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Article

H-V-Super-Strong-(a, d)-antimagic decomposition of complete bipartite graphs

Solomon Stalin Kumar[1,*]

[1] Department of Mathematics, The American College, Madurai - 625 002, Tamilnadu, India.

* **Correspondence:** sskumbas@gmail.com

Abstract: An H-(a,d)-antimagic labeling in a H-decomposable graph G is a bijection $f : V(G) \cup E(G) \to \{1, 2, ..., p + q\}$ such that $\sum f(H_1), \sum f(H_2), \cdots, \sum f(H_h)$ forms an arithmetic progression with difference d and first element a. f is said to be H-V-super-(a, d)-antimagic if $f(V(G)) = \{1, 2, ..., p\}$. Suppose that $V(G) = U(G) \cup W(G)$ with $|U(G)| = m$ and $|W(G)| = n$. Then f is said to be H-V-super-strong-(a, d)-antimagic labeling if $f(U(G)) = \{1, 2, ..., m\}$ and $f(W(G)) = \{m + 1, m + 2, ..., (m + n = p)\}$. A graph that admits a H-V-super-strong-(a, d)-antimagic labeling is called a H-V-super-strong-(a, d)-antimagic decomposable graph. In this paper, we prove that complete bipartite graphs $K_{m,n}$ are H-V-super-strong-(a, d)-antimagic decomposable with both m and n are even.

Keywords: H-decomposable graph, H-V-super magic labeling, complete bipartite graph.
Mathematics Subject Classification: 05C78, 05C70.

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1. Introduction

In this paper we consider only finite and simple undirected bipartite graphs. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$ respectively and we let $|V(G)| = p$ and $|E(G)| = q$. For graph theoretic notations, we follow [1, 2]. A labeling of a graph G is a mapping that carries a set of graph elements, usually vertices and/or edges into a set of numbers, usually integers. Many kinds of labeling have been studied and an excellent survey of graph labeling can be found in [3].

Although magic labeling of graphs was introduced by Sedlacek [4], the concept of vertex magic total labeling (VMTL) first appeared in 2002 in [5]. In 2004, MacDougall et al. [6] introduced the notion of super vertex magic total labeling (SVMTL). In 1998, Enomoto et al. [7] introduced the concept of super edge-magic graphs. In 2005, Sugeng and Xie [8] constructed some super edge-magic total graphs. The usage of the word "super" was introduced in [7]. The notion of a V-super vertex magic labeling was introduced by MacDougall et al. [6] as in the name of super vertex-magic total labeling and it was renamed as V-super vertex magic labeling by Marr and Wallis in [9] after referencing the article [10]. Most recently, Tao-ming Wang and Guang-Hui Zhang [11], generalized some results found in [10].

Hartsfield and Ringel [12] introduced the concept of an antimagic graph. In their terminology, an antimagic labeling is an edge-labeling of the graph with the integers $1, 2, \cdots, q$ so that the weight at each vertex is different from the weight at any other vertex. Bodendiek and Walther [13] defined the concept of an (a, d)-antimagic labeling as an edge-labeling in which the vertex weights forms an arithmetic progression starting from a and having common difference d. Bǎca et al. [14] introduced the notions of vertex-antimagic total labeling and (a, d)-vertex-antimagic total labeling. Simanjuntak et al [15] introduced the concept of (a, d)-antimagic graph. Sudarasana et al [16] studied the concept of super edge-antimagic total lableing of disconnected graphs.

A bijection f from $V(G) \cup E(G)$ to the integers $1, 2, ..., p + q$ is called a vertex-antimagic total labeling of G if the weights of vertices $\{w_f(x) = f(x) + \sum_{xy \in E(G)} f(xy),\ x \in V(G)\}$, are pairwise distinct. f is called an (a, d)-vertex-antimagic total labeling of G if the set of vertex weights $\{w_f(x)|x \in V(G)\} = \{a, a + d, \cdots, a + (p - 1)d\}$ for some integers a and d. f is said to be super-(a, d)-vertex-antimagic labeling if $f(V(G)) = \{1, 2, ..., p\}$. A graph G is called super-(a, d)-vertex-antimagic if it admits a super-(a, d)-vertex-antimagic labeling. A bijection f from $V(G) \cup E(G)$ to the integers $1, 2, ..., p + q$ is called an (a, d)-edge-antimagic total labeling of G if the edge weights $\{w(uv) = f(u) + f(v) + f(uv),\ uv \in E(G)\}$, forms an arithmetic sequence with the first term a and common difference d. f is said to be super-(a, d)-edge-antimagic labeling if $f(V(G)) = \{1, 2, ..., p\}$. A graph G is called super-(a, d)-edge-antimagic if it admits a super-(a, d)-edge-antimagic labeling.

A covering of G is a family of subgraphs $H_1, H_2, ..., H_h$ such that each edge of $E(G)$ belongs to at least one of the subgraphs H_i, $1 \le i \le h$. Then it is said that G admits an $(H_1, H_2, \cdots, H_h)$ covering. If every H_i is isomorphic to a given graph H, then G admits an H-covering. A family of subgraphs $H_1, H_2, \cdots, H_h$ of G is a H-decomposition of G if all the subgraphs are isomorphic to a graph H, $E(H_i) \cap E(H_j) = \emptyset$ for $i \neq j$ and $\cup_{i=1}^{h} E(H_i) = E(G)$. In this case, we write $G = H_1 \oplus H_2 \oplus \cdots \oplus H_h$ and G is said to be H-decomposable.

The notion of H-super magic labeling was first introduced and studied by Gutiérrez and Lladó [17] in 2005. They proved that some classes of connected graphs are H-super magic. Suppose G is H-decomposable. A total labeling $f : V(G) \cup E(G) \to \{1, 2, \cdots, p + q\}$ is called an H-magic labeling of G if there exists a positive integer k (called magic constant) such that for every copy H in the decomposition, $\sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e) = k$. A graph G that admits such a labeling is called a H-magic decomposable graph. An H-magic labeling f is called a H-V-super magic labeling if $f(V(G)) = \{1, 2, \cdots, p\}$. A graph that admits a H-V-super magic labeling is called a H-V-super magic decomposable graph. An H-magic labeling f is called a H-E-super magic labeling if $f(E(G)) = \{1, 2, \cdots, q\}$. A graph that admits a H-E-super magic labeling is called a H-E-super magic decomposable graph. The sum of all vertex and edge labels on H is denoted by $\sum f(H)$.

In 2001, Muntaner-Batle [18] introduced the concept of super-strong magic labeling of bipartite graph as in the name of special super magic labeling of bipartite graph and it was renamed as super-strong magic labeling by Marr and Wallis [9]. Marimuthu and Stalin Kumar [19] introduced the concept of H-V-super-strong magic decomposition and H-E-super-strong magic decomposition of complete bipartite graphs. Suppose G is a bipartite graph with vertex-sets V_1 and V_2 of sizes m and n respectively. An edge-magic total labeling of G is super-strong if the elements of V_1 receive labels $\{1, 2, ..., m\}$ and the elements of V_2 receive labels $\{m + 1, m + 2, ..., m + n\}$. Suppose G is H-decomposable and if $V(G) = U(G) \cup W(G)$ with $|U(G)| = m$ and $|W(G)| = n$. An H-V-super magic labeling f is called a H-V-super-strong magic if $f(U(G)) = \{1, 2, ..., m\}$ and $f(W(G)) = \{m + 1, m + 2, ..., (m + n = p)\}$. A graph that admits a H-V-super-strong magic labeling is called a H-V-super-strong magic decomposable graph. An H-E-super magic labeling f is called a H-E-super-strong magic labeling if if $f(U(G)) = \{q + 1, q + 2, ..., q + m\}$ and $f(W(G)) = \{q + m + 1, q + m + 2, ...,$

$(q+m+n = qp)\}$. A graph that admits a H-E-super-strong magic labeling is called a H-E-super-strong magic decomposable graph.

Suppose G is H-decomposable. A total labeling $f : V(G) \cup E(G) \rightarrow \{1, 2, \cdots, p+q\}$ is called an H-antimagic labeling of G if $\sum f(H_1), \sum f(H_2), \cdots, \sum f(H_h)$ are pairwise distinct. f is said to be H-(a, d)-antimagic if these numbers forms an arithmetic progression with difference d and first element a. A H-(a, d)-antimagic labeling f is called H-V-super-(a, d)-antimagic labeling if $f(V(G)) = \{1, 2, ..., p\}$. Suppose that $V(G) = U(G) \cup W(G)$ with $|U(G)| = m$ and $|W(G)| = n$. Then f is said to be H-V-super-strong-(a, d)-antimagic labeling if $f(U(G)) = \{1, 2, ..., m\}$ and $f(W(G)) = \{m + 1, m + 2, ..., (m + n = p)\}$. A graph that admits a H-V-super-strong-(a, d)-antimagic labeling is called a H-V-super-strong-(a, d)-antimagic decomposable graph. A H-(a, d)-antimagic labeling f is called H-E-super-(a, d)-antimagic labeling if $f(E(G)) = \{1, 2, ..., q\}$. f is said to be H-E-super-strong-(a, d)-antimagic labeling if $f(U(G)) = \{q + 1, q + 2, ..., q + m\}$ and $f(W(G)) = \{q + m + 1, q + m + 2, ..., (q + m + n = qp)\}$. A graph that admits a H-E-super-strong-(a, d)-antimagic labeling is called a H-E-super-strong-(a, d)-antimagic decomposable graph.

In 2012, Inayah et al. [20] studied magic and anti-magic H-decompositions and Zhihe Liang [21] studied cycle-super magic decompositions of complete multipartite graphs. In many of the results about H-magic graphs, the host graph G is required to be H-decomposable. Yoshimi Ecawa et al [22] studied the decomposition of complete bipartite graphs into edge-disjoint subgraphs with star components. The notion of star-subgraph was introduced by Akiyama and Kano in [23]. A subgraph F of a graph G is called a star-subgraph if each component of F is a star. Here by a star, we mean a complete bipartite graph of the form $K_{1,m}$ with $m \geq 1$. A subgraph F of a graph G is called a n-star-subgraph if $F \cong K_{1,n}$ with $2 \leq n < p$. Marimuthu and Stalin Kumar [24, 25] studied about the H-V-super magic decomposition and H-E-super magic decomposition of complete bipartite graphs.

2. Main Results

In this section, we consider the graphs $G \cong K_{m,n}$ and $H \cong K_{1,n}$, where $n \geq 1$ and both m and n are even. Clearly $p = m + n$ and $q = mn$.

Theorem 1. *Suppose $\{H_1, H_2, \cdots, H_m\}$ is a n-star-decomposition of G with both m and n are even. Then G is H-V-super-strong-(a, d)-antimagic decomposable with $a = 1 + \frac{n^2(m+3)+2n(2m+1)}{2}$ and $d = 1$.*

Proof. Let $U = \{u_1, u_2, \cdots, u_m\}$ and $V = \{v_1, v_2, \cdots, v_n\}$ be two stable sets of G. Let $\{H_1, H_2, \cdots, H_m\}$ be a n-star decomposition of G with both m and n are even, where each H_i is isomorphic to H, such that $V(H_i) = \{u_i, v_1, v_2, \cdots, v_n\}$ and $E(H_i) = \{u_iv_1, u_iv_2, \cdots, u_iv_n\}$, for all $1 \leq i \leq m$. Define a total labeling $f : V(G) \cup E(G) \rightarrow \{1, 2, \cdots, p + q\}$ by $f(u_i) = i$ and $f(v_j) = m + j$, for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

Case 1: $m \neq n$.
Now the edges of G can be labeled as shown in Table 1.
We prove the result for $n = k$ and the result follows for all $1 \leq k \leq m$.
From Table 1 and from definition of f, we get

$$\sum f(H_k) = f(u_k) + \sum_{i=1}^{n} f(v_i) + \sum_{i=1}^{n} f(u_kv_i) = k + \sum_{i=1}^{n}(m + i) + \sum_{i=1}^{n} f(u_kv_i).$$

Now,

$$\sum_{i=1}^{n} f(v_i) = (m + 1) + (m + 2) + \cdots + (m + n)$$

Table 1. The edge label of a n-star-decomposition of G if $m \neq n$..

f	v_1	v_2	...	v_{n-1}	v_n
u_1	$(m+n)$ $+m$	$(2m+n)$ $+1$	...	$(m+n)$ $+((n-1)m)$	$(m+n)$ $+((n-1)m+1)$
u_2	$(m+n)+$ $(m-1)$	$(2m+n)$ $+2$	...	$(m+n)$ $+((n-1)m-1)$	$(m+n)$ $+((n-1)m+2)$
u_3	$(m+n)+$ $(m-2)$	$(2m+n)$ $+3$	...	$(m+n)$ $((n-1)m-2)$	$(m+n)$ $+((n-1)m+3)$
$\vdots$	...	...	...	...	...
u_k	$(m+n)+$ $(m-(k-1))$	$(2m+n)$ $+k$	...	$(m+n)+((n-2)m)$ $+(m-(k-1))$	$(m+n)+(n-1)m$ $+k$
$\vdots$	...	...	...	...	...
u_{m-1}	$(m+n)+$ 2	$(2m+n)$ $+(m-1)$	...	$(m+n)$ $+((n-2)m+2)$	$(m+n)$ $+(mn-1)$
u_m	$(m+n)+$ 1	$(2m+n)$ $+m$	...	$(m+n)$ $+((n-2)m+1)$	$(m+n)$ $+mn$

$$= \quad mn + (1+2+\cdots+n) = mn + \frac{n(n+1)}{2}.$$

Also

$$\begin{aligned}
\sum_{i=1}^{n} f(u_k v_i) &= ((m+n)+(m-(k-1))) + ((m+n)+(m+k)) + \cdots \\
&\quad +((m+n)+(n-2)m+(m-(k-1))) + ((m+n)+(n-1)m+k) \\
&= ((2m+n)-(k-1)) + ((2m+n)+k) + ((4m+n)-(k-1)) + \\
&\quad ((4m+n)+k) + \cdots + (((n)m+n)-(k-1)) + (((n)m+n)+k) \\
&= 2((2m+n)+(4m+n)+\cdots+(nm+n)) + \frac{n}{2}(1) \\
&= 2((2m+2n+\cdots+nm) + \frac{n(n)}{2}) + \frac{n}{2} \\
&= 4m(1+2+\cdots+\frac{n}{2}) + \frac{2n^2+n}{2} = 4m(\frac{n(n+2)}{8}) + \frac{2n^2+n}{2} \\
&= \frac{mn^2+2mn+2n^2+n}{2} = \frac{n^2(m+2)+n(2m+1)}{2}.
\end{aligned}$$

Hence

$$\sum_{i=1}^{n} f(u_k v_i) = \frac{n^2(m+2)+n(2m+1)}{2}.$$

and is constant for all $1 \leq k \leq m$.
Using the above values, we get

$$\begin{aligned}
\sum f(H_k) &= k + mn + \frac{n(n+1)}{2} + \frac{n^2(m+2)+n(2m+1)}{2} \\
&= k + \frac{2mn+n^2+n+n^2(m+2)+n(2m+1)}{2} \\
&= k + \frac{n^2(m+3)+2n(2m+1)}{2}.
\end{aligned}$$

Table 2. The edge label of a n-star-decomposition of G if $m = n$.

f	v_1	v_2	...	v_{n-1}	v_n
u_1	$3n$	$3n+1$	...	$(n+1)n$	$(n+1)n+1$
u_2	$3n-1$	$3n+2$	...	$(n+1)n-1$	$(n+1)n+2$
u_3	$3n-2$	$3n+3$	...	$(n+1)n-2$	$(n+1)n+3$
$\vdots$	...	...	...	...	...
u_k	$3n-(k-1)$	$3n+k$	...	$(n+1)n-(k-1)$	$(n+1)n+k$
$\vdots$	...	...	...	...	...
u_{n-1}	$2n+2$	$4n-1$	...	$n(n)+2$	$(n+2)n-1$
u_n	$2n+1$	$4n$	...	$n(n)+1$	$(n+2)n$

for all $1 \le k \le m$. So, $\{\sum f(H_1), \sum f(H_2), \cdots, \sum f(H_m) = a, a+d, \cdots, a+(m-1)d\}$ forms an arithmetic progression with $a = (1 + \frac{n^2(m+3)+2n(2m+1)}{2})$ and common difference $d = 1$. Thus in this case, the graph G is a H-V-super-strong-(a,d)-antimagic decomposable.

Case 2: $m = n$.

Now the edges of G can be labeled as shown in Table 2.

We prove the result for $n = k$ and the result follows for all $1 \le k \le n$.

From Table 2 and from definition of f, we get

$$\sum f(H_k) = f(u_k) + \sum_{i=1}^{n} f(v_i) + \sum_{i=1}^{n} f(u_k v_i) = k + \sum_{i=1}^{n}(n+i) + \sum_{i=1}^{n} f(u_k v_i).$$

Now,

$$\begin{aligned} \sum_{i=1}^{n} f(v_i) &= (n+1) + (n+2) + \cdots + (n+n) = (n)n + (1+2+\cdots+n) \\ &= (n)n + \frac{n(n+1)}{2}. \end{aligned}$$

Also

$$\begin{aligned} \sum_{i=1}^{n} f(u_k v_i) &= (3n-(k-1)) + (3n+k) + (5n-(k-1)) + (5n+k) + \cdots \\ &\quad +((n+1)n-(k-1)) + ((n+1)n+k) \\ &= (3n+1) + 3n + (5n+1) + 5n + \cdots + ((n+1)n+1) + (n+1)n \\ &= 2(3n+5n+\cdots+(n+1)n) + \frac{n}{2}(1) \\ &= 2n(3+5+\cdots+(n+1)) + \frac{n}{2} \\ &= 2n((1+2+3+\cdots+(n+1)) - (2+4+6+\cdots+n) - 1) + \frac{n}{2} \\ &= 2n(\frac{(n+1)(n+2)}{2} - 2\frac{(\frac{n}{2})(\frac{n+1}{2})}{2} - 1) + \frac{n}{2} \\ &= 2(\frac{n^2+3n+2}{2} - \frac{(n^2+2n)}{4} - 1) + \frac{n}{2} \\ &= 2n(\frac{2n^2+6n+4-n^2-2n-4}{4} + \frac{n}{2} = \frac{n(n^2+4n+n)}{2} \\ &= \frac{n^3+2n^2+2n^2+n}{2} = \frac{n^2(n+2)+(n(2n+1)}{2}. \end{aligned}$$

Hence

$$\sum_{i=1}^{n} f(u_k v_i) \;=\; \frac{n^2(n+2)+n(2n+1)}{2}.$$

and is constant for all $1 \leq k \leq n$.
Using the above values, we get

$$\begin{aligned}\sum f(H_k) &= k+(n)n+\frac{n(n+1)}{2}+\frac{n^2(n+2)+n(2n+1)}{2}\\ &= k+\frac{2(n)n+n^2+n+n^2(n+2)+n(2n+1)}{2}\\ &= k+\frac{n^2(n+3)+2n(2n+1)}{2}.\end{aligned}$$

for all $1 \leq k \leq n$. So, $\{\sum f(H_1), \sum f(H_2), \cdots, \sum f(H_n) = a, a+d, \cdots, a+(n-1)d\}$ forms an arithmetic progression with $a = (1 + \frac{n^2(n+3)+2n(2n+1)}{2})$ and common difference $d = 1$. Thus in this case also, the graph G is a H-V-super-strong-(a, d)-antimagic decomposable. □

Theorem 2. *If a non-trivial H-decomposable graph* $G \cong K_{m,n}$ *is H-V-super-strong-*(a, d)*-antimagic decomposable graph with both m and n are even and if the sum of edge labels of a decomposition* H_j *is denoted by* $\sum f(E(H_j))$ *then* $\sum f(E(H_j))$ *is constant for all* $1 \leq j \leq m$ *and it is given by* $\sum f(E(H_j)) = \frac{n^2(m+2)+n(2m+1)}{2}$.

Proof. Suppose G is H-decomposable and possesses a H-V-super-strong-(a, d)-antimagic labeling f, then by Theorem 1, for each H_j in the H-decomposition of G, we get

$$\sum f(E(H_j)) \;=\; \sum_{i=1}^{n} f(u_j v_i) = \frac{n^2(m+2)+n(2m+1)}{2}$$

which is true for all $1 \leq j \leq m$. Thus $\sum f(E(H_j))$ is constant for all $1 \leq k \leq m$ and it is given by $\sum f(E(H_j)) = \frac{n^2(m+2)+n(2m+1)}{2}$. □

Theorem 3. *If a non-trivial H-decomposable graph* $G \cong K_{m,n}$ *is H-V-super-strong-*(a, d)*-antimagic decomposable graph with both m and n are even and if the sum of vertex labels of a decomposition* H_j *is denoted by* $\sum f(V(H_j))$ *then*
$\{\sum f(V(H_1)), \sum f(V(H_2)), \cdots, \sum f(V(H_m))\} = \{a, a+d, \cdots, a+(m-1)d\}$ *with* $a = (mn+1) + \frac{n(n+1)}{2}$ *and* $d = 1$.

Proof. Suppose G is H-decomposable and possesses a H-V-super-strong-(a, d)-antimagic labeling f, then by Theorem 1, for each H_j in the H-decomposition of G, we get

$$\begin{aligned}\sum f(V(H_j)) &= f(u_j)+\sum_{i=1}^{n} f(v_i) = j+\sum_{i=1}^{n}(m+i) = j+((m+1)+(m+2)+\cdots+(m+n))\\ &= j+mn+\frac{n(n+1)}{2}.\end{aligned}$$

which is true for all $1 \leq j \leq m$. Thus $\{\sum f(V(H_1)), \sum f(V(H_2)), \cdots,$
$\sum f(V(H_m))\} = \{a, a+d, \cdots, a+(m-1)d\}$ with $a = (mn+1) + \frac{n(n+1)}{2}$ and $d = 1$. □

Theorem 4. *Let* $G \cong K_{m,n}$ *be a H-decomposable graph with both m and n are even and if* $V(G) = U(G) \cup W(G)$ *with* $|U(G)| = m$ *and* $|W(G)| = n$. *let g be a bijection from* $V(G)$ *onto* $\{1, 2, \cdots, p\}$ *with* $g(U(G)) = \{1, 2, \cdots, m\}$ *and* $g(W(G)) = \{(m+1), (m+2), \cdots, (m+n = p)\}$ *then g can be extended to an H-V-super-strong-*(a, d)*-antimagic labeling if and only if* $\sum f(E(H_j))$ *is constant for all* $1 \leq j \leq m$ *and it is given by* $\sum f(E(H_j)) = \frac{n^2(m+2)+n(2m+1)}{2}$.

Proof. Suppose $G \cong K_{m.n}$ be a H-decomposable graph with both m and n are even and if $V(G) = U(G) \cup W(G)$ with $|U(G)| = m$ and $|W(G)| = n$. let g be a bijection from $V(G)$ onto $\{1, 2, \cdots, p\}$ with $g(U(G)) = \{1, 2, \cdots, m\}$ and $g(W(G)) = \{(m+1), (m+2), \cdots, (m+n = p)\}$. Assume that $\sum f(E(H_j))$ is constant for all $1 \leq j \leq m$ and it is given by $\sum f(E(H_j)) = \frac{n^2(m+2)+n(2m+1)}{2}$. Define $f : V(G) \cup E(G) \to \{1, 2, ..., p+q\}$ as $f(u_i) = g(u_i)$; $f(u_j) = g(u_j)$ for all $1 \leq i \leq m$; $1 \leq j \leq n$ and the edge labels are in either Table 1 (if $m \neq n$) or Table 2 (if $m = n$) then by Theorem 2.1, for each H_j in the H-decomposition of G, we get

$$\begin{aligned}\sum f(V(H_j)) &= f(u_j) + \sum_{i=1}^{n} f(v_i) = j + \sum_{i=1}^{n}(m+i) = j + ((m+1) + (m+2) + \cdots + (m+n)) \\ &= j + mn + \frac{n(n+1)}{2}.\end{aligned}$$

which is true for all $1 \leq j \leq m$. So, we have $\{\sum f(V(H_1)), \sum f(V(H_2)), \cdots,$
$\sum f(V(H_m))\} = \{a, a+d, \cdots, a+(m-1)d\}$ with $a = (mn+1) + \frac{n(n+1)}{2}$ and $d = 1$. Hence,

$$\begin{aligned}\sum f(H_j) &= \sum f(V(H_j)) + \sum f(E(H_j)) = (j + mn + \frac{n(n+1)}{2}) + (\frac{n^2(m+2) + n(2m+1)}{2}) \\ &= j + \frac{2mn + n^2 + n + n^2(m+2) + n(2m+1)}{2} = j + \frac{n^2(m+3) + 2n(2m+1)}{2}.\end{aligned}$$

for every H_j in the H-decomposition of G and for all $1 \leq j \leq m$. Thus we have, f is an H-V-super-strong-(a, d)-antimagic labeling.
Suppose g can be extended to an H-V-super-strong-(a, d)-antimagic labeling f of G with with $a = 1 + \frac{n^2(m+3)+2n(2m+1)}{2}$ and $d = 1$. Then by Theorem 2 $\sum f(E(H_j))$ is constant for all $1 \leq j \leq m$ and it is given by $\sum f(E(H_j)) = \frac{n^2(m+2)+n(2m+1)}{2}$. □

3. Conclusion

In this paper, we studied the H-V-super-strong-(a, d)-antimagic decomposition of $K_{m,n}$ with $n \geq 1$ and both m and n are even.

Conflict of Interest

The author declares no conflict of interests.

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