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Article

Description of the (≤ 3)-hypomorphic Posets and Bichains. Application to the (≤ 3)-reconstruction

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Abstract: Two binary structures $\Re$ and $\Re'$ on the same vertex set V are $(\leq k)$-hypomorphic for a positive integer k if, for every set K of at most k vertices, the two binary structures induced by $\Re$ and $\Re'$ on K are isomorphic. A binary structure $\Re$ is $(\leq k)$-reconstructible if every binary structure $\Re'$ that is $(\leq k)$-hypomorphic to $\Re$ is isomorphic to $\Re$. In this paper, we describe the pairs of (≤ 3)-hypomorphic posets and the pairs of (≤ 3)-hypomorphic bichains. As a consequence, we characterize the (≤ 3)-reconstructible posets and the (≤ 3)-reconstructible bichains. This answers a question suggested by Y. Boudabbous and C. Delhommé during a personal communication.

Keywords: Binary structure, Binary relation, Posets, Chain-modules, Hypomorphy, Reconstruction, Bichain, Linear-modules
Mathematics Subject Classification: 05C60, 05C20, 05C63

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1. Introduction

All binary relations considered in this paper are irreflexive binary relations. Two binary relations $\mathcal{R}_1$ and $\mathcal{R}_2$ on the same vertex set V are $(\leq k)$-hypomorphic for a positive integer k if, for every set K of at most k vertices, the two binary relations induced by $\mathcal{R}_1$ and $\mathcal{R}_2$ on K are isomorphic. A binary relation $\mathcal{R}$ is $(\leq k)$-reconstructible if every binary relation $\mathcal{R}'$ that is $(\leq k)$-hypomorphic to $\mathcal{R}$ is isomorphic to $\mathcal{R}$. G. Lopez showed in [1–3] that finite binary relations are (≤ 6)-reconstructible. This work was extended to the infinite case by J.G. Hagendorf in [4]. These works make essential use of difference classes introduced by Lopez [2, 3]. Based on the description by Lopez and C. Rauzy [5] of the difference classes of finite (≤ 4)-hypomorphic binary relations, Y. Boudabbous [6] provided a characterization of the (≤ 5)-reconstructible finite binary relations, that generalizes to (≤ 4)-reconstructibility. On the other hand, in [7] Boudabbous and C. Delhommé characterized the $(\leq k)$-reconstructible binary relations (finite or not), for each $k \geq 4$. For the (≤ 3)-reconstruction, Boudabbous and Lopez [8] characterized the finite binary relations that are (≤ 3)-reconstructible. Hagendorf [4] proved that every finite poset with at least 4 vertices is (≤ 3)-reconstructible. In [9] Boudabbous and Delhommé suggested the question about the characterization of the (≤ 3)-reconstruction of posets and bichains. In this paper, we give an answer to this question as follows. We first describe the pairs of (≤ 3)-hypomorphic posets and the pairs of (≤ 3)-hypomorphic bichains. As a consequence of these descriptions, we give a characterization of the (≤ 3)-reconstructible posets and bichains:

Theorem 1. *Two posets are* (≤ 3)*-hypomorphic if and only if they have the same maximal chain-modules, and they have the same corresponding quotient relation.*

Theorem 2. *Two bichains are* (≤ 3)*-hypomorphic if and only if they have the same partition into maximal linear-modules, the two corresponding quotients are equal, and each maximal linear-module has the same kind in both.*

We deduce the following (≤ 3)-reconstruction results.

Corollary 1. *A poset is* (≤ 3)*-reconstructible if and only if its chain-modules are finite.*

Corollary 2. *A bichain is* (≤ 3)*-reconstructible if and only if its linear-modules are finite.*

2. Preliminaries

Binary Structures

A binary structure is a pair $\mathfrak{R} := (V, (\mathcal{R}_i)_{i\in I})$ made of a set V and a family $(\mathcal{R}_i)_{i\in I}$ of binary relations on V. When $|I| = 1$, the binary structure is a binary relation. A binary relation $(V, \mathcal{R})$ is a *partially ordered set* (*order* or *poset*) if the relation $\mathcal{R}$ is an irreflexive, antisymmetric and transitive binary relation on V. The binary structure is a bichain when $|I| = 2$ and $\mathcal{R}_1$ and $\mathcal{R}_2$ are linear orderings (or simply chains). The substructure induced by $\mathfrak{R}$ on a subset A of V, simply called the restriction of $\mathfrak{R}$ to A, is the binary structure $\mathfrak{R} \upharpoonright A := (A, (\mathcal{R}_i \upharpoonright A)_{i\in I})$, where $\mathcal{R}_i \upharpoonright A := \mathcal{R}_i \cap A^2$ for all $i \in I$. Finally, a set M of vertices is a *module* [10, 11] (is an *interval* [12], is an *autonomous* set [13] or a clan [14]) of $\mathfrak{R}$, if for each $i \in I$,

$$((b,a) \in \mathcal{R}_i \Longleftrightarrow (b,a') \in \mathcal{R}_i) \text{ and } ((a,b) \in \mathcal{R}_i \Longleftrightarrow (a',b) \in \mathcal{R}_i) \text{ for any vertices } a, a', b \text{ with } a, a' \in M \text{ and } b \notin M.$$

The empty set, the singletons of V and the set V are modules of $\mathfrak{R}$ and said to be *trivial*. Notice for instance that if the two linear orderings of a bichain have a common *extremum* x (that may be a *minimum* for one and a *maximum* for the other one), then $V \setminus \{x\}$ is a module of the bichain.

The following is easy to check;

Lemma 1. *The collection* $\mathcal{M}$ *of modules of a binary structure* $\mathfrak{R}$ *with vertex set* V *satisfies the following properties;*

1. *It contains the empty set, the singletons and the vertex set (trivial modules).*
2. *It is closed under arbitrary intersection, i.e.* $\forall \mathcal{N} \subseteq \mathcal{M}$*:* $\cap\mathcal{N} \in \mathcal{M}$*(with the convention that* $\cap\emptyset = V$*).*
3. *It contains the union of any subcollection with a non-empty intersection, i.e.* $\forall \mathcal{N} \subseteq \mathcal{M}(\cap\mathcal{N} \neq \emptyset \Rightarrow \cup\mathcal{N} \in \mathcal{M})$.
4. *It is closed under balanced difference, i.e.* $\forall M, N \in \mathcal{M}(M \setminus N \neq \emptyset \Rightarrow N \setminus M \in \mathcal{M})$*.*

A *modular partition* of a binary structure $\mathfrak{R}$ is a partition $\mathcal{P}$ of its vertex set V into modules of $\mathfrak{R}$. Notice that the elements of such a partition are non-empty. The elements of $\mathcal{P}$ may be considered as the vertices of a new binary structure, the *quotient* $\mathfrak{R}/\mathcal{P}$ of $\mathfrak{R}$ by $\mathcal{P}$, defined as follows:
$\mathfrak{R}/\mathcal{P} := (\mathcal{P}, (\mathcal{R}_i/\mathcal{P})_{i\in I})$, where for all $i \in I$, $\mathcal{R}_i/\mathcal{P}$ is a binary relation defined on $\mathcal{P}$ by:

$$\textit{For all } A \neq B \in \mathcal{P}, (A,B) \in \mathcal{R}_i/\mathcal{P} \Leftrightarrow (a,b) \in \mathcal{R}_i, \ \textit{for any vertices } a, b \textit{ with } a \in A \textit{ and } b \in B.$$

Let $\mathfrak{R} := (V, (\mathcal{R}_i)_{i\in I})$ and $\mathfrak{R}' := (V', (\mathcal{R}'_i)_{i\in I})$ be two binary structures. A map $f : V \longrightarrow V'$ is an isomorphism from $\mathfrak{R}$ onto $\mathfrak{R}'$ if f is bijective and satisfies: $(x,y) \in \mathcal{R}_i$ if and only if $(f(x), f(y)) \in \mathcal{R}'_i$, for any $(x, y) \in V^2$, $i \in I$. The binary structure $\mathfrak{R}$ is isomorphic to $\mathfrak{R}'$ if there is some isomorphism from $\mathfrak{R}$ onto $\mathfrak{R}'$, which is denoted $\mathfrak{R} \simeq \mathfrak{R}'$. The binary structure $\mathfrak{R}$ is embeddable into $\mathfrak{R}'$, which is denoted $\mathfrak{R} \leq \mathfrak{R}'$, if $\mathfrak{R}$ is isomorphic to some restriction of $\mathfrak{R}'$.

We recall the basic notions of the *reconstruction* problems in the theory of relations that we apply to the case of binary structures. Let $\mathfrak{R}$ and $\mathfrak{R}'$ be two binary structures on the same set V and let k be a positive integer. The binary structure $\mathfrak{R}'$ is ($\leq k$)-*hypomorphic* to $\mathfrak{R}$ if for each subset A of V with at most k elements, the induced binary structures $\mathfrak{R}' \upharpoonright A$ and $\mathfrak{R} \upharpoonright A$ are isomorphic. The binary structure $\mathfrak{R}$ is ($\leq k$)-*reconstructible*, if each binary structure ($\leq k$)-hypomorphic to $\mathfrak{R}$ is isomorphic to $\mathfrak{R}$.

Lemma 2. *Consider two binary structures $\mathfrak{R}$ and $\mathfrak{R}'$ with the same vertex set and a common modular partition $\mathcal{M}$ such that $\mathfrak{R}/\mathcal{M} = \mathfrak{R}'/\mathcal{M}$. Then the following assertions hold.*

1. *If the restrictions $\mathfrak{R} \upharpoonright M$ and $\mathfrak{R}' \upharpoonright M$ are isomorphic for each member M of $\mathcal{M}$, then $\mathfrak{R}$ and $\mathfrak{R}'$ are isomorphic.*

2. *For $k \geq 1$, if $\mathfrak{R} \upharpoonright M$ and $\mathfrak{R}' \upharpoonright M$ are ($\leq k$)-hypomorphic for each $M \in \mathcal{M}$, then $\mathfrak{R}$ and $\mathfrak{R}'$ are ($\leq k$)-hypomorphic.*

Proof. Clearly, the first assertion is easy to check. Now, consider an integer $k \geq 1$ and let us prove the second assertion. Consider two binary structures $\mathfrak{R}$ and $\mathfrak{R}'$ with the same vertex set V and a common modular partition $\mathcal{M}$ such that $\mathfrak{R}/\mathcal{M} = \mathfrak{R}'/\mathcal{M}$, and $\mathfrak{R} \upharpoonright M$ and $\mathfrak{R}' \upharpoonright M$ are ($\leq k$)-hypomorphic for each $M \in \mathcal{M}$. Given a subset X of V with $|X| \leq k$, we will show that $\mathfrak{R} \upharpoonright X \simeq \mathfrak{R}' \upharpoonright X$. Let $X_{\mathcal{M}}$ denote the set $\{Y \cap X : Y \in \mathcal{M} \text{ and } Y \cap X \neq \emptyset\}$. Since $\mathcal{M}$ is a common modular partition of $\mathfrak{R}$ and $\mathfrak{R}'$ and $\mathfrak{R}/\mathcal{M} = \mathfrak{R}'/\mathcal{M}$, $X_{\mathcal{M}}$ is a common modular partition of $\mathfrak{R} \upharpoonright X$ and $\mathfrak{R}' \upharpoonright X$, and $(\mathfrak{R} \upharpoonright X)/X_{\mathcal{M}} = (\mathfrak{R}' \upharpoonright X)/X_{\mathcal{M}}$. Since $\mathfrak{R} \upharpoonright M$ and $\mathfrak{R}' \upharpoonright M$ are ($\leq k$)-hypomorphic for each $M \in \mathcal{M}$, $(\mathfrak{R} \upharpoonright X) \upharpoonright Y \simeq (\mathfrak{R}' \upharpoonright X) \upharpoonright Y$, for each element Y of $X_{\mathcal{M}}$. If $X_{\mathcal{M}}$ is a singleton $\{Y\}$, then $\mathfrak{R} \upharpoonright X \simeq \mathfrak{R}' \upharpoonright X$ because $\mathfrak{R} \upharpoonright X = (\mathfrak{R} \upharpoonright X) \upharpoonright Y$ and $\mathfrak{R}' \upharpoonright X = (\mathfrak{R}' \upharpoonright X) \upharpoonright Y$. Otherwise, by the first assertion applied to $\mathfrak{R} \upharpoonright X$ and $\mathfrak{R}' \upharpoonright X$, $\mathfrak{R} \upharpoonright X \simeq \mathfrak{R}' \upharpoonright X$. Thus, $\mathfrak{R}$ and $\mathfrak{R}'$ are ($\leq k$)-hypomorphic. □

3. (≤ 3)-hypomorphic Posets

Let $P := (V, \mathcal{R})$ be a poset. By $x <_P y$, we denote the fact that $(x, y) \in \mathcal{R}$. By $x \parallel_P y$, we denote the fact that $(x, y) \notin \mathcal{R}$ and $(y, x) \notin \mathcal{R}$. The *dual* of P is the poset denoted by P^* and defined on the set V as follows: $x <_{P^*} y$ if and only if $y <_P x$. Finally, remark that P and P^* have the same modules.

Notation 1. *Given a poset P on a vertex set V, let A and B be two disjoint subsets of V, and x be an element of $V \setminus A$. Write,*

- *$A <_P B$, if $a <_P b$ for all $a \in A$ and for all $b \in B$.*

- *$A \parallel_P B$, if $a \parallel_P b$ for all $a \in A$ and for all $b \in B$.*
  For $A = \{a\}$, $A <_P B$ (respectively $A \parallel_P B$) will be denoted simply by $a <_P B$ (respectively $a \parallel_P B$).
  For $B = \{b\}$, $A <_P B$ (respectively $A \parallel_P B$) will be denoted simply by $A <_P b$ (respectively $A \parallel_P b$).

- *$x \sim_P A$ (or simply $x \sim A$), if A is a module of $P \upharpoonright A \cup \{x\}$, i.e. $x <_P A$ or $A <_P x$ or $x \parallel_P A$.*

- *$x \nsim_P A$ (or simply $x \nsim A$), if A is not a module of $P \upharpoonright A \cup \{x\}$.*

3.1. Particular Posets

In this subsection, we present some useful particular posets.

$\mathcal{V}$-order

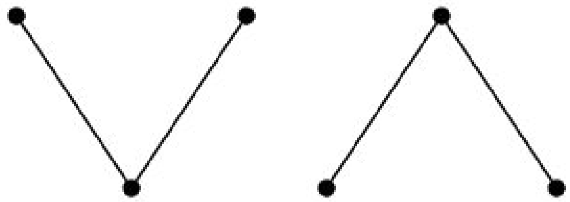

Figure 1. $\mathcal{V}$-orders

A 3-element order is called $\mathcal{V}$*-order* if it is isomorphic to one of the two orders illustrated in Figure 1.

Lozenge

A 4-element order is called a *lozenge* if it is isomorphic to the order illustrated in Figure 2.

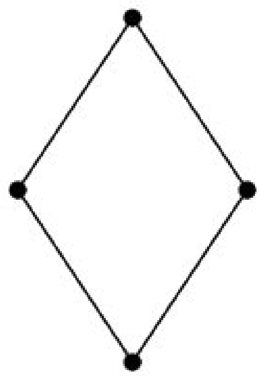

Figure 2. Lozenge

3.2. Posets and Maximal Chain-modules

Chain-modules

A *chain-module* M of a poset P is a module of P which is a chain.

In [7], the authors showed that the maximal chain-modules form a partition of the vertex set of a given binary relation and they showed the following lemma.

Lemma 3. *[7] Given a binary relation $\mathcal{R}$, the union of any collection of chain-modules containing a given vertex is a chain-module. Every non-empty chain-module of $\mathcal{R}$ is included in a unique maximal one. In particular, the maximal chain-modules form a partition of its vertex set, and in the corresponding quotient relation the chain-modules have at most one element.*

Notation 2. *Given a poset P on a vertex set V, and a proper module M of P, we consider the following useful subsets of $V \setminus M$;*

- $M_0 = \{a \in V \setminus M$ *such that* $a \parallel_P M\}$.
- $M^- = \{a \in V \setminus M$ *such that* $a <_P M\}$.
- $M^+ = \{a \in V \setminus M$ *such that* $M <_P a\}$.

Remark 1. *The set of non-empty elements of $\{M_0, M^+, M^-\}$ is a partition of $V \setminus M$.*

Proposition 1. *Given a poset P on a vertex set V such that P is not a chain, consider a maximal chain-module M of P and an element a of M^- (respectively M^+). Then at least one of the assertions below holds;*

1. There is $x \in M_0$ such that $P \restriction \{a, x, y\}$ is a $\mathcal{V}$-order, for all $y \in M$ (see Figure 3).

2. *There is $x \in M^- \setminus \{a\}$ (respectively $M^+ \setminus \{a\}$) such that $P \restriction \{a, x, y\}$ is a $\mathcal{V}$-order, for all $y \in M$ (see Figure 4).*

3. *There are $x \neq y \in M^- \setminus \{a\}$ (respectively $M^+ \setminus \{a\}$) such that $P \restriction \{a, x, y, z\}$ is a lozenge, for all $z \in M$ (see Figure 5).*

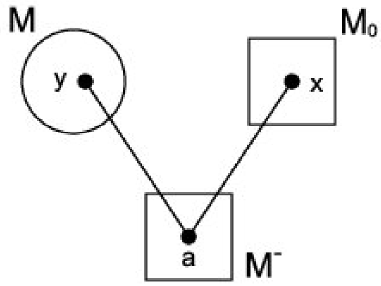

Figure 3. The First Assertion

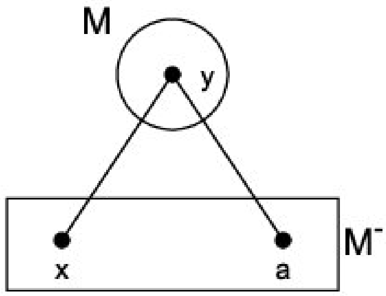

Figure 4. The Second Assertion

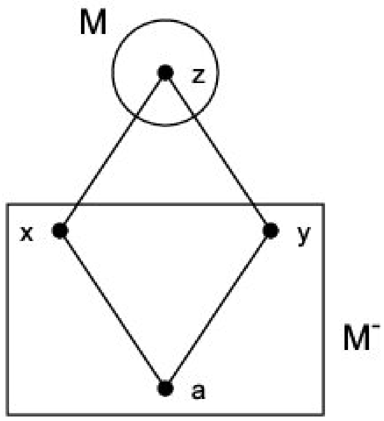

Figure 5. The Last Assertion

Proof. By Interchanging P and P^*, we may assume that $a \in M^-$. We distinguish the following two cases according to the comparability of a and the elements of M_0.

Case 1: There is $x \in M_0$ such that a and x are comparable. Since $a <_P M$ and $M \parallel_P x$, $a <_P x$. Therefore, $P \restriction \{a, x, y\}$ is a $\mathcal{V}$-order, for all $y \in M$, and thus assertion (1) holds.

Case 2: For all $z \in M_0$, $a \parallel_P z$; i.e. $a \parallel_P M_0$. Recall that $M \cup \{x\}$ is a chain by the definition of M^-. Using the maximality of the chain-module M in P, we conclude that $|M^-| \geq 2$. Now, according to the

comparability between a and the elements of $M^- \setminus \{a\}$, we will discuss the following two subcases.
Case 2.1: There is $x \in M^- \setminus \{a\}$ such that $x \parallel_P a$. In this case, $P \upharpoonright \{a, x, y\}$ is a $\mathcal{V}$-order, for all $y \in M$, and thus assertion (2) holds.
Case 2.2: For all $t \in M^- \setminus \{a\}$, a and t are comparable. Remark that $M \cup \{a\} \cup (\{a\}^+ \cap M^-) <_P M^+$ (because $M \cup M^- <_P M^+$) and $(\{a\}^- \cap M^-) <_P M \cup \{a\} \cup (\{a\}^+ \cap M^-)$. In addition, since $a <_P (\{a\}^+ \cap M^-)$, $a \parallel_P M_0$ and $M \parallel_P M_0$, $(\{a\}^+ \cap M^-) \parallel_P M_0$. Thus, $M \cup \{a\} \cup (\{a\}^+ \cap M^-) \parallel_P M_0$. It follows that, $M \cup \{a\} \cup (\{a\}^+ \cap M^-)$ is a module of P. Using the maximality of M in P, it follows that, $(\{a\}^+ \cap M^-) \cup \{a\}$ is not a chain of P. Therefore, $|(\{a\}^+ \cap M^-)| \geq 2$, and there are $x \neq y \in (\{a\}^+ \cap M^-)$ such that $x \parallel_P y$. Consequently, $P \upharpoonright \{a, x, y, z\}$ is a lozenge for all $z \in M$. Thus, assertion (3) holds. □

Remark 2. *By Proposition 1, each element of $M^+ \cup M^-$ belongs to a $\mathcal{V}$-order or a lozenge of P.*

The following useful remark is easily verified;

Remark 3. *Consider two (≤ 3)-hypomorphic posets P and P' on the same vertex set V and a subset X of V. If $P \upharpoonright X$ is a $\mathcal{V}$-order or a lozenge, then $P' \upharpoonright X = P \upharpoonright X$.*

Corollary 3. *Given two (≤ 3)-hypomorphic posets P and P' on the same vertex set V, consider a maximal chain-module M of P. Then $P' \upharpoonright \{a, x\} = P \upharpoonright \{a, x\}$, for all $x \in V \setminus M$ and $a \in M$.*

Proof. Since the result is clear if P is a chain, assume that $M \neq V$, and consider an element x of $V \setminus M$ and an element a of M. By Remark 1, $x \in M_0 \cup M^- \cup M^+$. Without loss of generality, by interchanging P and P^*, we may assume that x is an element of $M_0 \cup M^-$. First, assume that $x \in M_0$. Then, $P' \upharpoonright \{a, x\} = P \upharpoonright \{a, x\}$ because P and P' are (≤ 2)-hypomorphic. Second, assume that $x \in M^-$. By Remark 2, the vertex x belongs to a $\mathcal{V}$-order or a lozenge. In the first case, by Proposition 1, there is $b \in M_0 \cup M^-$ such that $P \upharpoonright \{a, b, x\}$ is a $\mathcal{V}$-order of P. Since the posets P and P' are (≤ 3)-hypomorphic and $P \upharpoonright \{a, b, x\}$ is a $\mathcal{V}$-order, $P' \upharpoonright \{a, b, x\} = P \upharpoonright \{a, b, x\}$ by Remark 3. In particular, $P' \upharpoonright \{a, x\} = P \upharpoonright \{a, x\}$. In the second case, by Proposition 1, there are $b \neq c \in (M^- \setminus \{x\})$ such that $P \upharpoonright \{a, b, c, x\}$ is a lozenge of P. Since P and P' are (≤ 3)-hypomorphic and $P \upharpoonright \{a, b, c, x\}$ is a lozenge, by Remark 3, $P' \upharpoonright \{a, b, c, x\} = P \upharpoonright \{a, b, c, x\}$. Consequently, $P' \upharpoonright \{a, x\} = P \upharpoonright \{a, x\}$. □

3.3. Proof of Theorem 1

Consider two (≤ 3)-hypomorphic posets P and P' with the same vertex set V. By Lemma 3, the maximal chain-modules of P and those of P' form two partitions of V. Denote by Q and Q' the partition of V in maximal chain-modules of P and P' respectively. Consider an element M of Q. Since P and P' are (≤ 3)-hypomorphic, M is a chain of P'. Now, consider an element x of $V \setminus M$. By Corollary 3, $P' \upharpoonright \{a, x\} = P \upharpoonright \{a, x\}$, for all $a \in M$. Moreover, $x \sim M$ in P because M is a module of P. It follows that, $x \sim M$ in P', and hence M is a chain-module of P'. Let M' be the element of Q' including M. By what precedes, M' is a chain-module of P. Thus, there is an element M'' of Q including M'. Consequently, $M'' = M' = M$, and hence $Q' = Q$. Finally, Corollary 3 implies that the quotient relations P/Q and P'/Q are equal.

Conversely, consider two posets P and P' on the same vertex set V having the same modular partition Q on maximal chain-modules such that $P/Q = P'/Q$. We have to prove that P and P' are (≤ 3)-hypomorphic. Let $M \in Q$. Since M is a chain of P and P', $P \upharpoonright M$ and $P' \upharpoonright M$ are (≤ 3)-hypomorphic. By Lemma 2, it follows that P and P' are (≤ 3)-hypomorphic.

Remark 4. *Theorem 1 can be obtained by the use of the difference classes introduced by G. Lopez [1–3], but our proof is self-contained.*

4. Bichains

4.1. Linear-vertex Subsets and Linear-modules

Given a bichain $\mathfrak{B} := (V, L_1, L_2)$, a vertex subset W is called *linear* if $L_2 \upharpoonright W = L_1 \upharpoonright W$ or $L_2 \upharpoonright W = L_1^* \upharpoonright W$. In the first case, W is *positive linear* and in the second it is *negative linear*. If in addition W is a module of $\mathfrak{B}$, speak of *linear-module*, positive linear-module and negative linear-module. If $L_2 = L_1$ or $L_2 = L_1^*$, speak of linear-bichain, positive linear-bichain and negative linear-bichain. Finally, we say that two linear-bichains have the same *kind* if they are both positive or both negative.

4.2. Maximal Linear-modules

We prove the following useful result by considering the linear-modules of a given bichain.

Lemma 4. *Given a bichain $\mathfrak{B}$, the union of any collection of linear-modules containing a given vertex is a linear-module. Every non-empty linear-module of $\mathfrak{B}$ is included in a unique maximal one. In particular, the maximal linear-modules of $\mathfrak{B}$ form a modular partition of $\mathfrak{B}$.*

Proof. Consider a bichain $\mathfrak{B} := (V, L_1, L_2)$. We start by the following two facts;
Fact 1: If M and N are two linear-modules such that $M \cap N \neq \emptyset$, then they are with the same kind.

Indeed, the result is immediate when $M \subseteq N$ or $N \subseteq M$. Now, assume that M and N overlap. In this case, $|M| \geq 2$ and $|N| \geq 2$. By Lemma 1, $N \setminus M$ and $M \setminus N$ are modules of $\mathfrak{B}$ and hence they are modules of both L_1 and L_2. To the contrary, suppose that M is positive and N is negative. It follows that, $|M \cap N| = 1$. Let denote by x the unique element of $M \cap N$. Observe that, x must be the smallest element or the largest one of the linear ordering $L_1 \upharpoonright N$ because $N \setminus M$ is a module of L_1. First, assume that x is the smallest element of the linear ordering $L_1 \upharpoonright N$, i.e $x <_{L_1} N \setminus M$. Since N is negative, it follows that x is also the smallest element of the linear ordering $L_2^* \upharpoonright N$, i.e $x <_{L_2^*} N \setminus M$. Recall that a linear ordering and its dual have the same modules. Since M and N are modules of both L_1 and L_2, M and N are also modules of L_1 and L_2^*. Therefore, the fact that $x <_{L_1} N \setminus M$ and $x <_{L_2^*} N \setminus M$, and M is a module of L_1 and L_2^* implies that $M \setminus N <_{L_1} N \setminus M$ and $M \setminus N <_{L_2^*} N \setminus M$. Since N is a module of L_1 and L_2^*, $M \setminus N <_{L_1} x$ and $M \setminus N <_{L_2^*} x$. Consequently, $M \setminus N <_{L_1} x$ and $x <_{L_2} M \setminus N$, which contradicts the fact that M is positive. Second, assume that x is the largest element of the linear ordering $L_1 \upharpoonright N$. By considering the linear orderings L_1^* and L_2 instead of L_1 and L_2^* in what precedes, we obtain a similar contradiction.
Fact 2: If M and N are two linear-modules with the same kind such that $M \cap N \neq \emptyset$, then $M \cup N$ is a linear-module with the same kind as M and N.

Indeed, we may assume that M and N overlap. Lemma 1 implies that $M \cup N$, $M \cap N$, $M \setminus N$ and $N \setminus M$ are modules of $\mathfrak{B}$ and hence they are modules of L_1. Thus, $N \setminus M <_{L_1} M \cap N$ or $M \cap N <_{L_1} N \setminus M$. First, assume that $N \setminus M <_{L_1} M \cap N$. Since M is a module of L_1, $N \setminus M <_{L_1} M \setminus N$. Moreover, the fact that N is a module of L_1 implies that $M \cap N <_{L_1} M \setminus N$. In other words, $N \setminus M <_{L_1} M \cap N <_{L_1} M \setminus N$. If M and N are both negative (respectively positive), then $M \setminus N <_{L_2} M \cap N <_{L_2} N \setminus M$ (respectively $N \setminus M <_{L_2} M \cap N <_{L_2} M \setminus N$). By transitivity of the linear ordering L_2, $M \setminus N <_{L_2} N \setminus N$ (respectively $N \setminus M <_{L_2} M \setminus N$). Thus, $M \cup N$ is a negative linear-module (respectively positive linear-module) of $\mathfrak{B}$. Second, assume that $M \cap N <_{L_1} N \setminus M$. The fact that M and N are modules of L_1 implies that $M \setminus N <_{L_1} N \setminus M$ and $M \setminus N <_{L_1} M \cap N$. In other words, $M \setminus N <_{L_1} M \cap N <_{L_1} N \setminus M$. If M and N are both negative (respectively positive), then $N \setminus M <_{L_2} M \cap N <_{L_2} M \setminus N$ (respectively $M \setminus N <_{L_2} M \cap N <_{L_2} N \setminus M$). By transitivity of the linear ordering L_2, $N \setminus M <_{L_2} M \setminus N$ (respectively $M \setminus N <_{L_2} N \setminus M$). Thus, $M \cup N$ is a negative linear-module (respectively positive linear-module) of $\mathfrak{B}$.

Now, let us prove the first assertion. Let denote by $\mathcal{U}$ the union of some collection C_x of linear-modules containing a given vertex x. We will prove that $\mathcal{U}$ is also a linear-module. Clearly, by

Lemma 1, $\mathcal{U}$ is a module of $\mathfrak{B}$. By Fact 1, all the members of C_x are with the same kind. First, assume that all the members of C_x are positive. We will prove that $\mathcal{U}$ is positive. Let $a \neq b \in \mathcal{U}$. There exist two members A and B of C_x such that $a \in A$ and $b \in B$. By Fact 2, $A \cup B$ is positive and hence $L_2 \restriction \{a, b\} = L_1 \restriction \{a, b\}$. It follows that, $L_2 \restriction \mathcal{U} = L_1 \restriction \mathcal{U}$. Second, assume that all the members of C_x are negative. We will prove that $\mathcal{U}$ is negative. Let $a \neq b \in \mathcal{U}$. There exist two members A and B of C_x such that $a \in A$ and $b \in B$. Fact 2 implies that $A \cup B$ is negative, and hence $L_2 \restriction \{a, b\} = L_1^* \restriction \{a, b\}$. It follows that, $L_2 \restriction \mathcal{U} = L_1^* \restriction \mathcal{U}$.

Since each singleton of the vertex set of $\mathfrak{B}$ is a non-empty linear-module of $\mathfrak{B}$, the last assertion is an immediate consequence of the second one. Finally, let us prove the second assertion. Consider a non-empty linear-module M of $\mathfrak{B}$. By the first assertion, the collection of all linear-modules including M is a linear-module with the same kind of M and thus it is the unique maximal one containing it. □

Notation 3.

1. *A finite bichain* $\mathfrak{B} := (\{x_1, \dots, x_n\}, L_1, L_2)$ *will be denoted by* $\begin{pmatrix} x_{\sigma_1(1)} & x_{\sigma_1(2)} & \dots & x_{\sigma_1(n)} \\ x_{\sigma_2(1)} & x_{\sigma_2(2)} & \dots & x_{\sigma_2(n)} \end{pmatrix}$, *where* σ_1 *and* σ_2 *are the permutations of the set* $\{1, \dots, n\}$ *such that* $L_1 := x_{\sigma_1(1)} < x_{\sigma_1(2)} < \dots < x_{\sigma_1(n)}$ *and* $L_2 := x_{\sigma_2(1)} < x_{\sigma_2(2)} < \dots < x_{\sigma_2(n)}$.

2. *Given a bichain* $\mathfrak{B} := (V, L_1, L_2)$, *let* M *be a vertex subset of* $\mathfrak{B}$ *and* x *be an element of* $V \setminus M$.
   - *Write* $x \sim M$, *if* M *is a module of* $\mathfrak{B} \restriction M \cup \{x\}$ *and* $x \nsim M$ *otherwise.*
   - *The set* $\{\{a, b\}, a \neq b \in V$ *such that* $L_2 \restriction \{a, b\} = L_1^* \restriction \{a, b\}\}$ *will be denoted simply by* $I(\mathfrak{B})$.

The lemma below is easily checked;

Lemma 5. *Let* $\mathfrak{B}$ *be a bichain with three vertices* x, y *and* z. *Then* $\mathfrak{B}$ *is isomorphic to one of the non-isomorphic following bichains,* $\begin{pmatrix} x & y & z \\ x & y & z \end{pmatrix}$, $\begin{pmatrix} x & y & z \\ z & y & x \end{pmatrix}$, $\begin{pmatrix} x & y & z \\ z & x & y \end{pmatrix}$, $\begin{pmatrix} x & y & z \\ x & z & y \end{pmatrix}$, $\begin{pmatrix} x & y & z \\ y & x & z \end{pmatrix}$ *and* $\begin{pmatrix} x & y & z \\ y & z & x \end{pmatrix}$.

Remark 5.

1. *Given a bichain* $\mathfrak{B}$ *on a set* V, $\mathfrak{B}$ *is positive linear (respectively negative linear) if and only if* $I(\mathfrak{B}) = \emptyset$ *(respectively* $I(\mathfrak{B})$ *is the set of all the 2-element subsets of* V*).*

2. *Given a positive integer* n, *up to isomorphism, there are a unique positive linear-bichain and a unique negative linear-bichain, with* n *vertices. These two linear-bichains are non isomorphic when* $n \geq 2$.

3. *Let* $\mathfrak{B}$ *and* $\mathfrak{B}'$ *be two bichains on a set* V. $\mathfrak{B}$ *and* $\mathfrak{B}'$ *are* (≤ 2)*-hypomorphic if and only if* $I(\mathfrak{B}) = I(\mathfrak{B}')$.

4. *A non linear-bichain on a 3-element set has a unique non trivial module.*

5. *Given a non linear-bichain* $\mathfrak{B}$ *on a set* $\{a, b, c\}$, *consider its unique non trivial module* M. *For each 2-element vertex subset* $X \neq M$, *there is a unique* $i \in \{1, 2\}$ *such that* $y \nsim_{L_i} X$ *where* $\{y\} = \{a, b, c\} \setminus X$.

Proof. The first assertion is easily checked by definitions of positive and negative linear-bichain. The second assertion is an immediate consequence of the first one. The third assertion follows from the definition of the (≤ 2)-hypomorphic bichains. The fourth and fifth assertions are easily checked by examining the four cases of non linear-bichains introduced in Lemma 5. □

The following useful corollary is immediately deduced from Remark 5;

Corollary 4. *Let $\mathfrak{B}$ be a linear-bichain on a set V, and $\mathfrak{B}'$ be a bichain on V. If $\mathfrak{B}'$ is (≤ 2)-hypomorphic to $\mathfrak{B}$, then $\mathfrak{B}'$ is linear with the same kind as $\mathfrak{B}$, and $\mathfrak{B}' \restriction X \simeq \mathfrak{B} \restriction X$ for each finite vertex subset X.*

Corollary 5. *Let $\mathfrak{B} := (\{x,y,z\}, L_1, L_2)$ be a bichain with three vertices. If $x \sim_{L_i} \{y,z\}$ and $x \nsim_{L_j} \{y,z\}$ where $\{i,j\} = \{1,2\}$, then $\mathfrak{B}$ is not linear.*

Proof. This result follows from the fact that if $\mathfrak{B}$ is a linear-bichain, then L_1 and L_2 have the same modules. □

Lemma 6. *Let $\mathfrak{B}$ and $\mathfrak{B}'$ be (≤ 3)-hypomorphic bichains with three vertices x, y and z. Then the following assertions hold.*

1. *If $\mathfrak{B} := \begin{pmatrix} x & y & z \\ x & y & z \end{pmatrix}$, then $\mathfrak{B}'$ is one of the six positive linear-bichains on $\{x,y,z\}$.*
2. *If $\mathfrak{B} := \begin{pmatrix} x & y & z \\ z & y & x \end{pmatrix}$, then $\mathfrak{B}'$ is one of the six negative linear-bichains on $\{x,y,z\}$.*
3. *If $\mathfrak{B} := \begin{pmatrix} x & y & z \\ z & x & y \end{pmatrix}$, then $\mathfrak{B}' = \mathfrak{B}$ or $\mathfrak{B}' := \begin{pmatrix} y & x & z \\ z & y & x \end{pmatrix}$.*
4. *If $\mathfrak{B} := \begin{pmatrix} x & y & z \\ x & z & y \end{pmatrix}$, then $\mathfrak{B}' = \mathfrak{B}$ or $\mathfrak{B}' := \begin{pmatrix} x & z & y \\ x & y & z \end{pmatrix}$.*
5. *If $\mathfrak{B} := \begin{pmatrix} x & y & z \\ y & x & z \end{pmatrix}$, then $\mathfrak{B}' = \mathfrak{B}$ or $\mathfrak{B}' := \begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$.*
6. *If $\mathfrak{B} := \begin{pmatrix} x & y & z \\ y & z & x \end{pmatrix}$, then $\mathfrak{B}' = \mathfrak{B}$ or $\mathfrak{B}' := \begin{pmatrix} x & z & y \\ z & y & x \end{pmatrix}$.*

Proof. First, let us prove the first and the second assertions. In that cases, observe that $\mathfrak{B}$ is a linear-bichain on $\{x,y,z\}$. Since $\mathfrak{B}'$ is (≤ 3)-hypomorphic to $\mathfrak{B}$, $\mathfrak{B}'$ is a linear-bichain with the same kind as $\mathfrak{B}$ by Corollary 4. Consequently, if $\mathfrak{B} := \begin{pmatrix} x & y & z \\ x & y & z \end{pmatrix}$, i.e. $\mathfrak{B}$ is a positive linear-bichain, then $\mathfrak{B}'$ is one of the six positive linear-bichains on $\{x,y,z\}$. If $\mathfrak{B} := \begin{pmatrix} x & y & z \\ z & y & x \end{pmatrix}$, i.e. $\mathfrak{B}$ is a negative linear-bichain, then $\mathfrak{B}'$ is one of the six negative linear-bichains on $\{x,y,z\}$.

Second, for the other assertions the proof follows immediately from Lemma 5 and the first assertion of Remark 5. For instance, let us prove the third assertion. Let $\mathfrak{B} := \begin{pmatrix} x & y & z \\ z & x & y \end{pmatrix}$. Since $\mathfrak{B}'$ is (≤ 3)-hypomorphic to $\mathfrak{B}$, $\mathfrak{B}' \simeq \mathfrak{B}$. Consequently, by Lemma 5, $\mathfrak{B}'$ is one of the following isomorphic bichains: $\begin{pmatrix} x & y & z \\ z & x & y \end{pmatrix}$, $\begin{pmatrix} x & z & y \\ y & x & z \end{pmatrix}$, $\begin{pmatrix} y & x & z \\ z & y & x \end{pmatrix}$, $\begin{pmatrix} y & z & x \\ x & y & z \end{pmatrix}$, $\begin{pmatrix} z & x & y \\ y & z & x \end{pmatrix}$ or $\begin{pmatrix} z & y & x \\ x & z & y \end{pmatrix}$. Moreover, Remark 5 implies that $I(\mathfrak{B}) = I(\mathfrak{B}')$. It follows that, $\mathfrak{B}' = \mathfrak{B}$ or $\mathfrak{B}' := \begin{pmatrix} y & x & z \\ z & y & x \end{pmatrix}$. □

The following useful corollary is immediately deduced from Lemma 6;

Corollary 6. *Let $\mathfrak{B} := (\{x,y,z\}, L_1, L_2)$ and $\mathfrak{B}' := (\{x,y,z\}, L'_1, L'_2)$ be (≤ 3)-hypomorphic bichains with three vertices such that $\mathfrak{B}$ is not linear where $\{x,y\}$ is its unique non trivial module. Then $\mathfrak{B} \restriction \{t,z\} = \mathfrak{B}' \restriction \{t,z\}$, for all $t \in \{x,y\}$.*

Notation 4. *Given a bichain $\mathfrak{B} := (V, L_1, L_2)$ on a set V, let M be a proper vertex subset and x be an element of $V \setminus M$. Write*

- $S^x_1 = \{y \in V \setminus (M \cup \{x\})$ *such that* $y \nsim_{L_1} M \cup \{x\}\}$.

- $S_2^x = \{y \in V \setminus (M \cup \{x\})$ *such that* $y \nsim_{L_2} M \cup \{x\}\}$.

Lemma 7. *Given a bichain* $\mathfrak{B} := (V, L_1, L_2)$*, a maximal linear-module* M *of* $\mathfrak{B}$*, and* $x \in V \setminus M$ *such that* $M \cup \{x\}$ *is linear, one of the following assertions holds.*

1. $S_1^x \setminus S_2^x \neq \emptyset$*, the bichain* $\mathfrak{B} \upharpoonright \{a, x, y\}$ *is not linear, and* $\{a, x\}$ *is not a module of* $\mathfrak{B} \upharpoonright \{a, x, y\}$*, for any vertices* a *and* y *where* $a \in M$ *and* $y \in S_1^x \setminus S_2^x$.

2. $S_2^x \setminus S_1^x \neq \emptyset$*, the bichain* $\mathfrak{B} \upharpoonright \{a, x, y\}$ *is not linear, and* $\{a, x\}$ *is not a module of* $\mathfrak{B} \upharpoonright \{a, x, y\}$*, for any vertices* a *and* y *where* $a \in M$ *and* $y \in S_2^x \setminus S_1^x$.

3. $S_1^x = S_2^x \neq \emptyset$ *and there are* $y \neq z \in S_1^x$ *with* $y <_{L_1} z$ *such that, for all* $a \in M$*, the bichain* $\mathfrak{B} \upharpoonright \{a, x, y, z\}$ *is one of the following bichains:* $\begin{pmatrix} a & y & z & x \\ a & z & y & x \end{pmatrix}$, $\begin{pmatrix} x & y & z & a \\ x & z & y & a \end{pmatrix}$, $\begin{pmatrix} a & y & z & x \\ x & y & z & a \end{pmatrix}$ *or* $\begin{pmatrix} x & y & z & a \\ a & y & z & x \end{pmatrix}$.

Proof. Consider a bichain $\mathfrak{B} := (V, L_1, L_2)$, and a maximal linear-module M of $\mathfrak{B}$. Let $x \in V \setminus M$ such that $M \cup \{x\}$ is linear, and $a \in M$. Since M is a maximal linear-module and $M \cup \{x\}$ is linear, $M \cup \{x\}$ is not a module of $\mathfrak{B}$. Thus, $S_1^x \cup S_2^x \neq \emptyset$. It follows that, $S_1^x \setminus S_2^x \neq \emptyset$ or $S_2^x \setminus S_1^x \neq \emptyset$ or $S_1^x = S_2^x \neq \emptyset$. For the first assertion, assume that $S_1^x \setminus S_2^x \neq \emptyset$. Let $y \in S_1^x \setminus S_2^x$. Clearly, $y \nsim_{L_1} \{a, x\}$ and $y \sim_{L_2} \{a, x\}$. Thus, $\{a, x\}$ is not a module of $\mathfrak{B} \upharpoonright \{a, x, y\}$. By Corollary 5, $\mathfrak{B} \upharpoonright \{a, x, y\}$ is not linear.

For the second assertion, assume that $S_2^x \setminus S_1^x \neq \emptyset$. Let $y \in S_2^x \setminus S_1^x$. Clearly, $y \sim_{L_1} \{a, x\}$ and $y \nsim_{L_2} \{a, x\}$. Thus, $\{a, x\}$ is not a module of $\mathfrak{B} \upharpoonright \{a, x, y\}$. By Corollary 5, $\mathfrak{B} \upharpoonright \{a, x, y\}$ is not linear.

For the third assertion, assume that $S_1^x = S_2^x \neq \emptyset$. Let $t \in V \setminus (M \cup S_1^x \cup \{x\})$. Let $i \in \{1, 2\}$. Since ($M <_{L_i} S_1^x <_{L_i} x$ or $x <_{L_i} S_1^x <_{L_i} M$) and $t \sim_{L_i} M \cup \{x\}$ because $t \notin S_1^x$, $t \sim_{L_i} M \cup S_1^x$ and $x \sim_{L_i} M \cup S_1^x$. It follows that, $M \cup S_1^x$ is a module of $\mathfrak{B}$. Since M is a maximal linear-module of $\mathfrak{B}$, $M \cup S_1^x$ is not linear. First, assume that $M \cup \{x\}$ is positive. We claim that there are $y \neq z \in S_1^x$ with $y <_{L_1} z$ such that $\{y, z\} \in I(\mathfrak{B} \upharpoonright S_1^x)$. Indeed, otherwise, $M \cup S_1^x$ is positive, and then it is a positive linear-module of $\mathfrak{B}$; witch contradicts the maximality of M. Thus, the bichain $\mathfrak{B} \upharpoonright \{a, x, y, z\}$ is one of the following bichains: $\begin{pmatrix} a & y & z & x \\ a & z & y & x \end{pmatrix}$, $\begin{pmatrix} x & y & z & a \\ x & z & y & a \end{pmatrix}$. Second, assume that $M \cup \{x\}$ is negative. We claim that there are $y \neq z \in S_1^x$ with $y <_{L_1} z$ such that $\{y, z\} \notin I(\mathfrak{B} \upharpoonright S_1^x)$. Indeed, otherwise, $M \cup S_1^x$ is negative, and then it is a negative linear-module of $\mathfrak{B}$; witch contradicts the maximality of M. Thus, the bichain $\mathfrak{B} \upharpoonright \{a, x, y, z\}$ is one of the following bichains: $\begin{pmatrix} a & y & z & x \\ x & y & z & a \end{pmatrix}$, $\begin{pmatrix} x & y & z & a \\ a & y & z & x \end{pmatrix}$. □

Lemma 8. *Let* $\mathfrak{B}$ *and* $\mathfrak{B}'$ *be* (≤ 3)*-hypomorphic bichains on the same vertex set* V*, and* M *be a maximal linear-module of* $\mathfrak{B}$*. Then* $\mathfrak{B}' \upharpoonright \{a, x\} = \mathfrak{B} \upharpoonright \{a, x\}$*, for any vertices* a, x *with* $a \in M$ *and* $x \in V \setminus M$.

Proof. Consider two (≤ 3)-hypomorphic bichains $\mathfrak{B} := (V, L_1, L_2)$ and $\mathfrak{B}' := (V, L'_1, L'_2)$. Let $x \in V \setminus M$ and $a \in M$. According to the linearity of $M \cup \{x\}$ in $\mathfrak{B}$, we distinguish the following two cases.
Case 1: Assume that $M \cup \{x\}$ is not linear.

In this case, $|M| \geq 2$. If M is positive, then $x <_{L_1} M \Leftrightarrow M <_{L_2} x$. If M is negative, then $x <_{L_1} M \Leftrightarrow x <_{L_2} M$. It follows that, for all $b \in M \setminus \{a\}$ $\mathfrak{B} \upharpoonright \{a, b, x\}$ is not linear and $\{a, b\}$ is the unique non trivial linear-module of $\mathfrak{B} \upharpoonright \{a, b, x\}$. Thus, Corollary 6 implies that $\mathfrak{B}' \upharpoonright \{a, x\} = \mathfrak{B} \upharpoonright \{a, x\}$.
Case 2: Assume that $M \cup \{x\}$ is linear.

Since $M \cup \{x\}$ is linear, Lemma 7 implies that one of its assertions holds. First, assume that assertion (1) holds. Let $y \in S_1^x \setminus S_2^x$. The bichain $\mathfrak{B} \upharpoonright \{a, x, y\}$ is not linear and $\{a, x\}$ is not a module of $\mathfrak{B} \upharpoonright \{a, x, y\}$. Thus, Corollary6 implies that $\mathfrak{B}' \upharpoonright \{a, x\} = \mathfrak{B} \upharpoonright \{a, x\}$. Second, assume that assertion (2) holds. Let $y \in S_2^x \setminus S_1^x$. The bichain $\mathfrak{B} \upharpoonright \{a, x, y\}$ is not linear and $\{a, x\}$ is not a module of $\mathfrak{B} \upharpoonright \{a, x, y\}$. Thus, Corollary6 implies that $\mathfrak{B}' \upharpoonright \{a, x\} = \mathfrak{B} \upharpoonright \{a, x\}$. Finally, assume that assertion

(3) holds. Let $y \neq z \in S_1^x$ with $y <_{L_1} z$ such that the bichain $\mathfrak{B} \restriction \{a, x, y, z\}$ is one of the following bichains: $\begin{pmatrix} a & y & z & x \\ a & z & y & x \end{pmatrix}$, $\begin{pmatrix} x & y & z & a \\ x & z & y & a \end{pmatrix}$, $\begin{pmatrix} a & y & z & x \\ x & y & z & a \end{pmatrix}$ or $\begin{pmatrix} x & y & z & a \\ a & y & z & x \end{pmatrix}$. Observe that, $\mathfrak{B} \restriction \{a, y, z\}$ and $\mathfrak{B} \restriction \{x, y, z\}$ are not linear-bichains and $\{y, z\}$ is the unique non trivial linear-module of $\mathfrak{B} \restriction \{a, y, z\}$ and $\mathfrak{B} \restriction \{x, y, z\}$. By Corollary 6, it follows that $\mathfrak{B}' \restriction \{a, t\} = \mathfrak{B} \restriction \{a, t\}$ and $\mathfrak{B}' \restriction \{t, x\} = \mathfrak{B} \restriction \{t, x\}$, for all $t \in \{y, z\}$. Now, let $t \in \{y, z\}$. Assume for instance that $\{a, x\}$ is positive and let $i \in \{1, 2\}$. If $a <_{L_1} t <_{L_1} x$ (respectively $x <_{L_1} t <_{L_1} a$), then $a <_{L'_i} t$ and $t <_{L'_i} x$ (respectively $x <_{L'_i} t$ and $t <_{L'_i} a$) because $\mathfrak{B}' \restriction \{a, t\} = \mathfrak{B} \restriction \{a, t\}$ and $\mathfrak{B}' \restriction \{t, x\} = \mathfrak{B} \restriction \{t, x\}$. Thus, by transitivity of the linear orderings L'_1 and L'_2 we deduce that $\mathfrak{B}' \restriction \{a, x\} = \mathfrak{B} \restriction \{a, x\}$. By the same reasoning in what precedes we obtain that $\mathfrak{B}' \restriction \{a, x\} = \mathfrak{B} \restriction \{a, x\}$, when $\{a, x\}$ is negative. □

4.3. Proof of Theorem 2

Consider two (≤ 3)-hypomorphic bichains $\mathfrak{B}$ and $\mathfrak{B}'$ on the same vertex set V. By Lemma 4, the maximal linear-modules of $\mathfrak{B}$ (respectively $\mathfrak{B}'$) form a partition of V. Denote by Q and Q' the partition of V into maximal linear-modules of $\mathfrak{B}$ and $\mathfrak{B}'$ respectively. Consider an element M of Q. First, we will prove that M is a linear-module of $\mathfrak{B}'$. Clearly, if $|M| = 1$, the result is obvious. Now, assume that $|M| \geq 2$. Since $\mathfrak{B}$ and $\mathfrak{B}'$ are (≤ 3)-hypomorphic and $\mathfrak{B} \restriction M$ is linear, by Corollary 4 $\mathfrak{B}' \restriction M$ is linear with the same kind as $\mathfrak{B} \restriction M$. Now, consider an element x of $V \setminus M$. By Lemma 8, $\mathfrak{B}' \restriction \{a, x\} = \mathfrak{B} \restriction \{a, x\}$, for all $a \in M$. Moreover, $x \sim M$ in $\mathfrak{B}$ because M is a module of $\mathfrak{B}$. It follows that, $x \sim M$ in $\mathfrak{B}'$ and hence M is a linear-module of $\mathfrak{B}'$. In conclusion, each element of Q is a linear-module of $\mathfrak{B}'$. By interchanging $\mathfrak{B}$ and $\mathfrak{B}'$, it follows that each element of Q' is a linear-module of $\mathfrak{B}$. Second, we will prove that M is also a maximal linear-module of $\mathfrak{B}'$. Let M' be an element of Q' including M. Since M' is a linear-module of $\mathfrak{B}$, there is an element M'' of Q including M'. Consequently, $M'' = M' = M$, and hence $M \in Q'$ which permits to conclude. Therefore, $Q' = Q$. Finally, by Lemma 8 the quotient bichains $\mathfrak{B}/Q$ and $\mathfrak{B}'/Q$ are equal.

Conversely, consider two bichains $\mathfrak{B}$ and $\mathfrak{B}'$ on the same vertex set V, having the same modular partition Q on maximal linear-modules such that $\mathfrak{B}/Q = \mathfrak{B}'/Q$, and each maximal linear-module has the same kind in both. We have to prove that $\mathfrak{B}$ and $\mathfrak{B}'$ are (≤ 3)-hypomorphic. Let $M \in Q$. Since $\mathfrak{B} \restriction M$ and $\mathfrak{B}' \restriction M$ are linear with the same kind, $\mathfrak{B} \restriction M$ and $\mathfrak{B}' \restriction M$ are (≤ 3)-hypomorphic. Thus, Lemma 2 implies that the bichains $\mathfrak{B}$ and $\mathfrak{B}'$ are (≤ 3)-hypomorphic.

5. (≤ 3)-reconstructibility of Posets and Bichains

5.1. (≤ 3)-reconstructibility of Posets

In this subsection we give a characterization of the (≤ 3)-reconstructible posets based on a description of the maximal chain-modules. We use essentially the following lemma which is an immediate consequence of Lemma 2 of [15] deduced from the study of Boudabbous and Delhommé [7].

Lemma 9. [15]

1. *The family of maximal chain-modules of a poset P is a modular partition of P.*

2. *If X is an infinite set, then there are at least two nonisomorphic chains on X.*

Corollary 7. *A poset is (≤ 3)-reconstructible if and only if its chain-modules are finite.*

Proof. Assume that a poset P has an infinite chain-module. By the first assertion of Lemma 9, the family $\mathcal{M}$ of maximal chain-modules of P forms a modular partition of P. Consider the family $\mathcal{F}$ of such maximal chain-modules having a same infinite cardinality $\mathcal{K}$. By the second assertion of Lemma 9 there are two nonisomorphic chains $\mathbf{c}_0$ and $\mathbf{c}_1$ with cardinality $\mathcal{K}$. Consider the following poset P_0 (respectively P_1) obtained from P by replacing $P \restriction A$ by a chain isomorphic to $\mathbf{c}_0$ (respectively $\mathbf{c}_1$), for each $A \in \mathcal{F}$. Now, let us prove that P_0 and P_1 are (≤ 3)-hypomorphic to P. Clearly, $\mathcal{M}$ is a common modular partition of P, P_0 and P_1 with $P/\mathcal{M} = P_0/\mathcal{M} = P_1/\mathcal{M}$. Moreover, for each $M \in \mathcal{M}$, $P \restriction M$, $P_0 \restriction M$ and $P_1 \restriction M$ are chains, and hence they are (≤ 3)-hypomorphic. Thus, Lemma 2 implies that P, P_0 and P_1 are (≤ 3)-hypomorphic. Consequently, by Theorem 1, $\mathcal{M}$ is a common partition into maximal chain-modules of P, P_0 and P_1. Since $\mathbf{c}_0$ and $\mathbf{c}_1$ are not isomorphic, P_0 and P_1 are not isomorphic. Therefore, at least one of the posets P_0 and P_1 is not isomorphic to P, and hence P is not (≤ 3)-reconstructible.

Conversely, let P be a poset with a vertex set V, and assume that its maximal chain-modules are finite. Let P' be a poset (≤ 3)-hypomorphic to P. We shall prove that P and P' are isomorphic. By Theorem 1, P and P' share the same maximal chain-modules, and they have the same corresponding quotient relation. Since the maximal chain-modules are finite, $P \restriction M$ and $P' \restriction M$ are isomorphic for each maximal chain-module M of P. Finally, by Lemma 2, P and P' are isomorphic. □

5.2. (≤ 3)-reconstructibility of Bichains

In this subsection we give a characterization of the (≤ 3)-reconstructible bichains based on a description of the maximal linear-modules.

Corollary 8. *A bichain is (≤ 3)-reconstructible if and only if its linear-modules are finite.*

Proof. Assume that a bichain $\mathfrak{B}$ has an infinite linear-module. By Lemma 4, the family $\mathcal{M}$ of maximal linear-modules of $\mathfrak{B}$ forms a modular partition of $\mathfrak{B}$. Consider the family $\mathcal{F}$ of such maximal linear-modules having a same infinite cardinality $\mathcal{K}$. By the second assertion of Lemma 9 there are two nonisomorphic linear orderings $\mathbf{c}_0$ and $\mathbf{c}_1$ with cardinality $\mathcal{K}$. Consider the following bichain $\mathfrak{B}$ (respectively $\mathfrak{B}_g$) obtained from $\mathfrak{B}$ by replacing $\mathfrak{B} \restriction A$ by a bichain isomorphic to $\begin{pmatrix}\mathbf{c}_0\\ \mathbf{c}_0\end{pmatrix}\Bigg($ respectively $\begin{pmatrix}\mathbf{c}_1\\ \mathbf{c}_1\end{pmatrix}$*Bigg*) for each A in $\mathcal{F}$ such that A is positive, and by a bichain isomorphic to $\begin{pmatrix}\mathbf{c}_0\\ \mathbf{c^*}_0\end{pmatrix}\Bigg($ respectively $\begin{pmatrix}\mathbf{c}_1\\ \mathbf{c^*}_1\end{pmatrix}\Bigg)$ for each A in $\mathcal{F}$ such that A is negative. Now, let us prove that $\mathfrak{B}_0$ and $\mathfrak{B}_1$ are (≤ 3)-hypomorphic to $\mathfrak{B}$. $\mathcal{M}$ is a common modular partition of $\mathfrak{B}$, $\mathfrak{B}_0$ and $\mathfrak{B}_1$ with $\mathfrak{B}/\mathcal{M} = \mathfrak{B}_0/\mathcal{M} = \mathfrak{B}_1/\mathcal{M}$. Moreover, for each $M \in \mathcal{M}$, the bichains $\mathfrak{B} \restriction M$, $\mathfrak{B}_0 \restriction M$ and $\mathfrak{B}_1 \restriction M$ are linear with the same kind, and hence they are (≤ 3)-hypomorphic. Thus, Lemma 2 implies that the bichains $\mathfrak{B}$, $\mathfrak{B}_0$ and $\mathfrak{B}_1$ are (≤ 3)-hypomorphic. Consequently, by Theorem 2, $\mathcal{M}$ is a common partition into maximal linear-modules of $\mathfrak{B}$, $\mathfrak{B}_0$ and $\mathfrak{B}_1$. Since the linear orderings $\mathbf{c}_0$ and $\mathbf{c}_1$ are not isomorphic, $\mathfrak{B}_0$ and $\mathfrak{B}_1$ are not isomorphic. Therefore, at least one of the bichains $\mathfrak{B}_0$ and $\mathfrak{B}_1$ is not isomorphic to $\mathfrak{B}$, and hence $\mathfrak{B}$ is not (≤ 3)-reconstructible.

Conversely, let $\mathfrak{B}$ be a bichain with a vertex set V, and assume that its maximal linear-modules are finite. Consider a bichain $\mathfrak{B}'$ (≤ 3)-hypomorphic to $\mathfrak{B}$ on the same vertex set V. We shall prove that $\mathfrak{B}$ and $\mathfrak{B}'$ are isomorphic. By Theorem 2, $\mathfrak{B}$ and $\mathfrak{B}'$ share the same maximal linear-modules, and they have the same corresponding quotient bichain and each maximal linear-module has the same kind in both. Since the maximal linear-modules of $\mathfrak{B}$ are finite, $\mathfrak{B} \restriction M \simeq \mathfrak{B}' \restriction M$ for each maximal linear-module M of $\mathfrak{B}$. Finally, by Lemma 2, $\mathfrak{B}$ and $\mathfrak{B}'$ are isomorphic. □

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Conflict of Interest

All authors declare that they have no conflicts of interest.

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