Ars Combinatoria, 156: 43–49
DOI:10.61091/ars156-5
http://www.combinatorialpress.com/ars
Received 17 August 2020, Accepted 10 December 2020, Published 27 July 2023

Combinatorial Press

Article

Equitable Edge Coloring of Splitting Graph of Some Classes of Wheel Graphs

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Abstract: The coloring of all the edges of a graph G with the minimum number of colors, such that the adjacent edges are allotted a different color is known as the proper edge coloring. It is said to be equitable, if the number of edges in any two color classes differ by atmost one. In this paper, we obtain the equitable edge coloring of splitting graph of W_n, DW_n and G_n by determining its edge chromatic number.

Keywords: splitting graph, equitable edge coloring, double wheel, gear
Mathematics Subject Classification: 05C15, 05C75.

1. Introduction

Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote the maximum degree of G by $\Delta(G)$. The first paper on edge coloring was written by Tait in 1880. Since all edges incident to the same vertex must be assigned different colors, obviously $\chi'(G) \geq \Delta(G)$.

In 1964, Vizing [1] proved that $\chi'(G) \leq \Delta(G) + 1$. In 1973, Meyer [2] presented the concept of equitable coloring and equitable chromatic number. The idea of equitable edge coloring was defined by Hilton and de Werra [3] in 1994. It is the assignment of colors to all the edges of G, so that the edges can be partitioned into color classes with the difference between any two classes is atmost one.

Sampathkumar et.al [4] introduced the concept of splitting graph of a graph G denoted by $S(G)$. It is framed by adding to each vertex v, a new vertex v' such that v' is adjacent to every vertex that is adjacent to v in G. We construct the splitting graph of some families of wheel graph and obtain the equitable edge coloring of such graphs.

In our day to day life many problems on optimization, network designing, scheduling problems, allocation of memory in computer networks and so on are related to edge coloring. For example, consider the time-tabling problem for the proper utilization of facilities, i.e., the minimum number of rooms needed at any one time can be scheduled by equitable edge coloring. In this paper, we determine the equitable edge chromatic number for splitting graph of W_n, DW_n and G_n.

2. Preliminaries

Theorem 1. *[1] Let G be a graph. Then $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.*

Definition 1. *[5] For any integer $n \geq 4$, the* wheel graph *W_n is the n-vertex graph obtained by joining a vertex v to each of the $n-1$ vertices $\{v_1, v_2, \ldots, v_n\}$ of the cycle graph C_{n-1}.*

Definition 2. *[6] A* double-wheel graph *DW_n of size n can be composed of $2C_n + K_1$, which consists of two cycles of size n, where the vertices of the two cycles are all connected to a common hub.*

Definition 3. *The* gear graph *G_n is the graph obtained from a wheel graph W_n by adding a vertex to each edge of the $n-1$ cycle in W_n.*

Definition 4. *[4] For each vertex v of a graph G, take a new vertex v'. Join v' to all vertices of G adjacent to v. The graph $S(G)$ thus obtained is called the* splitting graph *of G.*

Definition 5. *[3, 7, 8] An equitable edge coloring of a graph G is a k-proper edge coloring , if $\big||E_i| - |E_j|\big| \leq 1$, $i, j = 1, 2, \ldots, k$, where $E_i(G)$, $E_j(G)$ is the set of edges of color i and j repectively. The minimum of such k is called the equitable edge chromatic number of G and is denoted by $\chi'_=(G)$.*

Lemma 1. *[9] For any simple graph $G(V, E)$, $\chi'_=(G) \geq \Delta(G)$.*

Lemma 2. *[9] For any simple graph G and H, $\chi'_=(G) = \chi'(G)$ and if $H \subseteq G$ then $\chi'(H) \leq \chi'(G)$, where $\chi'(G)$ is the proper edge chromatic number of G.*

Lemma 3. *[8] Let G be a graph and let $k \geq 2$. If $k \nmid d(v)$ for each $v \in V(G)$, then G has an equitable edge-coloring with k colors.*

Lemma 4. *[8] Let G be a graph and let $k \geq 2$. If the k-core of G is a set of isolated vertices, then G has an equitable edge-coloring with k colors.*

Additional graph theory terminology used in this paper can be found in [9, 10].

In the following section, the equitable edge chromatic number of wheel W_n, double wheel DW_n and gear graph G_n are determined.

3. Equitable edge coloring of wheel, double wheel and gear graphs

Theorem 2. *The equitable edge chromatic number of splitting graph of wheel graph is $\chi'_=(S(W_n)) = 2(n-1)$, for $n \geq 5$.*

Proof. The wheel graph W_n consists of n vertices and $2(n-1)$ edges. Let

$$V(W_n) = \{v\} \bigcup \{v_i : 1 \leq i \leq n-1\} \text{ and}$$

$$E(W_n) = \{e_i : 1 \leq i \leq n-1\} \bigcup \{e_n\} \bigcup \{e_{i+n-1} : 2 \leq i \leq n-2\} \bigcup \{e_{2(n-1)}\}$$

where e_i is the edge vv_i $(1 \leq i \leq n-1)$, e_n is the edge v_1v_2, $e_{\{i+n-1\}}$ is the edge v_iv_{i+1} $(2 \leq i \leq n-2)$ and $e_{2(n-1)}$ is the edge $v_{n-1}v_1$.

For the construction of splitting graph of wheel graph, new vertices and edges are chosen. It consists of $2n$ vertices and $6(n-1)$ edges. Let

$$V(S(W_n)) = \{v\} \bigcup \{v'\} \bigcup \{v_i : 1 \leq i \leq n-1\} \bigcup \{v'_i : 1 \leq i \leq n-1\} \text{ and}$$

$$E(S(W_n)) = \{e_k\}, \text{ where } k = \begin{cases} i + j(n-1), 0 & \leq j \leq 3, 1 \leq i \leq n-1 \\ i + j(n-1), j = 4, & 1 \leq i \leq n-2 \\ i + j(n-1) - 2, j = 5, & 2 \leq i \leq n-1 \\ i + j(n-1) - 2, j = 6, & 1 \leq i \leq 2 \end{cases}$$

Table 1. Classification of edges of $S(W_n)$

Divisions of k	Values of j	Ranges of i	Edges between the vertices
$i + j(n-1)$	0	$1 \le i \le n-1$	vv_i
$i + j(n-1)$	1	$1 \le i \le n-2, i = n-1$	v_iv_{i+1},$v_{n-1}v_1$
$i + j(n-1)$	2	$1 \le i \le n-1$	vv'_i
$i + j(n-1)$	3	$1 \le i \le n-1$	$v'v_i$
$i + j(n-1)$	4	$1 \le i \le n-2$	$v_iv'_{i+1}$
$i + j(n-1) - 2$	5	$2 \le i \le n-1$	$v_iv'_{i-1}$
$i + j(n-1) - 2$	6	$i = 1, i = 2$	$v_1v'_{n-1}, v_{n-1}v'_1$

The edges $\{e_k\}$ between the vertices of the splitting graph of wheel graph are tabulated as follows: The coloring of edges is defined as

$$f : E(S(W_n)) \to C, \text{ where } C = \{1, 2, \ldots, 2(n-1)\}.$$

$$f(e_{i+k(n-1)}) = \begin{cases} i, \ k = 0, & 1 \le i \le n-1 \\ n-1, \ k = 1, & i = 1 \\ i-1, \ k = 1, & 2 \le i \le n-1 \\ i+n-1, & 2 \le k \le 3, \ 1 \le i \le n-1 \end{cases}$$

$$f(e_{i+4(n-1)}) = \begin{cases} n+2, & i = 1 \\ i+1, & 2 \le i \le n-2 \end{cases}$$

$$f(e_{i+5(n-1)-2}) = \begin{cases} i+n+1, & 2 \le i \le n-3 \\ n, & i = n-2 \\ 1, & i = n-1 \end{cases}$$

$$f(e_{i+6(n-1)-2}) = \begin{cases} 2, & i = 1 \\ n+1, & i = 2 \end{cases}$$

By this procedure, the assignment of colors to all the edges is attained with $2(n-1)$ colors. The edges of splitting graph of wheel graph are classified into independent color classes as $E(S(W_n)) = \{E_1, E_2, \ldots, E_{2(n-1)}\}$ such that $|E_1| = |E_2| = \ldots = \left|E_{2(n-1)}\right| = 3$ for any $n \ge 4$. Hence it is clear that $\left||E_i| - \left|E_j\right|\right| \le 1$ for $i \ne j$, thus satisfying equitablility condition. Therefore $\chi'_=(S(W_n)) \le 2(n-1)$. Since $\Delta = 2(n-1)$ and by lemma 1, it follows that $\chi'_=(S(W_n)) \ge \chi'(S(W_n)) \ge \Delta$. Hence $\chi'_=(S(W_n)) = 2(n-1)$. □

Theorem 3. *The equitable edge chromatic number of splitting graph of double wheel graph is* $\chi'_=(S(DW_n)) = 4(n-1)$*, for* $n \ge 5$.

Proof. The double wheel graph DW_n consists of $2n-1$ vertices and $4(n-1)$ edges.

$$\text{Let } V(DW_n) = \{v\} \bigcup \{v_i : 1 \le i \le n-1\} \bigcup \{u_i : 1 \le i \le n-1\}$$

and

$$\begin{aligned} E(DW_n) \ = \ & \{e_i : 1 \le i \le n-1\} \bigcup \{e_n\} \bigcup \{e_{i+n-1} : 2 \le i \le n-2\} \bigcup \{e_{2(n-1)}\} \bigcup \\ & \{e_{i+2(n-1)} : 1 \le i \le n-1\} \bigcup \{e_{3(n-1)+1}\} \bigcup \{e_{i+3(n-1)} : 2 \le i \le n-2\} \bigcup \{e_{4(n-1)}\} \end{aligned}$$

where e_i is the edge $vv_i\,(1 \le i \le n-1)$, e_n is the edge v_1v_2, e_{i+n-1} is the edge $v_iv_{i+1}\,(2 \le i \le n-2)$, $e_{2(n-1)}$ is the edge $v_{n-1}v_1$, $e_{i+2(n-1)}$ is the edge $vu_i\,(1 \le i \le n-1)$, $e_{3(n-1)+1}$ is the edge u_1u_2, $e_{i+3(n-1)}$ is the edge $u_iu_{i+1}\,(2 \le i \le n-2)$ and $e_{4(n-1)}$ is the edge $u_{n-1}u_1$.

The splitting graph of double wheel graph is constructed by adding new vertices and edges, which consists of $4n-2$ vertices and $12(n-1)$ edges. Let

$$\begin{aligned} V(S(W_n)) &= \{v\}\bigcup\{v'\}\bigcup\{v_i : 1 \le i \le n-1\}\bigcup\{v'_i : 1 \le i \le n-1\} \\ &\quad \bigcup\{u_i : 1 \le i \le n-1\}\bigcup\{u'_i : 1 \le i \le n-1\} \end{aligned}$$

and

$$E(S(DW_n)) = \{e_k\}, \text{where } k = \begin{cases} i + j(n-1), 0 & \le j \le 3, 6 \le j \le 9, \\ & 1 \le i \le n-1 \\ i + j(n-1), & j = 4, 10, 1 \le i \le n-2 \\ i + j(n-1) - 2, & j = 5, 11, 2 \le i \le n-1 \\ i + j(n-1) - 2, & j = 6, 12, 1 \le i \le 2 \end{cases}$$

The edges $\{e_k\}$ of the splitting graph of double wheel graph lying between the vertices are summarized as follows:

Table 2. Classification of edges of $S(DW_n)$

Divisions of k	Values of j	Ranges of i	Edges between the vertices
$i + j(n-1)$	0	$1 \le i \le n-1$	vv_i
$i + j(n-1)$	1	$1 \le i \le n-2, i = n-1$	v_iv_{i+1},$v_{n-1}v_1$
$i + j(n-1)$	2	$1 \le i \le n-1$	vv'_i
$i + j(n-1)$	3	$1 \le i \le n-1$	$v'v_i$
$i + j(n-1)$	4	$1 \le i \le n-2$	$v_iv'_{i+1}$
$i + j(n-1) - 2$	5	$2 \le i \le n-1$	$v_iv'_{i-1}$
$i + j(n-1) - 2$	6	$i = 1, i = 2$	$v_1v'_{n-1}, v_{n-1}v'_1$
$i + j(n-1)$	6	$1 \le i \le n-1$	vu_i
$i + j(n-1)$	7	$1 \le i \le n-2, i = n-1$	u_iu_{i+1},$u_{n-1}u_1$
$i + j(n-1)$	8	$1 \le i \le n-1$	vu'_i
$i + j(n-1)$	9	$1 \le i \le n-1$	$v'u_i$
$i + j(n-1)$	10	$1 \le i \le n-2$	$u_iu'_{i+1}$
$i + j(n-1) - 2$	11	$2 \le i \le n-1$	$u_iu'_{i-1}$
$i + j(n-1) - 2$	12	$i = 1, i = 2$	$u_1u'_{n-1}, u_{n-1}u'_1$

The coloring of edges is defined as

$$f : E(S(DW_n)) \to C, \text{ where } C = \{1, 2, \ldots, 4(n-1)\}.$$

$$f\left(e_{i+k(n-1)}\right) = \begin{cases} i, \ k = 0, & 1 \le i \le n-1 \\ n-1, \ k = 1, & i = 1 \\ i-1, \ k = 1, & 2 \le i \le n-1 \\ i+n-1, & 2 \le k \le 3, \ 1 \le i \le n-1 \end{cases}$$

$$f\left(e_{i+4(n-1)}\right) = \begin{cases} n+2, & i = 1 \\ i+1, & 2 \le i \le n-2 \end{cases}$$

$$f\left(e_{i+5(n-1)-2}\right)=\begin{cases} i+n+1, & 2\le i\le n-3\\ n, & i=n-2\\ 1, & i=n-1\end{cases}$$

$$f\left(e_{i+6(n-1)-2}\right)=\begin{cases} 2, & i=1\\ n+1, & i=2\end{cases}$$

$$f\left(e_{i+k(n-1)}\right)=\begin{cases} i+2(n-1),\ k=6, & 1\le i\le n-1\\ 3(n-1),\ k=7,\ i=1 & \\ i-1+2(n-1),\ k=7, & 2\le i\le n-1\\ i+3(n-1), & 8\le k\le 9,\ 1\le i\le n-1\end{cases}$$

$$f\left(e_{i+10(n-1)}\right)=\begin{cases} 3n, & i=1\\ i+1+2(n-1), & 2\le i\le n-2\end{cases}$$

$$f\left(e_{i+11(n-1)-2}\right)=\begin{cases} i-1+3n, & 2\le i\le n-3\\ 3n-2, & i=n-2\\ 2n-1, & i=n-1\end{cases}$$

$$f\left(e_{i+12(n-1)-2}\right)=\begin{cases} 2n, & i=1\\ 3n-1, & i=2\end{cases}$$

All the edges of this splitting graph are assigned colors by the above process with $4(n-1)$ different colors. The edges are classified into $E\left(S\left(DW_n\right)\right)=\{E_1,E_2,\ldots,E_{4(n-1)}\}$ such that each of the independent sets $|E_1|=|E_2|=\ldots=\left|E_{4(n-1)}\right|=3$ for any $n\ge 4$. It is clear that $\left||E_i|-\left|E_j\right|\right|\le 1$ for $i\ne j$, thus satisfying equitablility condition. Hence it is observed that $\chi'_{=}\left(S\left(DW_n\right)\right)\le 4(n-1)$. By lemma 1, it follows that $\chi'_{=}\left(S\left(DW_n\right)\right)\ge\chi'\left(S\left(DW_n\right)\right)\ge\Delta=4(n-1)$. Therefore $\chi'_{=}\left(S\left(DW_n\right)\right)=4(n-1)$. □

Theorem 4. *The equitable edge chromatic number of splitting graph of gear graph is* $\chi'_{=}\left(S\left(G_n\right)\right)=2(n-1)$, *for* $n\ge 4$.

Proof. The gear graph G_n consists of $2n-1$ vertices and $3(n-1)$ edges. Let

$$V\left(S\left(G_n\right)\right)=\{v\}\bigcup\{v_i:1\le i\le n-1\}\bigcup\{v'_i:1\le i\le n-1\}\text{ and}$$

$$E\left(G_n\right)\ =\ \{e_i:1\le i\le n-1\}\bigcup\{E'_i:1\le i\le n-1\}\bigcup\{E''_i:1\le i\le n-2\}\bigcup\{E''_{n-1}\}$$

where E_i is the edge $vv_i\,(1\le i\le n-1)$, E'_i is the edge $v_iv'_i\,(1\le i\le n-1)$, E''_i is the edge $v'_iv_{i+1}\,(1\le i\le n-2)$ and E'_{n-1} is the edge $v'_{n-1}v_1$.

The construction of splitting graph of gear graph consists of $4n-2$ vertices and $9(n-1)$ edges. Let

$$\begin{aligned}V\left(S\left(G_n\right)\right)\ =\ &\{v\}\bigcup\{v'\}\bigcup\{v_i:1\le i\le n-1\}\bigcup\{u_i:1\le i\le n-1\}\\ &\bigcup\{v'_i:1\le i\le n-1\}\bigcup\{u'_i:1\le i\le n-1\}\end{aligned}$$

and

$$E\left(S\left(G_n\right)\right)=\{E_k:k=i+j(n-1),1\le i\le n-1,0\le j\le 8\}$$

The edges $\{E_k\}$ connecting each of the vertices of the splitting graph of gear graph is obtained by substituting $k=i+j(n-1)$ and are tabulated as follows:

The edge coloring of splitting graph of gear graph is defined as

$$f:E\left(S\left(G_n\right)\right)\to C,\ \text{where the color set } C=\{1,2,\ldots,2(n-1)\}.$$

Table 3. Classification of edges of $S(G_n)$

Values of j	Ranges of i	Edges between the vertices
0	$1 \le i \le n-1$	vv_i
1	$1 \le i \le n-1$	v_iu_i
2	$1 \le i \le n-2, i = n-1$	u_iv_{i+1} , $u_{n-1}v_1$
3	$1 \le i \le n-1$	$v_iu'_i$
4	$1 \le i \le n-2, i = n-1$	$v_{i+1}u'_i$, $v_1u'_{n-1}$
5	$1 \le i \le n-1$	vv'_i
6	$1 \le i \le n-1$	$v'v_i$
7	$1 \le i \le n-1$	v'_iu_i
8	$1 \le i \le n-2, i = n-1$	$v'_{i+1}u_i$, v'_1u_{n-1}

$$f(E_i) = i, \qquad 1 \le i \le n-1$$

$$f(E_{i+(n-1)}) = \begin{cases} i+1, & 1 \le i \le n-2 \\ 1, & i = n-1 \end{cases}$$

$$f(E_{i+2(n-1)}) = \begin{cases} i, & 1 \le i \le n-2 \\ n-1, & i = n-1 \end{cases}$$

$$f(E_{i+3(n-1)}) = \begin{cases} i+2, & 1 \le i \le n-3 \\ i+3-n, & n-2 \le i \le n-1 \end{cases}$$

$$f(E_{i+4(n-1)}) = \begin{cases} i+1+n, & 1 \le i \le n-3 \\ n, & i = n-2 \\ n+1, & i = n-1 \end{cases}$$

$$f(E_{i+5(n-1)}) = i + (n-1), \ 1 \le i \le n-1$$

$$f(E_{i+6(n-1)}) = i + (n-1), \ 1 \le i \le n-1$$

$$f(E_{i+7(n-1)}) = \begin{cases} i+n, & 1 \le i \le n-2 \\ n, & i = n-1 \end{cases}$$

$$f(E_{i+8(n-1)}) = \begin{cases} i+(n-1), & 1 \le i \le n-2 \\ 2(n-1), & i = n-1 \end{cases}$$

By the above method of coloring all the edges of the splitting graph of gear graph are allocated with $2(n-1)$ colors. The edges are classified into $E(S(G_n)) = \{E_1, E_2, \ldots, E_{2(n-1)}\}$ such that each of the independent sets $|E_1| = |E_2| = \ldots = \left|E_{(n-1)}\right| = 4$ and $|E_n| = |E_{n+1}| = \ldots = \left|E_{2(n-1)}\right| = 5$ for all $n \ge 4$. Hence it is observed that $\left||E_i| - \left|E_j\right|\right| \le 1$ for $i \ne j$, satisfying equitablility condition. This implies $\chi'_=(S(G_n)) \le 2(n-1)$. By lemma 1, it follows that $\chi'_=(S(G_n)) \ge \chi'(S(G_n)) \ge \Delta = 2(n-1)$. Therefore $\chi'_=(S(G_n)) = 2(n-1)$. □

4. Conclusion

In this paper, the equitable edge chromatic number of wheel graph families such as wheel W_n, double wheel DW_n and gear graph G_n are obtained, the proofs provides an optimal solution to the equitable edge coloration of these splitting graphs. The equitable edge coloring in the area of splitting graphs and other product graphs is vast explorable and the focus on the determination of equitability in allocation can be extended to other families of graphs.

Acknowledgements

The authors sincerely thank the Referee for the careful reading, corrections and suggestions that have resulted in the improvement of the quality of this manuscript.

Conflict of Interest

The author declares no conflict of interests.

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