monomial ideal $I \subset S$ is of *Borel type*, see [1], if it satisfies the following condition: $(I : x_j^\infty) = (I : (x_1, \ldots, x_j)^\infty)$ for all $1 \le j \le n$. The Castelnuovo–Mumford regularity of I is the number $reg(I) = \max\{j - i | \beta_{ij}(I) \neq 0\}$, where $\beta_{ij}(I)$ are the graded Betti numbers of I. The regularity of monomial ideals of Borel type is extensively studied, see for instance [2–5].

Let G be a simple connected graph on the vertex–set $V(G) = \{x_1, \ldots, x_n\}$ and edge–set $E(G)$. There are a number of ways to study the algebraic properties of the graph by associating a monomial ideal to it, well known among all are edge–ideals. Recently, Anwar and Khalid in [3] introduced a new class of monomial ideals, namely; *elimination ideals* $I_D(G)$, associated to G. They showed that the elimination ideals are monomial ideals of Borel type. They gave the description of *Graphical Degree Stability* of graph G denoted by $\text{Stab}_d(G)$; a combinatorial measure associated to G. They gave a systematic procedure to compute the graphical degree stability, namely *Dominating Vertex Elimination Method* (DVE method). They computed the upper bound of the Castelnuovo–Mumford regularity of elimination ideals for complete graph, star graph, line graph, cyclic graph, fan graph, friendship graph and wheel graph.

Motivated from [3], we further extended this study to other family of graphs. We succeeded to obtain a sharp combinatorial bound for the Castelnuovo–Mumford regularity of elimination ideals associated to regular Harary graphs $H_{n-2,n}$ (see theorem 2). We obtain similar result for the join of complete graph and Path Graph; $K_n \vee P_m$ (see theorem 3) and for complete bipartite graph; $K_{m,n}$ (see theorem 4)

2. Background

Throughout in this paper, we consider G as a finite simple connected graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$ and $S = k[x_1, \ldots, x_n]$ be the associated polynomial ring over an infinite field k, also the edge set of G will be denoted by $E(G)$. As $|V(G)|$ is finite, we may use the set $[n] = \{1, 2, \ldots, n\}$ instead of $V(G)$ and we shall always use $[n]$ to label the vertices in figures.

Let $x_i \in V(G)$, then the number of edges incident to x_i is called the degree of x_i and is denoted by $\deg(x_i)$, if $\deg(x_i) \geq \deg(x_j)$ for all $x_j \in V(G)$, then x_i is called *dominating vertex* of G, and the set of all dominating vertices of G is called the *dominating set* of G, denoted as $D(G)$. A vertex $x_i \in V(G)$ with $\deg(x_i) = 0$ is called an *isolated vertex* of G. We call G a *scattered graph*, if it has at least one isolated vertex, otherwise, *non-scattered graph*. A finite sequence of nonnegative integers is called *graphical degree sequence* if it is a degree sequence of some graph. Throughout in this paper, we assume that $\deg(x_1) \geq \deg(x_2) \geq \cdots \geq \deg(x_n)$ in G. Now, we recall important definitions from [3].

Definition 1. *Let G_i be a graph and pick a vertex x in $D(G_i)$ such that $G_{i+1} := G_i - \{x\}$ is an induced, non-scattered subgraph of G_i. This method of obtaining an induced, non-scattered subgraph G_{i+1} from G_i by eliminating a vertex from the dominating set $D(G_i)$ is called the dominating vertex elimination method, the method is known as DVE method. See [3] for more details.*

Let $G := G_0$ be a graph with vertex set $[n]$, then the length of the maximum chain of induced, non scattered subgraphs of G obtained by successively using DVE method is called the *graphical degree stability* of G, and it is formally denoted by $\text{Stab}_d(G)$. In other words, if $G := G_0 \supset G_1 \supset \cdots \supset G_k$ is the maximum chain of induced, non scattered subgraphs of G then $\text{Stab}_d(G) = k$. Note that G_k is a subgraph of G with vertex set $[n - k]$.

Definition 2. *Let G_i be a graph with vertex set $V(G_i) = \{x_1, x_2, \ldots, x_i\}$ and having the degree sequence $(d_1, d_2, \ldots, d_i)$, then the ideal $Q_i := \langle x_1^{d_1}, x_2^{d_2}, \ldots, x_i^{d_i} \rangle$ is called the sequential ideal of G_i. Let $G := G_0 \supset G_1 \supset \cdots \supset G_k$ be the maximum chain of induced, non scattered subgraphs of G obtained by DVE method with $Stab_d(G) = k$, then $I_D(G) := \bigcap_{j=0}^{k} Q_j$ is called the elimination ideal of G.*

Now, we recall some definitions from graph theory.

Definition 3. *Let G and H be two graphs with mutually disjoint vertex sets $V(G) = \{u_1, u_2 \ldots, u_n\}$ and $V(H) = \{w_1, w_2, \ldots, w_m\}$. A graph, denoted by $G \vee H$, is called the join of G and H if $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{u_i w_j | u_i \in V(G),\ w_j \in V(H)\}$.*

Definition 4. *Let G be a graph with vertex set $V(G)$. A subset X of $V(G)$ is called an independent set if no two vertices of X are adjacent. A k-partite graph is a graph whose vertex set $V(G)$ can be partitioned into k distinct independent sets. A complete k-partite graph is a k-partite graph with every two vertices from distinct independent sets are adjacent. If k=2, then graph is bipartite.*

We conclude this section by recalling some important definitions and results regarding stable properties of ideals.

Let $I \subset S = k[x_1, \ldots, x_n]$ be a monomial ideal. For any monomial $u \in S$ set $m(u) := \max\{j |\ x_j | u\}$ and $m(I) = \max\{m(u) |\ u \in G(I)\}$, where $G(I)$ denotes the set of minimal monomial generators of I. The highest degree of monomial in $G(I)$ is denoted by $\deg(I)$. Also, $I_{\geq t}$ is the monomial ideal generated by monomials of I of degree $\geq$ t. A monomial ideal I is *stable* if for each monomial $u \in I$ we have $x_j . \frac{u}{x_{m(u)}} \in I$ for all $1 \leq j < m(u)$. We set $q(I) = m(I)(\deg(I) - 1) + 1$.

Eisenbud, Reeves and Totaro proved the following result in [6].

Theorem 1. Let I be a monomial ideal with $\deg(I) = d$ and $e \geq d$ be an integer such that $I_{\geq e}$ is stable, then $reg(I) \leq e$.

In [2], the authors gave the following bound for the regularity of Borel type ideals.

Proposition 1. Let I be a Borel type ideal, then $reg(I) \leq q(I)$.

Remark 1. As $Ass(S/I_D(G))$ is totally ordered under inclusion, therefore $I_D(G)$ is a Borel type ideal by [2, Theorem 2.2].

In [2], the authors proved the following:

Proposition 2. If I and J are two monomial ideals with $s \geq \deg(I)$ and $t \geq \deg(J)$ be two integers such that $I_{\geq s}$ and $J_{\geq t}$ are stable ideals, then $(I \cap J)_{\geq \max\{s,t\}}$ is stable ideal.

3. Main Results

In this section, we give our main results regarding the Castelnuovo–Mumford regularity of elimination ideals for different classes of graphs.

3.1. Regularity of Regular Harary Graph $H_{n-2,n}$

First, we recall the definition of Harary graph.

Definition 5. Harary graph $H_{k,n}$ is the smallest k-connected graph with n vertices. Let us have a set $V = \{x_1, x_2, \dots, x_n\}$ of n vertices, then the construction of Harary graphs are as follows:
Case I: If $k = 2m < n$ (*n may be even or odd*), then place all n vertices in a circle and join each vertex x_i to its m consecutive left vertices and to its m consecutive right vertices by drawing edges.
Case II: Let n is even. If $k = 2m + 1 < n$, then first construct $H_{2m,n}$ and then join each vertex x_i, $1 \leq i \leq \frac{n}{2}$, to its diametrically opposite vertex.
Case III: If both k and n are odd then first construct $H_{k-1,n}$, then join each vertex x_i, $1 \leq i \leq \frac{n-1}{2} + 1$, with vertex $x_{i+\frac{n-1}{2}}$.

Note that the graphs in Case I and Case II are regular. Also note that if $k = n - 1$ then Case I and Case II suggest that $H_{k,n}$ is a complete graph K_n. When n is even, the diametrically opposite vertex of x_i is given by:

$$\begin{cases} x_i \leftrightarrow x_{i+\frac{n}{2}} & if\ 1 \leq i \leq \frac{n}{2} \\ x_i \leftrightarrow x_{i-\frac{n}{2}} & if\ \frac{n}{2} + 1 \leq i \leq n \end{cases}$$

We are interested in computing the regularity of elimination ideal associated to $H_{n-2,n}$ when n is even and degree $k = n - 2$. We begin by computing the graphical degree stability of $H_{n-2,n}$.

Lemma 1. Let $H_{n-2,n}$ be a regular Harary graph with even vertices $n = 2r \geq 4$ and degree of each vertex is $n - 2$, then $\mathrm{Stab}_d(H_{n-2,n}) = n - 3$.

Proof. We prove it by induction on r for $n = 2r \geq 4$. For $r = 2$, $G_0 := H_{2,4}$ is a regular graph with degree sequence $(2, 2, 2, 2)$, so its dominating set will be $D(G_0) = \{x_1, x_2, x_3, x_4\}$. Now, pick vertex $x_1 \in D(G_0)$, after removing x_1 we get G_1 with degree sequence $(2, 1, 1)$. The process will stop at G_1 and $\mathrm{Stab}_d(H_{2,4}) = 1$.

Consider the result is true for $r = p$, i.e. $\mathrm{Stab}_d(H_{2p-2,2p}) = 2p - 3$.

Now take $r = p + 1$, and let $G_0 := H_{2p,2p+2}$. The degree sequence of G_0 is $\underbrace{(2p, \dots, 2p)}_{\text{(2p+2)-tuple}}$ with dominating set is $D(G_0) = \{x_1, x_2, \dots, x_{2p+2}\}$. Choose vertex x_1 from $D(G_0)$ and apply DVE method, we get G_1 with $D(G_1)$ solely consists of diametrically opposite vertex (see definition 5) of x_1 of degree

$2p$. All other vertices are of degree $p-1$. The degree sequence of G_1 will be $\underbrace{(2p, 2p-1, \ldots, 2p-1)}_{\text{(2p+1)-tuple}}$.

On removing x_1 (after relabeling of the vertices) we get $G_2 := H_{2p-2,2p}$. Now

$$\text{Stab}_d(H_{2p-2,2p}) = 2p-3$$

$$\Rightarrow \text{Stab}_d(H_{2p,2p+2}) = 2 + \text{Stab}_d(H_{2p-2,2p}) = (2p+2) - 3$$

which is required. □

Example 1. Consider $H_{4,6}$, here $n = 6$ and $k = 4$, see Figure 1.

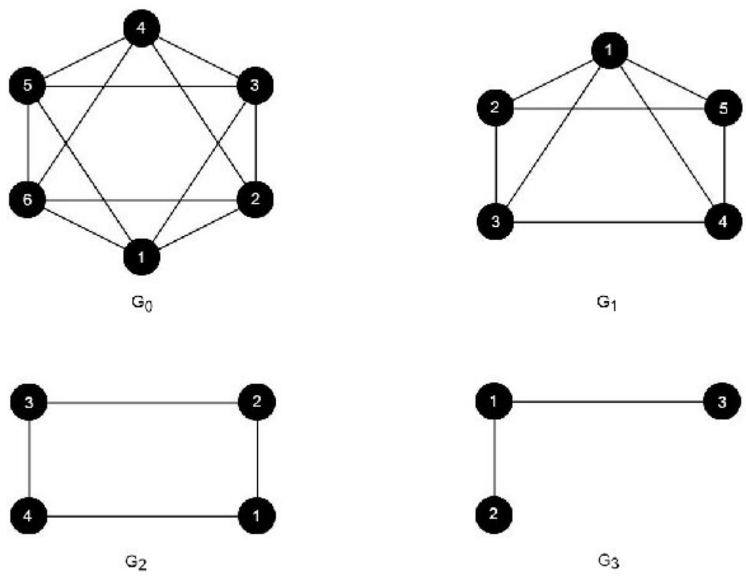

Figure 1. G_0, G_1, G_2, G_3

Corollary 1. Let $H_{n-2,n}$ be a regular Harary graph with even vertices $n \geq 4$ and degree of each vertex is $n-2$, then its sequential ideal is given as follows:

$$Q_i = \begin{cases} \langle x_1^{n-i-2}, x_2^{n-i-2}, \ldots, x_{n-i}^{n-i-2} \rangle & if\ i\ is\ even \\ \langle x_1^{n-i-1}, x_2^{n-i-2}, \ldots, x_{n-i}^{n-i-2} \rangle & if\ i\ is\ odd \end{cases}$$

where $0 \leq i \leq n-3$.

Proof. The proof follows from the definition of elimination ideal and lemma 1. □

Theorem 2. Let $H_{n-2,n}$ be a regular Harary graph with even vertices $n \geq 4$ and degree of each vertex is $n-2$, then $reg(I_D(H_{n-2,n})) \leq (n-1)(n-2) - 1$.

Proof. We shall discuss the two cases of corollary 1 separately.

Case 1. When $i \in \{0, 2, 4, \ldots, n-4\}$, the sequential ideal is given as $Q_i = \langle x_1^{a_1}, \ldots, x_{n-i}^{a_{n-i}} \rangle$ where $a_j = n-i-2$ for all $1 \leq j \leq n-i$. Let $\gamma(i) = a_i(a_i+1) - 1$ for all $i \in \{0, 2, 4, \ldots, n-4\}$. We shall show that $Q_{i_{\geq\gamma(i)}}$ is a stable ideal. Take $u \in Q_{i_{\geq\gamma(i)}}$, then $u = vx_k^{a_k}$ for some $1 \leq k \leq n-i$ where $v \in \langle x_1, \ldots, x_{n-i} \rangle^{\gamma(i)-a_k}$.

If $m(u) > k$, then $\frac{x_l u}{x_{m(u)}} = \frac{x_l v}{x_{m(u)}} x_k^{a_k} \in Q_{i_{\geq\gamma(i)}}$ for all $l < m(u)$. So, $Q_{i_{\geq\gamma(i)}}$ is stable.

If $m(u) = k$, then clearly $u \in \langle x_1, \ldots, x_{n-i} \rangle^{\gamma(i)}$ which is a stable ideal and $Q_{i_{\geq\gamma(i)}} \subseteq \langle x_1, \ldots, x_{n-i} \rangle^{\gamma(i)}$. It remains to show that $\langle x_1, \ldots, x_{n-i} \rangle^{\gamma(i)} \subseteq Q_{i_{\geq\gamma(i)}}$. Let $w \in \langle x_1, \ldots, x_{n-i} \rangle^{\gamma(i)}$ then $w = x_1^{\beta_1} x_2^{\beta_2} \cdots x_{n-i}^{\beta_{n-i}}$ with

$\beta_s \geq 0$ for all $1 \leq s \leq n-i$ and $\Sigma_{s=1}^{n-i}\beta_s \geq \gamma(i)$. Therefore, there exist at least one $r \in \{1, \ldots, n-i\}$ such that $\beta_r \geq a_r$ and $w = (x_1^{\beta_1} \cdots x_r^{\beta_r - a_r} \cdots x_{n-i}^{\beta_{n-i}})x_r^{a_r} \in Q_{i_{\geq\gamma(i)}}$, hence the result follows.

Case 2. When $i \in \{1, 3, 5, \ldots, n-3\}$, the sequential ideal then is given as $Q_i = \langle x_1^{a_1}, \ldots, x_{n-i}^{a_{n-i}} \rangle$ where

$$a_j = \begin{cases} n-i-1 & if\ j = 1 \\ n-i-2 & if\ 2 \leq j \leq n-i \end{cases}$$

Let $\gamma'(i) = a_i(a_i + 1)$ for all $i \in \{1, 3, 5, \ldots, n-3\}$, then we shall show that $Q_{i_{\geq\gamma'(i)}}$ is a stable ideal. Take $u \in Q_{i_{\geq\gamma'(i)}}$, then $u = vx_k^{a_k}$ for some $1 \leq k \leq n-i$ where $v \in \langle x_1, \ldots, x_{n-i} \rangle^{\gamma'(i) - a_k}$.

If $m(u) > k$, then $\frac{x_l u}{x_{m(u)}} = \frac{x_l v}{x_{m(u)}} x_k^{a_k} \in Q_{i_{\geq\gamma'(i)}}$ for all $l < m(u)$. So, $Q_{i_{\geq\gamma'(i)}}$ is stable.

If $m(u) = k$, then clearly $u \in \langle x_1, \ldots, x_{n-i} \rangle^{\gamma'(i)}$ which is stable ideal and $Q_{i_{\geq\gamma'(i)}} \subseteq \langle x_1, \ldots, x_{n-i} \rangle^{\gamma'(i)}$. We are to show that $\langle x_1, \ldots, x_{n-i} \rangle^{\gamma'(i)} \subseteq Q_{i_{\geq\gamma'(i)}}$. Let $w \in \langle x_1, \ldots, x_{n-i} \rangle^{\gamma'(i)}$ then $w = x_1^{\beta_1} x_2^{\beta_2} \cdots x_{n-i}^{\beta_{n-i}}$ with $\beta_s \geq 0$ for all $1 \leq s \leq n-i$ and $\Sigma_{s=1}^{n-i}\beta_s \geq \gamma'(i)$. Therefore, there exist at least one $r \in \{1, \ldots, n-i\}$ such that $\beta_r \geq a_r$ and $w = (x_1^{\beta_1} \cdots x_r^{\beta_r - a_r} \cdots x_{n-i}^{\beta_{n-i}})x_r^{a_r} \in Q_{i_{\geq\gamma'(i)}}$ and the result follows.

By lemma 1, $\text{Stab}_d(H_{n-2,n}) = n-3$, so the corresponding elimination ideal is given as $I_D(H_{n-2,n}) = \bigcap_{i=0}^{n-3} Q_i$. By proposition 2, $I_D(H_{n-2,n})$ is stable for γ_0, where

$$\gamma_0 = \max\{\gamma(i), \gamma'(j) | i \in \{0, 2, \ldots, n-4\}, j \in \{1, 3, \ldots, n-3\}\} = (n-1)(n-2) - 1$$

and by theorem 1 $reg(I_D(H_{n-2,n})) \leq (n-1)(n-2) - 1$. □

Remark 2. *In example 1,*
$D(G_0) = \{x_1, x_2, \ldots, x_6\}$ with $Q_0 = \langle x_1^4, x_2^4, \ldots, x_6^4 \rangle$ and $reg(Q_0) = 19$.
$D(G_1) = \{x_1\}$ with $Q_1 = \langle x_1^4, x_2^3, \ldots, x_5^3 \rangle$ and $reg(Q_1) = 12$
$D(G_2) = \{x_1, x_2, x_3, x_4\}$ with $Q_2 = \langle x_1^2, x_2^2, x_3^2, x_4^2 \rangle$ and $reg(Q_2) = 5$
$D(G_3) = \{x_1\}$ with $Q_3 = \langle x_1^2, x_2, x_3 \rangle$ and $reg(Q_3) = 2$

3.2. Regularity of $K_n \vee P_m$

In [3], following formula is given to compute the graphical degree stability of path graph:

Proposition 3. Let $P_m,\ m \geq 3$, be a path graph then:

$$\text{Stab}_d(P_m) = \begin{cases} \frac{m-3}{3} & if\ m \equiv 0\ (mod\ 3) \\ \frac{m-4}{3} & if\ m \equiv 1\ (mod\ 3) \\ \frac{m-2}{3} & if\ m \equiv 2\ (mod\ 3) \end{cases}$$

Lemma 2. Let $K_n, n \geq 2$ be a complete graph and $P_m,\ m \geq 4$ be a path graph then:

$$\text{Stab}_d(K_n \vee P_m) = n + \text{Stab}_d(P_m)$$

Proof. We shall prove it by induction on n. Let $n = 2$ and $m = 4$, then $G_0 := K_2 \vee P_4$ with degree sequence $(5, 5, 4, 4, 3, 3)$ and $D(G_0) = \{x_1, x_2\}$. Without loss of generality, remove $x_1 \in D(G_0)$ to get G_1 with the degree sequence $(4, 3, 3, 2, 2)$. So, $D(G_1) = \{x_1\}$ and on removing $x_1 \in D(G_1)$, we get $G_2 = P_4$.

$$\implies \quad \text{Stab}_d(K_2 \vee P_4) = 2 + \text{Stab}_d(P_4)$$

Suppose that result is true for $n = q$ and $m = r$, then $\text{Stab}_d(K_q \vee P_r) = q + \text{Stab}_d(P_r)$.

Consider $n = q+1$ and $m = r$ then $G_0 := K_{q+1} \vee P_r$ with degree sequence $(\underbrace{q+r, \ldots, q+r}_{\text{(q+1)-tuple}}, \underbrace{q+3, \ldots, q+3}_{\text{(r-2)-tuple}}, q+2, q+2)$ and $|V(G_0| = q+r+1$. Since $r \geq 4$, $D(G_0) = \{x_1, \ldots, x_{q+1}\}$

which are precisely the vertices that were initially belonged to K_{q+1}. As removing any vertex from K_{q+1} gives K_q, So without loss of generality pick $x_1 \in D(G_0)$ and on removing it, we get $G_1 = K_q \vee P_r$.

$$\implies \quad \text{Stab}_d(K_{q+1} \vee P_r) = 1 + \text{Stab}_d(K_q \vee P_r) = 1 + q + \text{Stab}_d(P_r)$$

which completes the proof. □

Corollary 2. Let $K_n, n \geq 2$ be a complete graph and $P_m, m \geq 4$ be a path graph, then the sequential ideal of $K_n \vee P_m$ is given as follows:

$$Q_i = \begin{cases} \langle x_1^{m+n-i-1}, \dots, x_{n-i}^{m+n-i-1}, x_{n-i+1}^{n-i+2}, \dots, x_{m+n-i-2}^{n-i+2}, x_{m+n-i-1}^{n-i+1}, x_{m+n-i}^{n-i+1} \rangle & if\ 0 \leq i \leq n-1 \\ \langle x_1^2, x_2^2, \dots, x_{m-3(i-n)-2}^2, x_{m-3(i-n)-1}, \dots, x_{m+n-i} \rangle & if\ n \leq i \leq n+p \end{cases}$$

where $p = \text{Stab}_d(P_m)$

Proof. The proof follows immediately from lemma 2 and [3, Proposition 3.10]. □

Theorem 3. Let $K_n, n \geq 2$ be a complete graph and $P_m, m \geq 4$ be a path graph then $reg(I_D(K_n \vee P_m)) \leq n^2 + 2n(m-1) + m - 1$.

Proof. We shall discuss the two cases of corollary 2 separately.

Case 1. When $0 \leq i \leq n-1$, the sequential ideal is given as $Q_i = \langle x_1^{a_1}, \dots, x_{m+n-i}^{a_{m+n-i}} \rangle$ where

$$a_j = \begin{cases} m+n-i+1 & if\ 1 \leq j \leq n-i \\ n-i+2 & if\ n-i+1 \leq j \leq m+n-i-2 \\ n-i+1 & if\ n-i+1 \leq j \leq m+n-i. \end{cases}$$

Let $\gamma(i) = (n-i)^2 + 2(m-1)(n-i) + m - 1$ for all $0 \leq i \leq n-1$. We shall show that $Q_{i_{\geq\gamma(i)}}$ is a stable ideal. Take $u \in Q_{i_{\geq\gamma(i)}}$, then $u = vx_k^{a_k}$ for some $1 \leq k \leq m+n-i$ where $v \in \langle x_1, \dots, x_{m+n-i} \rangle^{\gamma(i)-a_k}$.

If $m(u) > k$, then $\frac{x_l u}{x_{m(u)}} = \frac{x_l v}{x_{m(u)}} x_k^{a_k} \in Q_{i_{\geq\gamma(i)}}$ for all $l < m(u)$. So, $Q_{i_{\geq\gamma(i)}}$ is stable.

If $m(u) = k$, then clearly $u \in \langle x_1, \dots, x_{m+n-i} \rangle^{\gamma(i)}$ and $Q_{i_{\geq\gamma(i)}} \subseteq \langle x_1, \dots, x_{m+n-i} \rangle^{\gamma(i)}$. We are to show that $\langle x_1, \dots, x_{m+n-i} \rangle^{\gamma(i)} \subseteq Q_{i_{\geq\gamma(i)}}$. Let $w \in \langle x_1, \dots, x_{m+n-i} \rangle^{\gamma(i)}$ then $w = x_1^{\beta_1} x_2^{\beta_2} \cdots x_{m+n-i}^{\beta_{m+n-i}}$ with $\beta_s \geq 0$ for all $1 \leq s \leq m+n-i$ and $\Sigma_{s=1}^{m+n-i} \beta_s \geq \gamma(i)$. Therefore, there exist at least one $r \in \{1, \dots, m+n-i\}$ such that $\beta_r \geq a_r$ and $w = (x_1^{\beta_1} \cdots x_r^{\beta_r - a_r} \cdots x_{m+n-i}^{\beta_{m+n-i}}) x_r^{a_r} \in Q_{i_{\geq\gamma(i)}}$ and the result follows.

Case 2. When $n \leq i \leq n+p$, the sequential ideal is given as $Q_i = \langle x_1^{a_1}, \dots, x_{m+n-i}^{a_{m+n-i}} \rangle$ where

$$a_j = \begin{cases} 2 & if\ 1 \leq j \leq m-3(i-n)-2 \\ 1 & if\ m-3(i-n)-1 \leq j \leq m+n-i. \end{cases}$$

Let $\gamma'(i) = m - 3(i-n) - 1$ for all $n \leq i \leq n+p$, then we shall show that $Q_{i_{\geq\gamma'(i)}}$ is a stable ideal. Take $u \in Q_{i_{\geq\gamma'(i)}}$, then $u = vx_k^{a_k}$ for some $1 \leq k \leq m+n-i$ where $v \in \langle x_1, \dots, x_{m+n-i} \rangle^{\gamma'(i)-a_k}$.

If $m(u) > k$, then $\frac{x_l u}{x_{m(u)}} = \frac{x_l v}{x_{m(u)}} x_k^{a_k} \in Q_{i_{\geq\gamma'(i)}}$ for all $l < m(u)$. So, $Q_{i_{\geq\gamma'(i)}}$ is stable.

If $m(u) = k$, then clearly $u \in \langle x_1, \dots, x_{m+n-i} \rangle^{\gamma'(i)}$ and $Q_{i_{\geq\gamma'(i)}} \subseteq \langle x_1, \dots, x_{m+n-i} \rangle^{\gamma'(i)}$. We are to show that $\langle x_1, \dots, x_{m+n-i} \rangle^{\gamma'(i)} \subseteq Q_{i_{\geq\gamma'(i)}}$. Let $w \in \langle x_1, \dots, x_{m+n-i} \rangle^{\gamma'(i)}$ then $w = x_1^{\beta_1} x_2^{\beta_2} \cdots x_{m+n-i}^{\beta_{m+n-i}}$ with $\beta_s \geq 0$ for all $1 \leq s \leq m+n-i$ and $\Sigma_{s=1}^{m+n-i} \beta_s \geq \gamma'(i)$. Therefore, there exist at least one $r \in \{1, \dots, m+n-i\}$ such that $\beta_r \geq a_r$ and $w = (x_1^{\beta_1} \cdots x_r^{\beta_r - a_r} \cdots x_{m+n-i}^{\beta_{m+n-i}}) x_r^{a_r} \in Q_{i_{\geq\gamma'(i)}}$ and the result follows.

By lemma 2, $\text{Stab}_d(K_n \vee P_m) = n + p$, so the corresponding elimination ideal is given as $I_D(K_n \vee P_m) = \bigcap_{i=0}^{n+p} Q_i$, by proposition 2, $I_D(K_n \vee P_m)$ is stable for γ_0, where

$$\gamma_0 = \max\{\gamma(i), \gamma'(j) | 0 \leq i \leq n-1 \text{ and } n \leq j \leq n+p\} = n^2 + 2n(m-1) + m - 1$$

and by theorem 1, $reg(I_D(K_n \vee P_m)) \leq n^2 + 2n(m-1) + m - 1$. □

Example 2. Consider $K_3 \vee P_4$, here $n = 3$ and $m = 4$.
$D(G_0) = \{x_1, x_2, x_3\}$, $Q_0 = \langle x_1^6, x_2^6, x_3^6, x_4^5, x_5^5, x_6^4, x_7^4 \rangle$ and $reg(Q_0) = 30$, see Figure 2.

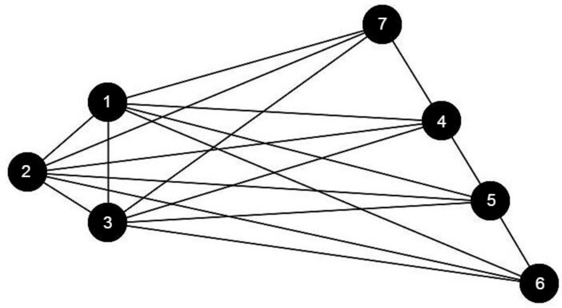

Figure 2. $G_0 = K_3 \vee P_4$

$D(G_1) = \{x_1, x_2\}$, $Q_1 = \langle x_1^5, x_2^5, x_3^4, x_4^4, x_5^3, x_6^3 \rangle$ and $reg(Q_1) = 19$, see Figure 3.

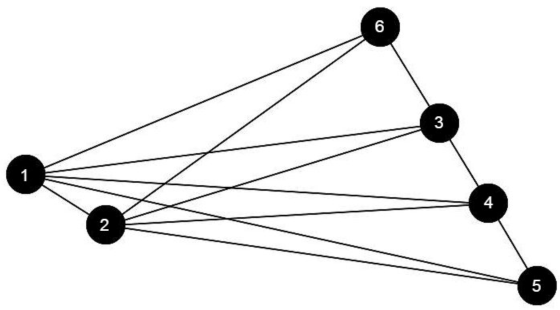

Figure 3. G_1

$D(G_2) = \{x_1\}$, $Q_2 = \langle x_1^4, x_2^3, x_3^3, x_4^2, x_5^2 \rangle$ and $reg(Q_2) = 10$, see Figure 4.

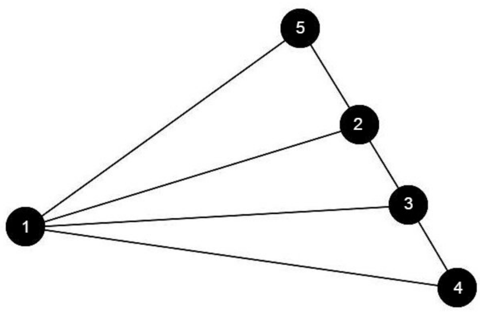

Figure 4. G_2

$D(G_3) = \{x_1, x_2\}$, $Q_3 = \langle x_1^2, x_2^2, x_3, x_4 \rangle$ and $reg(Q_3) = 3$, see Figure 5.

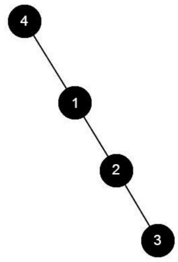

Figure 5. G_3

3.3. Regularity of complete bipartite graph $K_{m,n}$

We recall the result about graphical degree stability of complete bipartite graph from [3]:

Proposition 4. Let $K_{m,n}$ be a complete bipartite graph with $m \geq n$ then:

$$\text{Stab}_d(K_{m,n}) = n - 1$$

We generalize this result for complete n-partite graphs.

Lemma 3. Let $K_{m_1,\ldots,m_n}$ be a complete n-partite graph with $m_i \geq m_j$ for $1 \leq i < j \leq n$, then

$$\text{Stab}_d(K_{m_1,\ldots,m_n}) = m_n + m_{n-1} + \cdots + m_2 - 1$$

Proof. We prove it by induction. For $n = 2$, we have K_{m_1,m_2} with $m_1 \geq m_2$ then by proposition 4:

$$\text{Stab}_d(K_{m_1,m_2}) = m_2 - 1$$

Let the result is true for $n = k - 1$, i.e.

$$\text{Stab}_d(K_{m_1,\ldots,m_{k-1}}) = m_{k-1} + m_{k-2} + \cdots + m_2 - 1$$

with $m_i \geq m_j$ if $1 \leq i < j \leq k - 1$.

Consider $G_0 := K_{m_1,\ldots,m_k}$, be the complete $k-$partite graph and $V(G_0) = X_1 \cup X_2 \cup \cdots \cup X_k$, where each $X_r = \{x_{r_1}, x_{r_2}, \ldots, x_{r_{m_r}}\}$ is an independent set with $|X_r| = m_r,\ 1 \leq r \leq k$. Further $m_i \geq m_j$ if $1 \leq i < j \leq k$.

If $x \in X_r,\ 1 \leq r \leq k$ then degree of x would be $m_1 + \cdots + m_{r-1} + m_{r+1} + \cdots + m_k$. As $m_k \leq m_j$ for all $1 \leq j \leq k - 1$, hence $X_k \subseteq D(G_0)$. So, without loss of generality we pick the vertex $x_{k_{m_k}} \in X_k$, removing it will give us a new graph G_1 with dominating set $D(G_1) = X_k - \{x_{k_{m_k}}\}$ with degree of each vertex of $D(G_1)$ is still $m_1 + \cdots + m_{k-1}$. If $x \in X_r,\ 1 \leq r \leq k - 1$ then degree of x in G_1 would be $m_1 + \cdots + m_{r-1} + m_{r+1} + \cdots + m_k - 1$. Now pick $x_{k_{m_{k-1}}}$ from $D(G_1)$ and remove it so that we get new graph G_2 with dominating set $D(G_2) = X_k - \{x_{k_{m_k}}, x_{k_{m_{k-1}}}\}$ with degree of each vertex of $D(G_2)$ is still $m_1+\cdots+m_{k-1}$. If $x \in X_r,\ 1 \leq r \leq k-1$ then degree of x in G_2 would be $m_1+\cdots+m_{r-1}+m_{r+1}+\cdots+m_k-2$. Continue in this way we get $G_{m_k} := K_{m_1,\ldots,m_{k-1}}$. So,

$$\text{Stab}_d(K_{m_1,\ldots,m_k}) = m_k + \text{Stab}_d(K_{m_1,\ldots,m_{k-1}}) = m_k + m_{k-1} + \cdots + m_2 - 1$$

which completes the proof. □

Corollary 3. Let $K_{m,n}$ be a complete bipartite graph with $m \geq n$, then the sequential ideal is given as follows:

$$Q_i = \langle x_1^m, \ldots, x_{n-i}^m, x_{n-i+1}^{n-i}, \ldots, x_{m+n-i}^{n-i} \rangle$$

where $0 \leq i \leq n - 1$.

Proof. The proof follows immediately from lemma 4. □

Theorem 4. Let $K_{m,n}$ be a complete bipartite graph with $m \geq n$, then

$$reg(I_D(K_{m,n})) \leq m + (2m-1)(n-1).$$

Proof. By proposition 4, $\text{Stab}_d(K_{m,n}) = n-1$. The sequential ideal is given as $Q_i = \langle x_1^{a_1}, \dots, x_{m+n-i}^{a_{m+n-i}} \rangle$ for all $0 \leq i \leq n-1$ where

$$a_j = \begin{cases} m & if\ 1 \leq j \leq n-i \\ n-i & if\ n-i+1 \leq j \leq m+n-i. \end{cases}$$

Let $\gamma(i) = m + (2m-1)(n-i-1)$ for all $0 \leq i \leq n-1$, then we shall show that $Q_{i_{\geq \gamma(i)}}$ is a stable ideal. Take $u \in Q_{i_{\geq \gamma(i)}}$, then $u = vx_k^{a_k}$ for some $1 \leq k \leq m+n-i$ where $v \in \langle x_1, \dots, x_{m+n-i} \rangle^{\gamma(i)-a_k}$.

If $m(u) > k$, then $\frac{x_l u}{x_{m(u)}} = \frac{x_l v}{x_{m(u)}} x_k^{a_k} \in Q_{i_{\geq \gamma(i)}}$ for all $l < m(u)$. So, $Q_{i_{\geq \gamma(i)}}$ is stable.

If $m(u) = k$, then clearly $u \in \langle x_1, \dots, x_{m+n-i} \rangle^{\gamma(i)}$ and $Q_{i_{\geq \gamma(i)}} \subseteq \langle x_1, \dots, x_{m+n-i} \rangle^{\gamma(i)}$. We are to show that $\langle x_1, \dots, x_{m+n-i} \rangle^{\gamma(i)} \subseteq Q_{i_{\geq \gamma(i)}}$. Let $w \in \langle x_1, \dots, x_{m+n-i} \rangle^{\gamma(i)}$ then $w = x_1^{\beta_1} x_2^{\beta_2} \cdots x_{m+n-i}^{\beta_{m+n-i}}$ with $\beta_s \geq 0$ for all $1 \leq s \leq m+n-i$ and $\Sigma_{s=1}^{m+n-i} \beta_s \geq \gamma(i)$. Therefore, there exist at least one $r \in \{1, \dots, m+n-i\}$ such that $\beta_r \geq a_r$ and $w = (x_1^{\beta_1} \cdots x_r^{\beta_r - a_r} \cdots x_{m+n-i}^{\beta_{m+n-i}}) x_r^{a_r} \in Q_{i_{\geq \gamma(i)}}$ and the result follows.

By proposition 2, $I_D(K_{m,n}) = \bigcap_{i=0}^{n-1} Q_i$ is stable for γ_0, where

$$\gamma_0 = \max\{\gamma(i) | 0 \leq i \leq n-1\} = m + (2m-1)(n-1)$$

and by theorem 1, $reg(I_D(K_{m,n})) \leq m + (2m-1)(n-1)$. □

Example 3. Consider $K_{4,3}$, here $m = 4$ and $n = 3$.
$D(G_0) = \{x_1, x_2, x_3\}$, $Q_0 = \langle x_1^4, x_2^4, x_3^4, x_4^3, x_5^3, x_6^3, x_7^3 \rangle$ and $reg(Q_0) = 18$, see Figure 6.

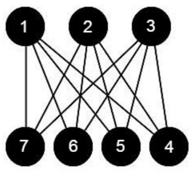

Figure 6. $G_0 = K_{4,3}$

$D(G_1) = \{x_1, x_2\}$, $Q_1 = \langle x_1^4, x_2^4, x_3^2, x_4^2, x_5^2, x_6^2 \rangle$ and $reg(Q_1) = 11$,see Figure 7.

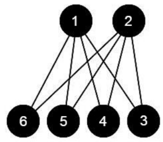

Figure 7. $G_1 = K_{4,2}$

$D(G_2) = \{x_1\}$, $Q_2 = \langle x_1^4, x_2, x_3, x_4, x_5 \rangle$ and $reg(Q_2) = 4$, see Figure 8.

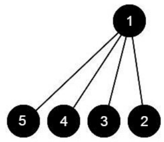

Figure 8. $G_2 = K_{4,1}$

Remark 3. As elimination ideals are of Borel type ideals and an upper bound for Borel type ideal were discussed in [2] and [5]. It is worthy to note that our given bounds are sharper than the one given in [2] and [5].

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Conflict of Interest

The author declares no conflict of interests.

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