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Article

Four Classes of Nonterminating ${}_3F_2(1)$-Series via Kummer and Thomae Transformations

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Abstract: By employing Kummer and Thomae transformations, we examine four classes of nonterminating ${}_3F_2(1)$-series with five integer parameters. Several new summation formulae are established in closed form.

Keywords: Hypergoemetric series, Kummer transformation, Thomae transformation
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1. Introduction and Motivation

Given an indeterminate x and an integer $n \in \mathbb{N}_0$, the rising factorial is defined by

$$(x)_0 \equiv 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{for} \quad n > 0.$$

Following Bailey [1], the classical hypergeometric series reads as

$${}_{1+p}F_p\left[\begin{matrix} a_0, a_1, \cdots, a_p \\ b_1, \cdots, b_p \end{matrix}\middle| z\right] = \sum_{k=0}^{\infty} \frac{(a_0)_k(a_1)_k\cdots(a_p)_k}{k!(b_1)_k\cdots(b_p)_k} z^k.$$

For $z = 1$, the series converges only when the real part of the sum of the numerator parameters is less that of the denominator parameters.

There exist numerous summation formulae of hypergeometric series in the literature (see for example [2–8]). By means of the algebro–geoemtric approach, Asakura–Otsubo–Terasoma [9] examined following exotic ${}_3F_2(1)$-series

$${}_3F_2\left[\begin{matrix} x, 1-x, y \\ 1, 1+y \end{matrix}\middle| 1\right] \quad \text{for} \quad x, y \in \mathbb{Q} \tag{1}$$

and proved, when $y = \frac{1}{2}$ and $x \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\}$ the following elegant formulae:

$${}_3F_2\left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, \frac{3}{2} \end{matrix}\middle| 1\right] = \frac{4}{\pi} G,$$

$${}_3F_2\left[\begin{matrix} \frac{1}{4}, \frac{3}{4}, \frac{1}{2} \\ 1, \frac{3}{2} \end{matrix}\middle| 1\right] = \frac{4}{\pi} \ln(1+\sqrt{2});$$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{3},\frac{2}{3},\frac{1}{2}\\ 1,\frac{3}{2}\end{matrix}\middle|1\right]=\frac{3\sqrt{3}}{\pi}\ln 2, $$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},\frac{1}{2}\\ 1,\frac{3}{2}\end{matrix}\middle|1\right]=\frac{3\sqrt{3}}{2\pi}\ln(2+\sqrt{3}). $$

By employing the integral representations, the authors [10] succeeded in not only reviewing the above identities, but also evaluating further series for $y=\frac{1}{2}$ and $x\in\{\frac{1}{5},\frac{2}{5},\frac{1}{8},\frac{3}{8},\frac{1}{10},\frac{3}{10},\frac{1}{12},\frac{5}{12}\}$. Moreover, Chen K-W [11] and the authors [12] extended these results to the following series by introducing five extra integer parameters

$$ {}_3F_2\left[\begin{matrix}x+a,1-x+c,y+e\\ 1+b,1+y+d\end{matrix}\middle|1\right] \quad \text{with} \quad \{a,b,c,d,e\}\subset\mathbb{Z} \tag{2} $$

provided that $b\geq 0$ and $a+c+e\leq b+d$ such that the series is well–defined and convergent.

Recall that for the nonterminating ${}_3F_2(1)$-series, there are two fundamental transformations named after Thomae and Kummer (cf. [1]§3.2 and Page 98)

$$ {}_3F_2\left[\begin{matrix}a,c,e\\ b,d\end{matrix}\middle|1\right]={}_3F_2\left[\begin{matrix}b-a,d-a,\Delta\\ c+\Delta,\ e+\Delta\end{matrix}\middle|1\right]\frac{\Gamma(\Delta)\Gamma(b)\Gamma(d)}{\Gamma(a)\Gamma(c+\Delta)\Gamma(e+\Delta)}, \tag{3} $$

$$ {}_3F_2\left[\begin{matrix}a,c,e\\ b,d\end{matrix}\middle|1\right]={}_3F_2\left[\begin{matrix}a,b-c,b-e\\ a+\Delta,\ b\end{matrix}\middle|1\right]\frac{\Gamma(\Delta)\Gamma(d)}{\Gamma(a+\Delta)\Gamma(d-a)}, \tag{4} $$

where $\Delta=b+d-a-c-e$ denotes the parameter excess. The objective of this paper is to investigate the following four classes of ${}_3F_2(1)$-series represented by the respective examples (in the right column):

$$ \text{[A]}\quad {}_3F_2\left[\begin{matrix}a,\ c,\ e-y\\ b+x,d-x\end{matrix}\middle|1\right]:\quad {}_3F_2\left[\begin{matrix}1,1,\frac{1}{2}\\ \frac{3}{2},\frac{3}{2}\end{matrix}\middle|1\right]=2G. $$

$$ \text{[B]}\quad {}_3F_2\left[\begin{matrix}a,c+x,e+x+y\\ b+x,d+y\end{matrix}\middle|1\right]:\quad {}_3F_2\left[\begin{matrix}1,\frac{1}{3},\frac{5}{6}\\ \frac{4}{3},\frac{3}{2}\end{matrix}\middle|1\right]=2\ln 2. $$

$$ \text{[C]}\quad {}_3F_2\left[\begin{matrix}a+y,c+x+y,e-x+y\\ b+y,d+y\end{matrix}\middle|1\right]:\quad {}_3F_2\left[\begin{matrix}\frac{1}{2},\frac{2}{3},\frac{4}{3}\\ \frac{3}{2},\frac{3}{2}\end{matrix}\middle|1\right]=\frac{3\sqrt{3}}{4}\ln(2+\sqrt{3}). $$

$$ \text{[D]}\quad {}_3F_2\left[\begin{matrix}a+x,c+x,e-y\\ b,d+x\end{matrix}\middle|1\right]:\quad {}_3F_2\left[\begin{matrix}\frac{1}{4},\frac{1}{4},\frac{1}{2}\\ 1,\frac{5}{4}\end{matrix}\middle|1\right]=\frac{\Gamma^2(\frac{1}{4})}{2\pi^{3/2}}\ln(1+\sqrt{2}), $$

where $x,y\in\mathbb{Q}$ and $\{a,b,c,d,e\}\subset\mathbb{Z}$ such that the series are not only well–defined and nonterminating, but also convergent and irreducible to known ${}_2F_1(1)$-series.

Their evaluations will be fulfilled by making use of Kummer and Thomae transformations in conjunction with the closed formulae for the series (2) obtained in [12] via the linearization method (cf. [13, 14]). The remaining part of the paper will be divided into four sections, dedicated separately to evaluations of the afore described four classes of ${}_3F_2(1)$-series [A], [B], [C] and [D].

2. Evaluation of the ${}_3F_2$(1)-Series in Class [A]

Performing the parameter replacements in Thomae transformation (3)

$$ e\to e-y,\ b\to b+x,\ d\to d-x:\Delta\to\sigma+y \quad \text{with} \quad \sigma=b+d-a-c-e, $$

we can state the resulting equation as the transformation formula below.

Theorem 1.

$${}_3F_2\left[\begin{matrix}a,\ c,\ e-y\\ b+x, d-x\end{matrix}\middle|1\right] = {}_3F_2\left[\begin{matrix}b-a+x, d-a-x, \sigma+y\\ \sigma+e, \sigma+c+y\end{matrix}\middle|1\right]\frac{\Gamma(b+x)\Gamma(d-x)\Gamma(\sigma+y)}{\Gamma(a)\Gamma(\sigma+e)\Gamma(\sigma+c+y)}.$$

This formula is valid for two variables $\{x, y\}$ and five integer parameters $\{a, c, e, b, d\}$ subject to conditions $a > 0$, $c > 0$, $\sigma + y > 0$ and $\sigma + e > 0$ such that both series are not only well–defined and convergent, but also nonterminating and irreducible to known ${}_2F_1(1)$-series.

Observe that the ${}_3F_2(1)$-series on the right-hand side of Theorem 1 has the same parameter structure as the exotic ${}_3F_2(1)$-series displayed in (2). By applying the summation formulae obtained in [10, 12], we can further evaluate, in closed form, the following ${}_3F_2(1)$-series in class (A), specified by $y = \frac{1}{2}$ and $x \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\}$.

- $\boxed{x = y = \frac{1}{2}}$

$${}_3F_2\left[\begin{matrix}1, 1, -\frac{1}{2}\\ \frac{1}{2}, \frac{5}{2}\end{matrix}\middle|1\right] = \frac{3}{8}(3-2G),$$

$${}_3F_2\left[\begin{matrix}1, 1, \frac{1}{2}\\ \frac{3}{2}, \frac{3}{2}\end{matrix}\middle|1\right] = 2G,$$

$${}_3F_2\left[\begin{matrix}1, 2, \frac{3}{2}\\ \frac{5}{2}, \frac{7}{2}\end{matrix}\middle|1\right] = \frac{15}{4}(6G-5).$$

- $\boxed{x = \frac{1}{3},\ y = \frac{1}{2}}$

$${}_3F_2\left[\begin{matrix}1, 1, \frac{1}{2}\\ \frac{4}{3}, \frac{5}{3}\end{matrix}\middle|1\right] = \frac{8\ln 2}{3},$$

$${}_3F_2\left[\begin{matrix}1, 2, \frac{1}{2}\\ \frac{7}{3}, \frac{8}{3}\end{matrix}\middle|1\right] = \frac{8}{21}(9-8\ln 2),$$

$${}_3F_2\left[\begin{matrix}1, 2, \frac{3}{2}\\ \frac{8}{3}, \frac{10}{3}\end{matrix}\middle|1\right] = \frac{14}{9}(32\ln 2-21).$$

- $\boxed{x = \frac{1}{4},\ y = \frac{1}{2}}$

$${}_3F_2\left[\begin{matrix}1, 1, -\frac{1}{2}\\ \frac{7}{4}, \frac{9}{4}\end{matrix}\middle|1\right] = \frac{3}{7\sqrt{2}}\{8\ln(1+\sqrt{2})-3\sqrt{2}\},$$

$${}_3F_2\left[\begin{matrix}1, 1, \frac{1}{2}\\ \frac{5}{4}, \frac{7}{4}\end{matrix}\middle|1\right] = \frac{3}{\sqrt{2}}\ln(1+\sqrt{2}),$$

$${}_3F_2\left[\begin{matrix}1, 2, \frac{3}{2}\\ \frac{9}{4}, \frac{11}{4}\end{matrix}\middle|1\right] = \frac{9}{8\sqrt{2}}\{7\sqrt{2}-5\ln(1+\sqrt{2})\}.$$

- $\boxed{x = \frac{1}{6},\ y = \frac{1}{2}}$

$${}_3F_2\left[\begin{matrix}1, 1, -\frac{1}{2}\\ \frac{7}{6}, \frac{11}{6}\end{matrix}\middle|1\right] = \frac{5}{64}\{2+3\sqrt{3}\ln(2+\sqrt{3})\},$$

$${}_3F_2\left[\begin{matrix}1, 1, \frac{1}{2}\\ \frac{7}{6}, \frac{11}{6}\end{matrix}\middle|1\right] = \frac{5}{2\sqrt{3}}\ln(2+\sqrt{3}),$$

$${}_3F_2\left[\begin{matrix}1, 1, \frac{3}{2}\\ \frac{11}{6}, \frac{13}{6}\end{matrix}\middle|1\right] = \frac{35}{9}\{3-\sqrt{3}\ln(2+\sqrt{3})\}.$$

3. Evaluation of the ${}_3F_2(1)$-Series in Class [B]

Alternatively, under the parameter settings

$$c \to c+x,\ e \to e+x+y,\ b \to b+x,\ d \to d+y : \Delta \to \sigma - x \quad \text{with} \quad \sigma = b+d-a-c-e;$$

Thomae transformation (3) becomes the following one.

Theorem 2.

$$ {}_3F_2\left[\begin{matrix} a, c+x, e+x+y \\ b+x, d+y \end{matrix}\middle| 1\right] = {}_3F_2\left[\begin{matrix} b-a+x, \sigma-x, d-a+y \\ \sigma+c, \sigma+e+y \end{matrix}\middle| 1\right] \frac{\Gamma(b+x)\Gamma(\sigma-x)\Gamma(d+y)}{\Gamma(a)\Gamma(\sigma+c)\Gamma(\sigma+e+y)}.$$

In order that the above two ${}_3F_2(1)$-series are not only well–defined and convergent, but also nonterminating and irreducible to known ${}_2F_1(1)$-series, the two variables $\{x, y\}$ and five integer parameters $\{a, c, e, b, d\}$ should satisfy the conditions $a > 0$, $b > c$, $\sigma > x$ and $\sigma + c > 0$. Evaluating the exotic ${}_3F_2(1)$-series on the right-hand side of Theorem 2 by the summation formulae shown in [10, 12], we find the following identities for the ${}_3F_2(1)$-series in class (B), specified by $y = \frac{1}{2}$ and $x \in \left\{\frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{6}, \frac{5}{6}\right\}$.

- $x = \frac{1}{3},\ y = \frac{1}{2}$

$$ {}_3F_2\left[\begin{matrix} 1, -\frac{2}{3}, -\frac{1}{6} \\ \frac{1}{3}, \frac{3}{2} \end{matrix}\middle| 1\right] = \frac{2}{7}(3 + 2\ln 2),$$

$$ {}_3F_2\left[\begin{matrix} 1, \frac{1}{3}, \frac{5}{6} \\ \frac{4}{3}, \frac{3}{2} \end{matrix}\middle| 1\right] = 2\ln 2.$$

- $x = \frac{2}{3},\ y = \frac{1}{2}$

$$ {}_3F_2\left[\begin{matrix} 1, -\frac{1}{3}, -\frac{5}{6} \\ \frac{2}{3}, \frac{1}{2} \end{matrix}\middle| 1\right] = \frac{2}{9}(12 - 5\ln 2),$$

$$ {}_3F_2\left[\begin{matrix} 1, \frac{2}{3}, \frac{7}{6} \\ \frac{5}{3}, \frac{3}{2} \end{matrix}\middle| 1\right] = 4\ln 2.$$

- $x = \frac{1}{4},\ y = \frac{1}{2}$

$$ {}_3F_2\left[\begin{matrix} 1, \frac{1}{4}, \frac{3}{4} \\ \frac{5}{4}, \frac{3}{2} \end{matrix}\middle| 1\right] = \sqrt{2}\ln(1+\sqrt{2}),$$

$$ {}_3F_2\left[\begin{matrix} 1, \frac{5}{4}, \frac{3}{4} \\ \frac{9}{4}, \frac{3}{2} \end{matrix}\middle| 1\right] = \frac{5}{\sqrt{2}}\{\sqrt{2} - \ln(1+\sqrt{2})\}.$$

- $x = \frac{3}{4},\ y = \frac{1}{2}$

$$ {}_3F_2\left[\begin{matrix} 1, -\frac{1}{4}, -\frac{3}{4} \\ \frac{3}{4}, \frac{1}{2} \end{matrix}\middle| 1\right] = \frac{1}{2\sqrt{2}}\{5\sqrt{2} - 3\ln(1+\sqrt{2})\},$$

$$ {}_3F_2\left[\begin{matrix} 1, \frac{3}{4}, \frac{5}{4} \\ \frac{7}{4}, \frac{3}{2} \end{matrix}\middle| 1\right] = 3\sqrt{2}\ln(1+\sqrt{2}).$$

- $x = \frac{1}{6},\ y = \frac{1}{2}$

$$ {}_3F_2\left[\begin{matrix} 1, -\frac{5}{6}, -\frac{1}{3} \\ \frac{1}{6}, \frac{3}{2} \end{matrix}\middle| 1\right] = \frac{1}{8}\{6 + 5\sqrt{3}\ln(2+\sqrt{3})\},$$

$$ {}_3F_2\left[\begin{matrix} 1, \frac{1}{6}, \frac{2}{3} \\ \frac{7}{6}, \frac{3}{2} \end{matrix}\middle| 1\right] = \frac{\sqrt{3}}{2}\ln(2+\sqrt{3}).$$

- $\boxed{x = \frac{5}{6},\ y = \frac{1}{2}}$

$$ {}_3F_2\left[\begin{matrix}1, -\frac{1}{6}, \frac{1}{3}\\ \frac{5}{6}, \frac{3}{2}\end{matrix}\middle|1\right] = \frac{1}{4}\{6 - \sqrt{3}\ln(2+\sqrt{3})\}, $$

$$ {}_3F_2\left[\begin{matrix}1, \frac{5}{6}, \frac{4}{3}\\ \frac{11}{6}, \frac{3}{2}\end{matrix}\middle|1\right] = \frac{5\sqrt{3}}{2}\ln(2+\sqrt{3}). $$

4. Evaluation of the ${}_3F_2(1)$-Series in Class [C]

Performing the parameter replacements in Kummer transformation (4)

$$ a \to a+y,\ c \to c+x+y,\ e \to e-x+y,\ b \to b+y,\ d \to d+y : \Delta \to \sigma - y, $$

with $\sigma = b+d-a-c-e$; we can state the resulting equation as the transformation formula below.

Theorem 3.

$$ {}_3F_2\left[\begin{matrix}a+y, c+x+y, e-x+y\\ b+y, d+y\end{matrix}\middle|1\right] = {}_3F_2\left[\begin{matrix}b-e+x, b-c-x, a+y\\ \sigma+a, b+y\end{matrix}\middle|1\right]\frac{\Gamma(d+y)\Gamma(\sigma-y)}{\Gamma(d-a)\Gamma(\sigma+a)}. $$

The above formula is valid for two variables $\{x, y\}$ and five integer parameters $\{a, c, e, b, d\}$ subject to conditions $a < b$, $a < d$, $\sigma > y$ and $\sigma + a > 0$ such that both series are not only well–defined and convergent, but also nonterminating and irreducible to known ${}_2F_1(1)$-series. Since the ${}_3F_2(1)$-series on the right-hand side of Theorem 3 can be evaluated by the summation formulae given in [10, 12], we derive further closed formulae below for exotic ${}_3F_2(1)$-series in class (C) with the two variables being specified by $y = \frac{1}{2}$ and $x \in \left\{\frac{1}{3}, \frac{1}{4}, \frac{1}{6}\right\}$.

- $\boxed{x = \frac{1}{3},\ y = \frac{1}{2}}$

$$ {}_3F_2\left[\begin{matrix}-\frac{1}{2}, -\frac{1}{6}, \frac{1}{6}\\ \frac{1}{2}, \frac{1}{2}\end{matrix}\middle|1\right] = \frac{3+\ln 2}{2\sqrt{3}}, $$

$$ {}_3F_2\left[\begin{matrix}-\frac{1}{2}, \frac{5}{6}, \frac{1}{6}\\ \frac{1}{2}, \frac{3}{2}\end{matrix}\middle|1\right] = \frac{15-4\ln 2}{8\sqrt{3}}, $$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2}, \frac{5}{6}, \frac{7}{6}\\ \frac{3}{2}, \frac{3}{2}\end{matrix}\middle|1\right] = \frac{3\sqrt{3}\ln 2}{2}, $$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2}, \frac{11}{6}, \frac{7}{6}\\ \frac{3}{2}, \frac{5}{2}\end{matrix}\middle|1\right] = \frac{9\sqrt{3}}{40}(3+4\ln 2). $$

- $\boxed{x = \frac{1}{4},\ y = \frac{1}{2}}$

$$ {}_3F_2\left[\begin{matrix}-\frac{1}{2}, -\frac{1}{4}, -\frac{3}{4}\\ \frac{1}{2}, \frac{1}{2}\end{matrix}\middle|1\right] = \frac{1}{2}\{3\ln(1+\sqrt{2}) - \sqrt{2}\}, $$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2}, \frac{3}{4}, \frac{5}{4}\\ \frac{3}{2}, \frac{3}{2}\end{matrix}\middle|1\right] = 2\ln(1+\sqrt{2}), $$

$$ {}_3F_2\left[\begin{matrix}\frac{3}{2}, \frac{3}{4}, \frac{5}{4}\\ \frac{5}{2}, \frac{5}{2}\end{matrix}\middle|1\right] = 8\{2\sqrt{2} - 3\ln(1+\sqrt{2})\}, $$

$$ {}_3F_2\left[\begin{matrix}\frac{3}{2}, \frac{7}{4}, \frac{5}{4}\\ \frac{5}{2}, \frac{7}{2}\end{matrix}\middle|1\right] = 12\{5\ln(1+\sqrt{2}) - 3\sqrt{2}\}. $$

- $\boxed{x = \frac{1}{6},\ y = \frac{1}{2}}$

$$ {}_3F_2\left[\begin{matrix} -\frac{1}{2}, -\frac{1}{3}, -\frac{2}{3} \\ \frac{1}{2}, \frac{1}{2} \end{matrix}\middle| 1\right] = \frac{1}{6}\{8\sqrt{3}\ln(2+\sqrt{3}) - 15\}, $$

$$ {}_3F_2\left[\begin{matrix} -\frac{1}{2}, \frac{2}{3}, \frac{1}{3} \\ \frac{1}{2}, \frac{3}{2} \end{matrix}\middle| 1\right] = \frac{1}{12}\{12 - \sqrt{3}\ln(2+\sqrt{3})\}, $$

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}, \frac{2}{3}, \frac{4}{3} \\ \frac{3}{2}, \frac{3}{2} \end{matrix}\middle| 1\right] = \frac{3\sqrt{3}}{4}\ln(2+\sqrt{3}), $$

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}, \frac{5}{3}, \frac{1}{3} \\ \frac{3}{2}, \frac{3}{2} \end{matrix}\middle| 1\right] = \frac{3}{8}\{6 - \sqrt{3}\ln(2+\sqrt{3})\}. $$

5. Evaluation of the ${}_3F_2(1)$-Series in Class [D]

Finally, under the parameter replacements

$$ a \to a+x,\ c \to c+x,\ e \to e-y,\ d \to d+x : \Delta \to \sigma - x + y \quad \text{with} \quad \sigma = b+d-a-c-e, $$

the transformation corresponding to Kummer's (4) is given by the theorem below.

Theorem 4.

$$ {}_3F_2\left[\begin{matrix} a+x, c+x, e-y \\ b, d+x \end{matrix}\middle| 1\right] = {}_3F_2\left[\begin{matrix} a+x, b-c-x, b-e+y \\ b, \sigma+a+y \end{matrix}\middle| 1\right] \frac{\Gamma(d+x)\Gamma(\sigma-x+y)}{\Gamma(d-a)\Gamma(\sigma+a+y)}. $$

In order that the above two series are not only well–defined and convergent, but also nonterminating and irreducible to known ${}_2F_1(1)$-series, the following conditions $a < d$, $c < d$, $\sigma > x - y$ and $b > 0$ should be imposed on the two variables $\{x, y\}$ and five integer parameters $\{a, c, e, b, d\}$. By employing the summation formulae in [10, 12], we establish further identities below for the exotic ${}_3F_2(1)$-series in class (D) with $y = \frac{1}{2}$ and $x \in \left\{\frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{6}, \frac{5}{6}\right\}$.

- $\boxed{x = \frac{1}{3},\ y = \frac{1}{2}}$

$$ {}_3F_2\left[\begin{matrix} -\frac{2}{3}, -\frac{2}{3}, -\frac{1}{2} \\ 1, \frac{1}{3} \end{matrix}\middle| 1\right] = \frac{\Gamma(\frac{1}{3})\Gamma(\frac{1}{6})}{4\sqrt{3}\pi^{3/2}}(5 - 6\ln 2), $$

$$ {}_3F_2\left[\begin{matrix} \frac{1}{3}, \frac{1}{3}, \frac{1}{2} \\ 1, \frac{4}{3} \end{matrix}\middle| 1\right] = \frac{\Gamma(\frac{1}{3})\Gamma(\frac{1}{6})}{\sqrt{3}\pi^{3/2}}\ln 2. $$

- $\boxed{x = \frac{2}{3},\ y = \frac{1}{2}}$

$$ {}_3F_2\left[\begin{matrix} -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{2} \\ 1, \frac{2}{3} \end{matrix}\middle| 1\right] = \frac{\sqrt{3}\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}{\pi^{3/2}}(4 - 3\ln 2), $$

$$ {}_3F_2\left[\begin{matrix} \frac{2}{3}, \frac{2}{3}, \frac{1}{2} \\ 1, \frac{5}{3} \end{matrix}\middle| 1\right] = \frac{4\sqrt{3}\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}{\pi^{3/2}}\ln 2. $$

- $\boxed{x = \frac{1}{4},\ y = \frac{1}{2}}$

$$ {}_3F_2\left[\begin{matrix} -\frac{3}{4}, -\frac{3}{4}, -\frac{1}{2} \\ 1, \frac{1}{4} \end{matrix}\middle| 1\right] = \frac{\Gamma^2(\frac{1}{4})}{12\pi^{3/2}}\{7\sqrt{2} - 12\ln(1+\sqrt{2})\}, $$

$$ {}_3F_2\left[\begin{matrix} \frac{1}{4}, \frac{1}{4}, \frac{1}{2} \\ 1, \frac{5}{4} \end{matrix}\middle| 1\right] = \frac{\Gamma^2(\frac{1}{4})}{2\pi^{3/2}}\ln(1+\sqrt{2}). $$

- $\boxed{x = \frac{3}{4},\ y = \frac{1}{2}}$

$$ {}_3F_2\left[\begin{matrix} -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{2} \\ 1, \frac{3}{4} \end{matrix}\middle| 1\right] = \frac{\Gamma^2(\frac{3}{4})}{\pi^{3/2}}\{5\sqrt{2} - 4\ln(1+\sqrt{2})\}, $$

$$ {}_3F_2\left[\begin{matrix} \frac{3}{4}, \frac{3}{4}, \frac{1}{2} \\ 1, \frac{7}{4} \end{matrix}\middle| 1\right] = \frac{6\Gamma^2(\frac{3}{4})}{\pi^{3/2}}\ln(1+\sqrt{2}). $$

- $\boxed{x = \frac{1}{6},\ y = \frac{1}{2}}$

$$ {}_3F_2\left[\begin{matrix} -\frac{5}{6}, -\frac{5}{6}, \frac{1}{2} \\ 1, \frac{1}{6} \end{matrix}\middle| 1\right] = \frac{\Gamma(\frac{1}{6})\Gamma(\frac{1}{3})}{15\pi^{3/2}}\{6 + 5\sqrt{3}\ln(2+\sqrt{3})\}, $$

$$ {}_3F_2\left[\begin{matrix} \frac{1}{6}, \frac{1}{6}, \frac{1}{2} \\ 1, \frac{7}{6} \end{matrix}\middle| 1\right] = \frac{\Gamma(\frac{1}{6})\Gamma(\frac{1}{3})}{2\sqrt{3}\pi^{3/2}}\ln(2+\sqrt{3}). $$

- $\boxed{x = \frac{5}{6},\ y = \frac{1}{2}}$

$$ {}_3F_2\left[\begin{matrix} -\frac{1}{6}, -\frac{1}{6}, \frac{1}{2} \\ 1, \frac{5}{6} \end{matrix}\middle| 1\right] = \frac{\Gamma(\frac{5}{6})\Gamma(\frac{2}{3})}{\pi^{3/2}}\{6 - \sqrt{3}\ln(2+\sqrt{3})\}, $$

$$ {}_3F_2\left[\begin{matrix} \frac{5}{6}, \frac{5}{6}, \frac{1}{2} \\ 1, \frac{11}{6} \end{matrix}\middle| 1\right] = \frac{5\sqrt{3}\Gamma(\frac{5}{6})\Gamma(\frac{2}{3})}{2\pi^{3/2}}\ln(2+\sqrt{3}). $$

Conflict of Interest

The authors declare no conflict of interest.

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