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Article

Extremal Regular Graphs of Given Chromatic Number

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Abstract: We define an extremal $(r|\chi)$-graph as an r-regular graph with chromatic number χ of minimum order. We show that the Turán graphs $T_{ak,k}$, the antihole graphs and the graphs $K_k \times K_2$ are extremal in this sense. We also study extremal Cayley $(r|\chi)$-graphs and we exhibit several $(r|\chi)$-graph constructions arising from Turán graphs.

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1. Introduction

An *r-regular* graph is a simple finite graph such that each of its vertices has degree r. Regular graphs are one of the most studied classes of graphs; especially those with symmetries such as Cayley graphs. Let Γ be a finite group and let $X = \{x_1, x_2, \dots, x_t\}$ a generating set for Γ such that $X = X^{-1}$ with $1_\Gamma \notin X$; a *Cayley* graph $Cay(\Gamma, X)$ has vertex set consisting of the elements of Γ and two vertices g and h are adjacent if $gx_i = h$ for some $1 \le i \le t$. Cayley graphs are regular but there exist non-Cayley vertex-transitive graphs. The Petersen graph is a classic example of this fact.

The *girth* of a graph is the size of its shortest cycle. An (r, g)*-graph* is an r-regular graph of girth g. An (r, g)*-cage* is an (r, g)-graph of smallest possible order. The *diameter* of a graph is the largest length between shortest paths of any two vertices. An $(r; D)$*-graph* is an r-regular graph of diameter D.

While the cage problem asks for the constructions of cages, the *degree-diameter problem* asks for the construction of $(r; D)$-graphs of maximum order. Both of them are open and active problems (see [1, 2]) in which, frequently, it is considered the restriction to Cayley graphs, see [3, 4].

In this paper, we study a similar problem using a well-known parameter of coloration instead of girth or diameter. A *k-coloring* of a graph G is a partition of its vertices into k independent sets. The *chromatic number* $\chi(G)$ *of* G is the smallest number k for which there exists a k-coloring of G.

We define an $(r|\chi)$*-graph* as an r-regular graph of chromatic number χ. In this work, we investigate the $(r|\chi)$-graphs of minimum order. We also consider the case of Cayley $(r|\chi)$-graphs.

The remainder of this paper is organized as follows: In Section 2 we show the existence of $(r|\chi)$-graphs, we define $n(r|\chi)$ as the order of the smallest $(r|\chi)$-graph, and similarly, we define $c(r|\chi)$ as the order of the smallest Cayley $(r|\chi)$-graph. We also exhibit lower and upper bounds on the orders of the extremal graphs. We show that the Turán graphs $T_{ak,k}$, antihole graphs (the complements of cycles) and $K_k \times K_2$ are Cayley $(r|\chi)$-graphs of order $n(r|\chi)$ for some r and χ. To prove that $K_k \times K_2$

are extremal we use instances of the Reed's Conjecture for which it is true. In Section 3 we only consider non-Cayley graphs. We give another upper bound for $n(r|\chi)$ and we exhibit two families of $(r|\chi)$-graphs with a few number of vertices which are extremal for some values of r and χ. Finally, in Section 4 we study the small values $2 \leq r \leq 10$ and $2 \leq \chi \leq 6$. We obtain a full table of extremal $(r|\chi)$-graphs except for the pair $(6|6)$.

2. Cayley $(r|\chi)$-graphs

It is known that for any graph G, $1 \leq \chi(G) \leq \Delta + 1$ where Δ is the maximum degree of G. Therefore, for any $(r|\chi)$-graph we have that

$$1 \leq \chi \leq r + 1.$$

Suppose that G is a $(r|1)$-graph. Hence G is the empty graph, then $r = 0$. Therefore, the extremal graph is the trivial graph. We can assume that $2 \leq \chi \leq r + 1$.

Next, we prove that for any r and χ such that $2 \leq \chi \leq r + 1$, there exists a Cayley $(r|\chi)$-graph G.

We recall that the (n, k)-*Turán graph* $T_{n,k}$ is the complete k-partite graph on n vertices whose partite sets are as nearly equal in cardinality as possible, i.e., it is formed by partitioning a set of $n = ak + b$ vertices (with $0 \leq b < k$) into the partition of independent sets $(V_1, V_2, \ldots, V_b, V_{b+1}, \ldots, V_k)$ with order $|V_i| = a + 1$ if $1 \leq i \leq b$ and $|V_i| = a$ if $b + 1 \leq i \leq k$. Every vertex in V_i has degree $a(k-1) + b - 1$ for $1 \leq i \leq b$ and every vertex in V_i has degree $a(k-1) + b$ for $b + 1 \leq i \leq k$. The (n, k)-Turán graph has chromatic number k, and size (see [5])

$$\left\lfloor \frac{(k-1)n^2}{2k} \right\rfloor.$$

Lemma 1. *The (ak, k)-Turán graph $T_{ak,k}$ is a Cayley graph.*

Proof. Let Γ be the group $\mathbb{Z}_a \times \mathbb{Z}_k$ and $X = \{(i, j) : 0 \leq i < a, 0 < j < k\}$. Then, the graph $Cay(\Gamma, X)$ is isomorphic to $T_{ak,k}$. □

Before to continue, we recall some definitions. Given two graphs H_1 and H_2, the *cartesian product* $H_1 \square H_2$ is defined as the graph with vertex set $V(H_1) \times V(H_2)$ and two vertices (u, u') and (v, v') are adjacent if either $u = v$ and u' is adjacent with v' in H_2, or $u' = v'$ and u is adjacent with v in H_1. The following proposition appears in [6].

Proposition 1. *The cartesian product of two Cayley graphs is a Cayley graph.*

On the other hand, the chromatic number of $H_1 \square H_2$ is the maximum between $\chi(H_1)$ and $\chi(H_2)$, see [7]. Now we can prove the following theorem.

Theorem 1. *For any r and χ such that $2 \leq \chi \leq r + 1$, there exists a Cayley $(r|\chi)$-graph.*

Proof. Let $r = a(\chi - 1) + b$ where $a \geq 1$ and $0 \leq b < \chi - 1$. Consider the Cayley graph $H_1 = T_{a\chi,\chi}$. The graph H_1 has chromatic number χ and it is an $a(\chi - 1)$-regular graph of order $a\chi$.

Additionally, consider the graph $H_2 = T_{b+1,b+1} = K_{b+1}$. The graph H_2 has chromatic number $b + 1 < \chi$ and it is a b-regular graph of order $b + 1$.

Therefore, the graph $G = H_1 \square H_2$ is a Cayley graph by Proposition 1 such that it has chromatic number

$$\max\{\chi(H_1), \chi(H_2)\} = \chi,$$

regularity r and order $a\chi(b + 1)$. □

Now, we define $n(r|\chi)$ as the order of the smallest $(r|\chi)$-graph and $c(r|\chi)$ as the order of the smallest Cayley $(r|\chi)$-graph. Hence,

$$r + 1 \leq n(r|\chi) \leq c(r|\chi) \leq a\chi(b + 1)$$

where $r = a(\chi - 1) + b$ with $a \geq 1$ and $0 \leq b < \chi - 1$.

To improve the lower bound we consider the (n, χ)-Turán graph $T_{n,\chi}$. Suppose G is an $(r|\chi)$-graph. Let ς be a χ-coloring of G resulting in the partition $(V_1, V_2, \ldots, V_\chi)$ with $|V_i| = a_i$ for $1 \leq i \leq \chi$. Then the largest possible size of G occurs when G is a complete χ-partite graph with partite sets $(V_1, V_2, \ldots, V_\chi)$ and the cardinalities of these partite sets are as equal as possible. This implies that

$$\frac{nr}{2} \leq \left\lfloor \frac{(\chi - 1)n^2}{2\chi} \right\rfloor \leq \frac{(\chi - 1)n^2}{2\chi},$$

since G has size $rn/2$. After some calculations we get that

$$\frac{r\chi}{\chi - 1} \leq n.$$

Theorem 2. *For any* $2 \leq \chi \leq r + 1$,

$$\left\lceil \frac{r\chi}{\chi - 1} \right\rceil \leq n(r|\chi) \leq c(r|\chi) \leq \frac{r - b}{\chi - 1}\chi(b + 1)$$

where $\chi - 1 | r - b$ *with* $0 \leq b < \chi - 1$.

An $(r|\chi)$-graph G of $n(r|\chi)$ vertices is called *extremal* $(r|\chi)$*-graph*. Similarly, a Cayley $(r|\chi)$-graph G of $c(r|\chi)$ vertices is called *extremal Cayley* $(r|\chi)$*-graph*. When $\chi - 1|r$ the lower bound and the upper bound of Theorem 2 are equal. We have the following corollary.

Corollary 1. *The Cayley graph* $T_{a\chi,\chi}$ *is an extremal* $(a(\chi - 1)|\chi)$*-graph.*

In the remainder of this paper we exclusively work with $b \neq 0$, that is, when $\chi - 1$ is not a divisor of r.

2.1. Antihole graphs

A *hole graph* is a cycle of length at least four. An *antihole graph* is the complement G^c of a hole graph G. Note that a hole graph and its antihole graph are both connected if and only if their orders are at least five. In this subsection we prove that antihole graphs of order n are extremal $(r|\chi)$-graphs for any n at least six. There are two cases depending of the number of vertices.

1. $G = C_n^c$ for $n = 2k$ and $k \geq 3$.

   The graph G has regularity $r = 2k - 3$ and chromatic number $\chi = k$. Any $(2k - 3|k)$-graph has an even number of vertices and at least $\frac{r\chi}{\chi-1} = \frac{(2k-3)k}{k-1} = 2k - \frac{k}{k-1}$ vertices.

   If $k > 2$, then $\frac{k}{k-1} < 2$. Therefore we have the following result:

   $$n(2k - 3, k) = c(2k - 3, k) = 2k$$

   for all $k \geq 3$.

2. $G = C_n^c$ for $n = 2k - 1$ and $k \geq 4$.

   The graph G has regularity $r = 2k - 4$ and chromatic number $\chi = k$. Any $(2k - 4|k)$-graph has at least $\frac{r\chi}{\chi-1} = \frac{(2k-4)k}{k-1} = 2k - 2 - \frac{2}{k-1}$ vertices.

   If $k - 1 > 2$, we have that $\frac{2}{k-1} < 1$. Therefore

   $$2k - 2 \leq n(2k - 4, k) \leq c(2k - 4, k) \leq 2k - 1$$

   for all $k \geq 4$.

Suppose that G is a $(2k-4|k)$-graph of $2k-2$ vertices. Then $G = ((k-1)K_2)^c$, i.e., G is the complement of a matching of $k-1$ edges. Then $\chi(G) = k-1$, a contradiction. Therefore

$$n(2k-4,k) = c(2k-4,k) = 2k-1$$

for all $k \geq 4$.

Therefore, we have the following theorem.

Theorem 3. *The antihole graphs of order $n \geq 6$ are extremal $(n-3|\lceil \frac{n}{2} \rceil)$-graphs.*

A hole graph is also considered a 2-*factor* since is a spanning 2-regular graph. For short, we denote the disjoint union of j cycles of lenght i as jC_i.

Let G be an union of cycles

$$a_3C_3 \cup a_4C_4 \cup \ldots \cup a_{2t}C_{2t}$$

for $a_i \geq 0$ with $i \in \{3,4,\ldots,2t\}$. Note that the complement G^c of G is the join of the complement of cycles.

Theorem 4. *The graph $(a_3C_3 \cup a_4C_4 \cup \ldots \cup a_{2t}C_{2t})^c$ is extremal if $a_5 + a_7 + \cdots + a_{2t-1} + 1 < a_3$.*

Proof. Let $G^c = (a_3C_3 \cup a_4C_4 \cup \ldots \cup a_{2t}C_{2t})^c$. The graph G^c has order $n = 3a_3 + 4a_4 + \cdots + 2ta_{2t}$, regularity $r = n-3$ and chromatic number $\chi = a_3 + 2a_4 + 3a_5 + 3a_6 + \cdots + ta_{2t-1} + ta_{2t}$ since the the chromatic numbers of C_3^c, C_4^c, C_5^c, ...,C_i^c are $1,2,3,\ldots,\lceil i/2 \rceil$ respectively.

Any $(r|\chi)$-graph has at least $\frac{r\chi}{\chi-1} = r + \frac{r}{\chi-1} = n - \frac{3\chi-n}{\chi-1}$ vertices for $r = n-3$. If $\frac{3\chi-n}{\chi-1} < 1$ then G^c is extremal, that is, when

$$2\chi + 1 < n,$$

i.e. when

$$a_5 + a_7 + \cdots + a_{2t-1} + 1 < a_3.$$

□

Moreover, we have the following results.

Theorem 5. *Since C_n^c is extremal then*

1. *When n is even, if $G = (a_3C_3 \cup a_4C_4 \cup \ldots \cup a_{2t}C_{2t})^c$ is a graph of order n such that $a_5 + a_7 + \cdots + a_{2t-1} = a_3$, then G is extremal.*
2. *When n is odd, if $G = (a_3C_3 \cup a_4C_4 \cup \ldots \cup a_{2t}C_{2t})^c$ is a graph of order n such that $a_5 + a_7 + \cdots + a_{2t-1} = a_3 + 1$, then G is extremal.*

Corollary 2. *Since the antihole graphs of order $n \geq 8$ are $(r|\chi)$-graphs, then there exist many non-isomorphic extremal $(r|\chi)$-graphs (not necessarily Cayley).*

For instance, there are three extremal $(5,4)$-graphs, namely, C_8^c, $(2C_4)^c$ and $(C_3 \cup C_5)^c$. See also Table 1.

2.2. The case of $r = \chi$

In this subsection, we discuss the case of $r = \chi = k$, i.e., the $(k|k)$-graphs of minimum order. We have the following bounds so far:

$$\left\lceil \frac{k^2}{k-1} \right\rceil = k + 1 \leq n(k|k) \leq 2k.$$

We prove that the upper bound is correct except for $k = 4$ and maybe for $k = 6, 8, 10, 12$. To achieve it, we assume that there exist $(k|k)$-graphs of order $n \leq 2k - 2$, that is

$$\left\lceil \frac{n}{2} \right\rceil < k = \chi. \tag{1}$$

Now, we use a bound for the chromatic number arising from the Reed's Conjecture, see [8]. We recall the clique number $\omega(G)$ of a graph G is the largest k for which G has a complete subgraph of order k.

Conjecture 1. *For every graph G,*

$$\chi(G) \leq \left\lceil \frac{\omega(G) + 1 + \Delta(G)}{2} \right\rceil.$$

It is known that the conjecture is true for graphs satisfying Equation 1, see [9]. It follows that $k \leq \omega(G) + 1$ for any $(k|k)$-graph G of order $n \leq 2k - 2$, that is, $\omega(G) = k$ or $\omega(G) = k - 1$.

Case 1: $\omega(G) = k$.

Let H_1 be a clique of G and $H_2 = G \setminus V(H_1)$. There is a set of k edges from $V(H_1)$ and $V(H_2)$. Therefore, if $t = n - k \leq k - 2$ is the order of H_2 and $m = (kt - k)/2$ is the number of edges in H_2, then

$$m \leq \binom{t}{2}.$$

We obtain that $k \leq t$, a contradiction.

Case 2: $\omega(G) = k - 1$.

Let H_1 be a clique of G and $H_2 = G \setminus V(H_1)$. There is a set of $2(k - 1)$ edges from $V(H_1)$ to $V(H_2)$. Therefore, if $t = n - (k - 1) \leq k - 1$ is the order of H_2 and $m = (kt - 2(k - 1))/2$ is the number of edges in H_2, then

$$m \leq \binom{t}{2}.$$

We obtain that $k \leq t + 1$, hence, $k = t + 1$ and n has to be $2k - 2$. Since every vertex v in $V(H_2)$ has degree k in G, v has at least two neighbours in H_1. By symmetry, G is the union of two complete graphs K_{k-1} with the addition of two perfect matchings between them. Its complement is a $(k - 3)$-regular bipartite graph. Any perfect matching of G^c induce a $(k - 1)$-coloring in G, a contradiction.

We have the following results.

Lemma 2. *For any $k \geq 3$,*

$$2k - 1 \leq n(k|k) \leq c(k|k) \leq 2k.$$

If k is odd then the order of any k-regular graph is even, therefore:

Corollary 3. *For any $k \geq 3$ an odd number, $n(k|k) = c(k|k) = 2k$.*

We have that C_7^c is the extremal (4|4)-graph. Next, assume that $k \geq 6$ is an even number and there exists a $(k|k)$-graph G of $n = 2k - 1$ vertices. Owing to the fact that $\chi(G) \leq n - \alpha(G) + 1$ where $\alpha(G)$ is the independence number of G, we get that $\alpha(G) \leq k$.

In [9] was proved that the Reed's conjecture holds for graphs of order n satisfying $\chi > \frac{n+3-\alpha}{2}$. In the case of the graph G, we have that

$$\frac{n + 3 - \alpha(G)}{2} \leq \frac{k}{2} + 1 < k.$$

It follows that $\omega(G) \leq k \leq \omega(G) + 1$. Newly, we have two cases:

Case 1: $\omega(G) = k$.

As we saw before, let H_1 be a clique of G and $H_2 = G \setminus V(H_1)$. There is a set of k edges from $V(H_1)$ and $V(H_2)$. Therefore, if $t = k - 1$ is the order of H_2 and $m = (kt - k)/2$ is the number of edges in H_2, then

$$m \leq \binom{t}{2}.$$

We obtain that $k \leq t$, a contradiction.

Case 2: $\omega(G) = k - 1$.

In [10] was proved that every graph satisfies

$$\chi \leq \left\{ \omega, \Delta - 1, \left\lceil \frac{15 + \sqrt{48n + 73}}{4} \right\rceil \right\}.$$

Hence, for the graph G we have that $k \leq \left\lceil \frac{15+\sqrt{96k+25}}{4} \right\rceil$. After some calculations we get that $k = 6, 8, 10, 12$, otherwise, $k > \left\lceil \frac{15+\sqrt{96k+25}}{4} \right\rceil$.

Finally, we have the following theorem.

Theorem 6. *For any $k \geq 3$ such that $k \notin \{4, 6, 8, 10, 12\}$,*

$$n(k|k) = c(k|k) = 2k.$$

Moreover, if $k = 4$ then $n(k|k) = c(k|k) = 2k - 1$ and if $k \in \{6, 8, 10, 12\}$ then

$$2k - 1 \leq n(k|k) \leq c(k|k) \leq 2k.$$

We point out that if there exists an extremal $(k|k)$-graph G of $2k - 1$ vertices for $k \in \{6, 8, 10, 12\}$, then G has clique number $\omega = k - 1$, a clique H_1 of order ω for which $G \setminus V(H_1)$ has $\frac{k}{2} - 1$ edges, G is Hamiltonian-connected and it has independence number $\alpha(G)$ such that $\alpha(G) \in \{k/4, \dots, k/2 + 1\}$, see [10].

3. Non-Cayley constructions

In this section we improve the upper bound of $n(r|\chi)$ given on Theorem 2 by exhibiting a construction of graphs not necessarily Cayley. We assume that r is not a multiple of $\chi - 1$, therefore $2 \leq \chi \leq r$. Additionally, we show two more constructions which are tight for some values.

3.1. Upper bound

To begin with, take the Turán graph $T_{n,\chi}$, for $n = a\chi + b$, $0 < b < \chi$ with $r = a(\chi - 1) + b$ and the partition $(V_1, V_2, \dots, V_b, V_{b+1}, \dots, V_\chi)$ such that $|V_i| = a + 1$ if $1 \leq i \leq b$ and $|V_i| = a$ if $b + 1 \leq i \leq \chi$. Every vertex in V_i for $1 \leq i \leq b$ has degree $r - 1$ and every vertex in V_i for $b + 1 \leq i \leq \chi$ has degree r.

Next, we define the graph $G_{n,\chi}$ as the graph formed by two copies G_1 and G_2 of $T_{n,\chi}$ with the addition of a matching between the vertices of degree $r - 1$ of G_1 and the vertices of degree $r - 1$ of G_2 in the natural way. In consequence, the graph $G_{n,\chi}$ is an r-regular graph of order $2n$ and chromatic number χ. To obtain its chromatic number, suppose that $T_{n,\chi}$ has the vertex partition V_i, then the vertices of V_i have the color i in G_1 and the vertices of V_i are colored $i + 1 \bmod \chi$ in G_2. Hence $\chi = \chi(G_1) \leq \chi(G_{n,\chi}) \leq \chi$ and then $\chi(G_{n,\chi}) = \chi$.

Theorem 7. *For $2 \leq \chi \leq r + 1$, then*

$$\left\lceil \frac{r\chi}{\chi - 1} \right\rceil \leq n(r|\chi) \leq \min \left\{ 2 \left\lfloor \frac{r\chi}{\chi - 1} \right\rfloor, \frac{r - b}{\chi - 1}\chi(b + 1) \right\},$$

where $\chi - 1 | r - b$ with $0 \leq b < \chi$.

3.2. The graph $T^*_{n,\chi}$

In this subsection we give a better construction for some values of r and χ. Consider the $(a\chi+b,\chi)$-Turán graph $T_{a\chi+b,\chi}$ such that $\chi > b \geq 0$ and partition $(V_1, \dots, V_{\chi-b}, \dots, V_\chi)$ for $\chi \geq 3$, $|V_i| = a_i = a \geq 2$ with $i \in \{1, \dots, \chi - b\}$ and $|V_i| = a_i = a + 1 \geq 3$ with $i \in \{\chi - b + 1, \dots, \chi\}$.

We claim that a is even or $\chi - b$ is even. To prove it, assume that a and $\chi - b$ are odd. Hence, if b is even, then χ is odd, $n = a\chi + b$ is odd and r is odd, a contradiction. If b is odd, then χ is even, $n = a\chi + b$ is odd and r is odd, newly, a contradiction.

Now, we define the graph $T^*_{n,\chi}$ of regularity $r = a(\chi - 1) + b - 1$ as follows: If $\chi - b$ is even, the removal of a perfect matching between X_i and X_{i+1} for all $i \in \{1, 3, \dots, \chi - b - 1\}$ of $T_{n,\chi}$ produces $T^*_{n,\chi}$. If $\chi - b \geq 3$ is odd then a is even, therefore, the removal of a perfect matching between X_i and X_{i+1} for all $i \in \{4, 6 \dots, \chi - b - 1\}$ and a perfect matching between V'_1 and V''_2, V'_2 and V''_3, and V'_3 and V''_1 where $V_i \setminus V'_i = V''_i$ is a set of $a/2$ vertices for $i \in \{1, 2, 3\}$, of $T_{n,\chi}$ produces $T^*_{n,\chi}$.

The graphs $T^*_{n,\chi}$ improve the upper bound given in Theorem 7 for some numbers n and χ:

$$\frac{r\chi}{\chi - 1} = a\chi + b - \frac{\chi - b}{\chi - 1} \leq a\chi + b.$$

Hence, if $\frac{\chi-b}{\chi-1} < 1$, the construction gives extremal graphs, that is, when

$$1 < b.$$

Theorem 8. *Let $\chi \geq 3$, $\chi \geq b > 1$ and $a \geq 2$. Then the graph $T^*_{a\chi+b,\chi}$ defined above is an extremal $(a(\chi - 1) + b - 1|\chi)$-graph when $\chi - b$ is even or $a > 2$ is even.*

3.3. The graph $G_{a,c,t}$

Consider the (at, t)-Turán graph $T_{at,t}$ with partition $(V_1, \dots, V_t)$. Now, we define the graph $G_{a,c,t}$ with $1 \leq c < a$ as follows: consider two parts of $(V_1, \dots, V_t)$, e.g. V_1 and V_2, and c vertices of these two parts $\{u_1, \dots, u_c\} \subseteq V_1$ and $\{v_1, \dots, v_c\} \subseteq V_2$.

The removal of the edges u_iv_j for $i, j \in \{1, \dots, c\}$ when $i \neq j$ (all the edges between $\{u_1, \dots, u_c\}$ and $\{v_1, \dots, v_c\}$ except for a matching) and the addition of the edges u_iu_j and v_iv_j for $i, j \in \{1, \dots, c\}$ when $i \neq j$ (all the edges between the vertices u_i and all the edges between the vertices v_i) results in the graph $G_{a,c,t}$.

The graph $G_{a,c,t}$ is a $a(t - 1)$-regular graph of order at. Its chromatic number is $t + c - 1$ because the partition

$$(V_1 \setminus \{u_2, \dots, u_c\}, V_2 \setminus \{v_1, \dots, v_{c-1}\}, V_2, \dots, V_t, \{u_2, v_1\}, \dots, \{u_c, v_{c-1}\})$$

is a proper coloring with $t + c - 1$ colors. Moreover, the graph $G_{a,c,t}$ has a clique of $t + c - 1$ vertices, namely, the vertices $\{u_1, \dots, u_c, x_2, \dots, x_t\}$ where $x_i \in V_i$ for $i \in \{3, \dots, t\}$ and $x_2 \in V_2 \setminus \{v_1 \dots, v_c\}$.

The graphs $G_{a,c,t}$ improve the upper bound given in Theorem 2:

$$\frac{t + c - 1}{t + c - 2}a(t - 1) = at - a\frac{c - 1}{t + c - 2} \leq at.$$

Hence, if $a\frac{c-1}{t+c-2} < 1$, the construction gives extremal graphs, that is, when

$$(a - 1)(c - 1) < t - 1.$$

Theorem 9. *Let $a, t \geq 2$ and $a > c \geq 1$. The graph $G_{a,c,t}$ defined above is an extremal $(a(t - 1)|at)$-graph when $(a - 1)(c - 1) < t - 1$.*

$r \setminus \chi$	2	3	4	5	6
2	$T_{4,2}$	$T_{3,3}$	-	-	-
3	$T_{6,2}$	C_6^c	$T_{4,4}$	-	-
4	$T_{8,2}$	$T_{6,3}$	C_7^c	$T_{5,5}$	-
5	$T_{10,2}$	$G_{5,2,2}$	$C_8^c, (2C_4)^c,$ $(C_3 \cup C_5)^c$	$K_5 \times K_2$	$T_{6,6}$
6	$T_{12,2}$	$T_{9,3}$	$T_{8,4}$	$C_9^c, (C_4 \cup C_5)^c$	?
7	$T_{14,2}$	$T^*_{12,3}$	$T^*_{10,4}$	$C_{10}^c, (C_4 \cup C_6)^c$ $(C_3 \cup C_7)^c$	$(2C_5)^c$
8	$T_{16,2}$	$T_{12,3}$	$G_{4,2,3}$	$T_{10,5}$	$C_{11}^c, (C_4 \cup C_7)^c$ $(C_5 \cup C_6)^c$
9	$T_{18,2}$	$T^{**}_{16,3}$	$T_{12,4}$	$T^*_{12,5}$	$C_{12}^c, (2C_6)^c, (3C_4)^c$ $(C_3 \cup C_4 \cup C_5)^c$ $(C_3 \cup C_9)^c$
10	$T_{20,2}$	$T_{15,3}$	$T^*_{14,4}$	$T^*_{13,5}$	$T_{12,6}$

Table 1. Extremal $(r|\chi)$-graphs.

4. Small values

In this section we exhibit extremal $(r|\chi)$-graphs of small orders. These exclude the extremal graphs given before. Table 1 shows the extremal $(r|\chi)$-graphs for $2 \leq r \leq 10$ and $2 \leq \chi \leq 6$.

4.1. *Extremal* (5|3)*-graph*

Suppose that G is an extremal (5|3)-graph of order 8, i.e., its order equals the lower bound given in Theorem 2. Then its complement is 2 regular. That is, G^c is C_8 or $C_5 \cup C_3$ or $C_4 \cup C_4$. By Theorem 5, the complement of C_8 or $C_5 \cup C_3$ or $C_4 \cup C_4$ has chromatic number 4. Since G is 5-regular, a (5|3)-graph of order 9 does not exist and therefore 10 is the best possible. The graph $G_{5,2,2}$ is an extremal (5|3)-graph with 10 vertices.

4.2. *Extremal* $(7|\chi)$*-graphs for* $\chi = 3, 6$

Let G be an extremal (7|3)-graph. Its order is at least 11. Since its degree is odd, its order is at least 12. The graph $T^*_{12,3}$ is an extremal (7|3)-graph.

Now, suppose that G is an extremal (7|6)-graph. G has at least 9 vertices. Newly, because it has an odd regularity, G has at least 10 vertices. If this is the case, its complement is a 2 regular graph. The graph $(2C_5)^c$ has chromatic number 6. It is unique and it is Cayley.

4.3. *Extremal* (9|3)*-graph*

Any (9|3)-graph has 14 vertices, i.e., its order equals the lower bound given in Theorem 2. Suppose that there exist at least one of degree 14. Let (V_1, V_2, V_3) a partition by independent sets. Some of the parts, V_1, has at least five vertices. Since the graph is 9-regular, V_1 has exactly 5 vertices. The induced graph of V_2 and V_3 is a bipartite regular graph of an odd number of vertices, a contradiction. Then, any (9|3)-graph has at least 16 vertices.

Consider the graph $T_{16,3}$ with partition (U, V, W) and the sets partition are $U = \{u_1, u_2, u_3, u_4, u_5\}$, $V = \{v_1, v_2, v_3, v_4, v_5\}$, $W = \{w_1, w_2, w_3, w_4, w_5, w_6\}$. The removal of the edges

$$\{w_1v_1, v_1u_1, u_1w_4, w_2v_2, v_2u_2, u_2w_5, w_3v_3, v_3u_3, u_3w_6, u_4v_4, v_4u_5, u_5v_5, v_5u_4\}$$

is the graph $T^{**}_{16,3}$ which is the extremal (9|3)-graph.

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Conflict of Interest

The author declares no conflict of interests.

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