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Article

Independent Fixed Connected Geodetic Number of a Graph

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Abstract: In this paper we introduce the concept of independent fixed connected geodetic number and investigate its behaviours on some standard graphs. Lower and upper bounds are found for the above number and we characterize the suitable graphs achieving these bounds. We also define two new parameters connected geo-independent number and upper connected geo-independent number of a graph. Few characterization and realization results are formulated for the new parameters. Finally an open problem is posed.

Keywords: Independent fixed connected geodetic set, Independent fixed connected geodetic number, Connected Geo-independent number, Upper connected Geo-independent number
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1. Introduction

The introduction of Graph Theory is a revolution in the field of Mathematics. Various concepts were made easily understandable by its simple expression through graphical models. By a graph G we mean V, the set of vertices; E, the set of edges together with a binary operation of association. We refer to [1–3] for basic graph theoretic terms. In G, a shortest $x-y$ path is also known as $x-y$ geodesic. The *distance* $d(x,y)$ is defined as the number of edges of an $x-y$ geodesic in G. For any two vertices x and y in G, the *closed interval* $I[x,y]$ is the collection of vertices on an $x-y$ geodesic. The *closed interval* $I[S,S']$, where $S,S' \subseteq V(G)$, is defined as the union of subintervals $I[x,y]$ for some $x \in S$ and $y \in S'$. i.e., $I[S,S'] = \bigcup_{x\in S, y\in S'} I[x,y]$. A vertex v in G is called an *extreme vertex* or *simplicial vertex* if the subgraph induced by its adjacent vertices is complete.

A set $S \subseteq V(G)$ is called a *geodetic set* or *geodomination set* if every vertex of G is on some $x-y$ geodesic where $x,y \in S$. The minimum cardinality of a geodetic set of G is called as the *geodetic number* of G, denoted by $g(G)$ [4–8]. If S is a geodetic set of G and $\langle S\rangle$ is connected, then S is called the *connected geodetic set* of G. Its minimum order is named as the *connected geodetic number* of G, denoted by $cg(G)$. A connected geodetic set of cardinality $cg(G)$ is called a cg-set of G [9]. Again parameters upper connected geodetic number and forcing connected geodetic number were defined and investigated in [10]. Santhakumaran and Titus first introduced the vertex geodomination number in [11] and further studied in [12, 13]. For any vertex x in G, a set $S \subseteq V(G)$ is called an *x-geodominating set* of G if every vertex v in G is on an $x-y$ geodesic for some y in S. The minimum cardinality of an x-geodominating set of G is defined as the *x-geodomination number* of G, denoted by $g_x(G)$. An x-geodominating set of cardinality $g_x(G)$ is called a g_x-set of G. A *connected x-geodominating set* of G is an x-geodominating set S such that $\langle S \rangle$ is connected. The minimum cardinality of a connected x-geodominating set of G is the *connected x-geodomination number* of G and is denoted by $cg_x(G)$. A connected x-geodominating set of cardinality $cg_x(G)$ is called a cg_x-set of G [14].

Let $e = xy$ be any edge of a connected graph G of order atleast 3. A set S of vertices of G is an *e-geodominating set* of G if every vertex of G is either on an $x-u$ geodesic or on an $y-u$ geodesic for some element u in S. The minimum cardinality of an e-geodominating set of G is defined as the *e-geodomination number* of G and is denoted by $g_e(G)$ or $g_{xy}(G)$. An e-geodominating set of cardinality $g_e(G)$ is called a g_e-set of G [15].

It is clear that x-geodominating set of G is obtained by fixing a vertex x in G and e-geodominating set of G is obtained by fixing an edge $e = xy$ in G. Based on these concepts we defined a new parameter called independent fixed geodomination number (or independent fixed geodetic number) in [16]. Let S be an independent set of a connected graph G of order atleast 2. Let S' be a subset of $V(G)$. If each vertex v in G is on an $x-y$ geodesic for some $x \in S$ and $y \in S'$, then S' is an *S-fixed geodetic set* of G . The *S-fixed geodetic number* $g_s(G)$ of G is the minimum cardinality of an S-fixed geodetic set of G. The *independent fixed geodetic number* of G is $g_{if}(G) = min\{g_s(G)\}$, where the minimum is taken over all independent sets S in G. An independent fixed geodetic set of cardinality $g_{if}(G)$ is described as g_{if}-set of G. We too further proceed to infer about connectedness of an S-fixed geodetic set of G, where S is an independent set in a connected graph G. In the computation of independent fixed connected geodetic number, the succeeding theorems will be employed.

Theorem 1. *[16] For any connected graph G,* $1 \leq g_{if}(G) \leq p-1$.

Theorem 2. *[16] Let G be a connected graph. Then* $g_{if}(G) = 1$ *if and only if there is an independent set S and its eccentric vertex y such that every vertex of G is on an* $x-y$ *geodesic for some* $x \in S$.

Theorem 3. *[16] For any complete graph* K_p, $g_{if}(K_p) = p-1$.

2. Independent fixed connected geodetic number

Definition 1. *Let S be an independent set of a connected graph G of order atleast 2. An S-fixed connected geodetic set of G is an S-fixed geodetic set S′ such that* $\langle S' \rangle$ *is connected. The S-fixed connected geodetic number* $cg_s(G)$ *of G is the minimum cardinality of an S-fixed connected geodetic set of G. The independent fixed connected geodetic number* $cg_{if}(G)$ *of G is defined as* $cg_{if}(G) = min\{cg_s(G)\}$*, where the minimum is taken over all independent sets S in G. An independent fixed connected geodetic set of cardinality* $cg_{if}(G)$ *is called a* cg_{if}*-set of G.*

Example 1. *Consider a graph G as shown in Figure 1. The Table 1 gives the independent sets S, their corresponding minimum S-fixed geodetic sets and minimum S-fixed connected geodetic sets, the S-fixed geodetic numbers* $g_s(G)$ *and the S-fixed connected geodetic numbers* $cg_s(G)$*. Then the independent fixed geodetic number of G is* $g_{if}(G) = min\{g_s(G)\} = 2$ *and the independent fixed connected geodetic number of G is* $cg_{if}(G) = min\{cg_s(G)\} = 3$.

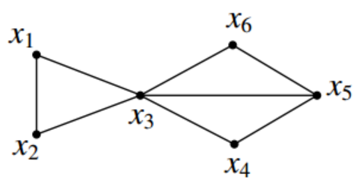

Figure 1. GS.

Independent set S	Minimum S-fixed geodetic set	S-fixed geodetic number g_sG	Minimum S-fixed connected geodetic set	S-fixed connected geodetic number cg_sG
$\{x_1\}$	$\{x_2, x_4, x_5, x_6\}$	4	$\{x_2, x_3, x_4, x_5, x_6\}$	5
$\{x_2\}$	$\{x_1, x_4, x_5, x_6\}$	4	$\{x_1, x_3, x_4, x_5, x_6\}$	5
$\{x_3\}$	$\{x_1, x_2, x_4, x_5, x_6\}$	5	$\{x_1, x_2, x_3, x_4, x_5, x_6\}$	6
$\{x_4\}$	$\{x_1, x_2, x_6\}$	3	$\{x_1, x_2, x_3, x_6\}$	4
$\{x_5\}$	$\{x_1, x_2, x_4, x_6\}$	4	$\{x_1, x_2, x_3, x_4, x_6\}$	5
$\{x_6\}$	$\{x_1, x_2, x_4\}$	3	$\{x_1, x_2, x_3, x_4\}$	4
$\{x_1, x_4\}$	$\{x_2, x_6\}$	2	$\{x_2, x_3, x_6\}$	3
$\{x_1, x_5\}$	$\{x_2, x_4, x_6\}$	3	$\{x_2, x_3, x_4, x_6\}$	4
$\{x_1, x_6\}$	$\{x_2, x_4\}$	2	$\{x_2, x_3, x_4\}$	3
$\{x_2, x_4\}$	$\{x_1, x_6\}$	2	$\{x_1, x_3, x_6\}$	3
$\{x_2, x_5\}$	$\{x_1, x_4, x_6\}$	3	$\{x_1, x_3, x_4, x_6\}$	4
$\{x_2, x_6\}$	$\{x_1, x_4\}$	2	$\{x_1, x_3, x_4\}$	3
$\{x_4, x_6\}$	$\{x_1, x_2, x_5\}$	3	$\{x_1, x_2, x_3, x_5\}$	4
$\{x_1, x_4, x_6\}$	$\{x_2, x_5\}$	2	$\{x_2, x_3, x_5\}$	3
$\{x_2, x_4, x_6\}$	$\{x_1, x_5\}$	2	$\{x_1, x_3, x_5\}$	3

Table 1. Table 2.1

Theorem 4. *In a connected graph G,* $1 \leq g_{if}(G) \leq cg_{if}(G) \leq p-1$.

Proof. For any independent set S in a connected graph G, every S-fixed connected geodetic set is an S-fixed geodetic set of G, we have $g_{if}(G) \leq cg_{if}(G)$. Then by Theorem 1, we have $1 \leq g_{if}(G) \leq cg_{if}(G) \leq p-1$. □

We intend to characterize the graphs that realize the bounds in Theorem 4. For that we use the following definition.

Definition 2. *[16] Let G be a connected graph of order atleast 2. Let* $S \subset V(G)$ *and* $y \in V(G) - S$. *The distance between the set S and the vertex y is* $d(S, y) = min\{d(x, y) : x \in S\}$. *The eccentricity of the set S is* $e(S) = max\{d(S, y) : y \in V(G) - S\}$. *An eccentric vertex of S is a vertex v of G with* $d(S, v) = e(S)$.

Theorem 5. *Let G be a connected graph of order atleast 2. Then* $cg_{if}(G) = 1$ *if and only if there is an independent set S and its eccentric vertex y such that every vertex of G is on an* $x - y$ *geodesic for some* $x \in S$.

Proof. The result follows from Theorems 2 and 4. □

The following theorem gives $cg_{if}(G)$ for some standard graphs.

Theorem 6. (*i*) *If* $G = T$ *or* $K_{m,n}$, *then* $cg_{if}(G) = 1$.

(*ii*) *If* $G = C_p$, *then* $cg_{if}(G)$ *is 1 or 2 according as p is even or odd.*

(*iii*) *If* $G = K_1 + \cup m_j K_j$ *with G is neither a complete graph nor a star, then* $cg_{if}(G) = j - 1$ *or* $\sum m_j(j-1) + 1$ *according as* $\sum_{j\geq 2} m_j = 1$ *or* $\sum_{j\geq 2} m_j \geq 2$.

In view of Theorem 4, we proceed to characterize graphs G with $cg_{if}(G) = p - 1$.

Theorem 7. *Let G be a connected graph of order* $p \geq 2$. *Then* $cg_{if}(G) = p - 1$ *if and only if* $G = K_p$.

Proof. Let $cg_{if}(G) = p-1$. Claim that $G = K_p$. If not, then G has a diametral path $P : u_0, u_1, \ldots, u_d$ of length $d \geq 2$. It is clear that u_0 and u_d are non-cut vertices of G. If u_0 or u_d is an end vertex of G, then $S = \{u_0, u_d\}$ is an independent set of G and $S' = V(G) - S$ is an S-fixed connected geodetic set of G. Hence $cg_{if}(G) \leq p-2$, but this leads to a contradiction. If both u_0 and u_d are non-end vertices of G, then there exist atleast two $u_0 - u_d$ paths in G. If atleast one vertex other than u_0 and u_d is common for any two $u_0 - u_d$ paths in G, then $S = \{u_0, u_d\}$ is an independent set of G and $S' = V(G) - S$ is an S-fixed connected geodetic set of G. Hence $cg_{if}(G) \leq p-2$, but this leads to a contradiction. Suppose that there exist two $u_0 - u_d$ paths, say P_1 and P_2, has no common vertices other than u_0 and u_d. Let z be an adjacent vertex of u_0 in P_1. If z is a cut vertex of G, then let H be a component of $G - z$ with u_0 not in H. Let z_1 be a vertex farthest away from z in H. Then it is clear that $S = \{z_1, u_d\}$ is an independent set of G and $S' = V(G) - S$ is an S-fixed connected geodetic set of G. Hence $cg_{if}(G) \leq p-2$, but this leads to a contradiction. If z is not a cut vertex and $\{u_o, z\}$ is not a vertex-cut of G, then $S = \{u_0\}$ is an independent set of G and $S' = V(G) - \{u_0, z\}$ is an S-fixed connected geodetic set of G. Hence $cg_{if}(G) \leq p-2$, but this leads to a contradiction. If z is not a cut vertex and $\{u_0, z\}$ is a vertex-cut of G, then there exists another $u_0 - z$ path, say Q, of length atleast 2. Let w be an adjacent vertex of u_0 on Q. If w is a cut vertex of G, then let H_1 be a component of $G - w$ with u_0 not in H_1. Let w_1 be a vertex farthest away from w in H_1. Then it is clear that $S = \{w_1, u_d\}$ is an independent set of G and $S' = V(G) - S$ is an S-fixed connected geodetic set of G. Hence $cg_{if}(G) \leq p-2$, but this leads to a contradiction. If w is not a cut vertex and $\{w, u_d\}$ is not a vertex-cut of G, then $S = \{w, u_d\}$ is an independent set of G and $S' = V(G) - S$ is an S-fixed connected geodetic set of G. Hence $cg_{if}(G) \leq p-2$, but this leads to a contradiction. If w is not a cut vertex and $\{w, u_d\}$ is a vertex-cut of G, then $S = \{w\}$ is an independent set of G and $S' = V(G) - \{w, u_0\}$ is an S-fixed connected geodetic set of G. Hence $cg_{if}(G) \leq p-2$, but this leads to a contradiction. Converse is clear from Theorems 3 and 4. □

Now we proceed to characterize graphs G with $cg_{if}(G) = p-2$. For that we require the definition of a special graph $K_m \leftarrow P_r \rightarrow K_n$.

Definition 3. *The graph $G = K_m \leftarrow P_r \rightarrow K_n$ is obtained from two complete graphs K_m, K_n and a path P_r by joining every vertex in K_m with an end vertex of P_r and joining every vertex in K_n with the other end vertex of P_r.*

The graph $G = K_m \leftarrow P_r \rightarrow K_n$ is shown in Figure 2.

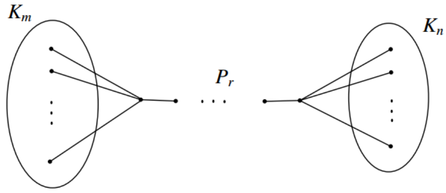

Figure 2. G.

Theorem 8. *Let G be a connected graph of order $p \geq 3$. Then $cg_{if}(G) = p-2$ if and only if G is either P_3 or $K_m \leftarrow P_r \rightarrow K_n$ ($m, n \geq 2$ and $r \geq 1$).*

Proof. Let $cg_{if}(G) = p-2$. If $p = 3$, then by Theorems 7 and 6(i) we conclude that $G = P_3$. If $p = 4$, then G is either $K_4, K_2 + \bar{K}_2, K_1 + (K_1 \cup K_2), K_{1,3}, P_4$ or C_4. If $G = K_4$, then by Theorem 7, $cg_{if}(G) = 3 = p-1$, but this leads to a contradiction. If $G = K_2 + \bar{K}_2$, then G has two simplicial vertices, say x and y. It is obvious that $S = \{x\}$ is an independent set of G and $S' = \{y\}$ is the minimum S-fixed connected geodetic set of G. Hence $cg_{if}(G) = 1 = p-3$, but this leads to a contradiction. If $G = K_1 + (K_1 \cup K_2), K_{1,3}, P_4$ or C_4, then by Theorem 6, $cg_{if}(G) = 1 = p-3$, but this leads to a contradiction.

Now, let $p \geq 5$. Since $cg_{if}(G) = p-2$, by Theorem 7, we have $G \neq K_p$. First, we show that every vertex of G is either a cut vertex or a simplicial vertex. Incase there exists a vertex, say x, in G which is neither a cut vertex nor a simplicial vertex, then x lies on a cycle in G. Let C be a largest chordless cycle containing the vertex x in G. We consider three cases.

Case (1) Length of the cycle C is more than 3 and degree of x is 2.

Let y and z be the adjacent vertices of x on C. Since C is a chordless cycle, y and z are non-simplicial vertices of G.

Subcase (1) $deg\ y = deg\ z = 2$. Let $S = \{x\}$ be an independent set of G. If C is an even cycle, then y and z is on an $x-x_1$ geodesic, where x_1 is an eccentric vertex of x on C; and if C is an odd cycle, then y is on an $x-x_1$ geodesic and z is on an $x-x_2$ geodesic, where x_1 and x_2 are the eccentric vertices of x on C. Then it is clear that $S' = V(G) - \{x,y,z\}$ is an S-fixed geodetic set of G. Since $deg\ y = deg\ z = deg\ x = 2$, $\langle S' \rangle$ is connected. Hence S' is an S-fixed connected geodetic set of G and so $cg_{if}(G) \leq p-3$, but this leads to a contradiction.

Subcase (2) $deg\ y \geq 3$ and $deg\ z \geq 3$. Since $deg\ x = 2$, $G - x$ is a connected graph. If y and z are cut vertices of G, then let G_1 and G_2 be the components of $G-y$ and $G-z$, respectively, with the vertex x not in both G_1 and G_2. Let y_1 be a vertex farthest away from y in G_1 and let z_1 be a vertex farthest away from z in G_2. Then it is vivid that $S = \{x, y_1, z_1\}$ is an independent set of G and $S' = V(G) - S$ is an S-fixed connected geodetic set of G. Hence $cg_{if}(G) \leq p-3$, but this leads to a contradiction.

If y and z are non-cut vertices of G and $\{y,z\}$ is a vertex-cut of $G - x$, then y, x and z are the consecutive vertices of another chordless cycle, say C', in G. Let $z' \neq x$ be an adjacent vertex of z on C'. If z' is a non-cut vertex of G, then $S = \{x, z'\}$ is an independent set of G and $S' = V(G) - \{x, z, z'\}$ is an S-fixed connected geodetic set of G. Hence $cg_{if}(G) \leq p-3$, but this leads to a contradiction. If z' is a cut vertex of G, then let G_3 be a component of $G - z'$ with the vertex z not in G_3. Let z'' be a vertex farthest away from z' in G_3. Then $S_1 = \{x, z''\}$ is an independent set of G and $S'_1 = V(G) - \{x, z, z''\}$ is an S-fixed connected geodetic set of G. Hence $cg_{if}(G) \leq p-3$, but this leads to a contradiction.

If y and z are non-cut vertices of G and $\{y,z\}$ is not a vertex-cut of $G - x$. It is clear that $S = \{x\}$ is an independent set of G and $S' = V(G) - \{x,y,z\}$ is an S-fixed connected geodetic set of G and so $cg_{if}(G) \leq p-3$, but this leads to a contradiction. If either y or z is a cut vertex of G, then by the arguments similar to the above, we get a contradiction.

Subcase (3) $deg\ y = 2$ and $deg\ z \geq 3$ ((or) $deg\ y \geq 3$ and $deg\ z = 2$). If z is a cut vertex of G, then let G_1 be a component of $G - z$ with the vertex x not in G_1. Let z' be a vertex farthest away from z in G_1. Then $S = \{x, z'\}$ is an independent set of G and $S' = V(G) - \{x, y, z'\}$ is an S-fixed connected geodetic set of G and so $cg_{if}(G) \leq p-3$, but this leads to a contradiction.

If z is not a cut vertex of G, then clearly $S = \{x\}$ is an independent set of G and $S' = V(G) - \{x,y,z\}$ is an S-fixed connected geodetic set of G. Hence $cg_{if}(G) \leq p-3$, but this leads to a contradiction.

Case (2) Length of the cycle C is more than 3 and degree of x is more than 2.

Subcase (1) $\{x,y\}$ and $\{x,z\}$ are non vertex-cuts of G. Then it is clear that $S = \{x\}$ is an independent set of G and $S' = V(G) - \{x,y,z\}$ is an S-fixed connected geodetic set of G. Hence $cg_{if}(G) \leq p-3$, but this leads to a contradiction.

Subcase (2) $\{x,y\}$ and $\{x,z\}$ are vertex-cuts of G. Then G has two more cycles, say C_1 and C_2, with xy an edge of C_1 and xz an edge of C_2. Let $x' \neq y$ be an adjacent vertex of x on C_1 and let $x'' \neq z$ be an adjacent vertex of x on C_2. If x' is a cut vertex of G, then take $a = x'_1$, where x'_1 is a vertex farthest away from x' in a component H of $G - x'$ with x not a vertex of H. Otherwise, take $a = x'$. Similarly, if x'' is a cut vertex of G, then take $b = x''_1$, where x''_1 is a vertex farthest away from x'' in a component H_1 of $G - x''$ with x not a vertex of H_1. Otherwise, take $b = x''$. It is clear that $S = \{a, b\}$ is an independent set of G and $S' = V(G) - \{x, a, b\}$ is an S-fixed connected geodetic set of G. It implies $cg_{if}(G) \leq p-3$, but this leads to a contradiction.

Subcase (3) $\{x,y\}$ is a vertex-cut and $\{x,z\}$ is not a vertex-cut of G ((or) $\{x,y\}$ is not a vertex-cut and $\{x,z\}$ is a vertex-cut of G). Then G has one more chordless cycle, say C_1, with xy an edge of C_1. Let

$x' \neq y$ be an adjacent vertex of x on C_1. If x' is adjacent to y in G and x' is not a cut vertex of G, then $S = \{x'\}$ is an independent set of G and $S' = V(G) - \{x, x', z\}$ is an S-fixed connected geodetic set of G and so $cg_{if}(G) \leq p - 3$, but this leads to a contradiction. If x' is adjacent to y in G and x' is a cut vertex of G, then by an argument similar to Subcase (3) of Case (1), $S = \{x, x''\}$ is an independent set of G and $S' = V(G) - \{x, x'', z\}$ is an S-fixed connected geodetic set of G. Hence $cg_{if}(G) \leq p - 3$, but this leads to a contradiction.

Similarly, if x' is a non-adjacent vertex of y on C_1 and x' is a non-cut vertex of G, then $S = \{x\}$ and $S' = V(G) - \{x, x', z\}$. If x' is a non-adjacent vertex of y and x' is a cut vertex of G, then $S = \{x, x''\}$ and $S' = V(G) - \{x, x'', z\}$, where x'' is a vertex farthest away from x' in a component H of $G - x'$ with x not a vertex of H. In both cases, S is an independent set of G and S' is an S-fixed connected geodetic set of G and hence $cg_{if}(G) \leq p - 3$, but this leads to a contradiction.

Case (3) Length of the cycle C is 3.

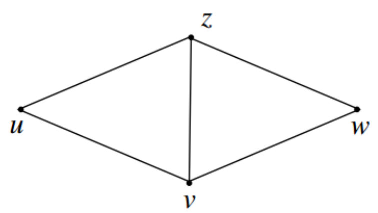

Figure 3. G.

Then the graph H given in Figure 2.3 is a subgraph of G. If not, every block of G is either K_2 or K_3 and so every vertex of G is either a cut vertex or a simplicial vertex, which is a contradiction to our assumption. Here we take the vertex u as x and the cycle u, v, w, z, u in H as C and continue the procedure exactly similar to Case (1) and Case (2), we get $cg_{if}(G) \leq p - 3$, but this leads to a contradiction.

Hence in all the three cases we get a contradiction and so every vertex in G is either a cut vertex or a simplicial vertex. Since $cg_{if}(G) = p - 2$, by Theorem 7, G is a non-complete graph. Hence G has atleast one cut vertex. Let $Q = \{u_1, u_2, \ldots, u_a\}$ be the set of all cut vertices of G. We consider two cases.

Case (1) G has exactly one cut vertex, say u_1.

Let $G_1, G_2, \ldots, G_t (t \geq 2)$ be the components of $G - u_1$. If $t \geq 3$, then $S = \{x_1, x_2, \ldots, x_t\}$, where $x_i \in G_i (1 \leq i \leq t)$, is an independent set of G and $S' = V(G) - S$ is an S-fixed connected geodetic set of G and so $cg_{if}(G) \leq p - 3$, but this leads to a contradiction. Hence $t = 2$. Now claim that each component $G_i (1 \leq i \leq 2)$ has atleast two vertices. If G_1 has exactly one vertex, say v, then $S = \{v, z\}$, where $z \in G_2$, is an independent set of G. It is clear that $S' = V(G) - \{v, z, u_1\}$ is an S-fixed connected geodetic set of G and so $cg_{if}(G) \leq p - 3$, but this leads to a contradiction. Hence G has exactly one cut vertex and two components with each component having atleast two vertices. Hence $G = K_m \leftarrow P_r \rightarrow K_n$ ($m, n \geq 2$ and $r = 1$).

Case (2) G has two or more cut vertices.

Let $R = \{z_1, z_2, \ldots, z_l\} (l \geq 0)$ be the set of all cut vertices of degree ≥ 3 in G. If $l = 0$, then G is a path and so by Theorem 6(i), $cg_{if}(G) = 1 < p - 3$, but this leads to a contradiction. Now, let $l \geq 1$ and let $S = \{x_1, x_2, \ldots, x_b\}$ ($b \geq max\{2, l\}$), where x_i is a simplicial vertex of G in the i^{th} component of $G - R$. If $l \geq 3$, then clearly S is an independent set of G and $S' = V(G) - S$ is an S-fixed connected geodetic set of G and so $cg_{if}(G) \leq p - 3$, but this leads to a contradiction. If $l = 1$, then clearly S is an independent set of G and $S' = V(G) - \{S \cup (Q - R)\}$ is an S-fixed connected geodetic set of G. Hence $cg_{if}(G) \leq p - 3$, but this leads to a contradiction. Hence $l = 2$. If $G - R$ has 3 or more components, then S is an independent set of G and $S' = V(G) - S$ is an S-fixed connected geodetic set of G and so $cg_{if}(G) \leq p - 3$, but this leads to a contradiction. Hence $G - R$ has exactly 2 components. Since every vertex of G is either a cut vertex or a simplicial vertex, the components of $G - R$ are complete graphs. Since $l = 2$, the two components of $G - R$ are complete graphs with atleast two vertices and $\langle R\rangle$ is a path. Hence $G = K_m \leftarrow P_r \rightarrow K_n$ ($m, n \geq 2$ and $r \geq 1$).

Conversely, let $G = P_3$ or $K_m \leftarrow P_r \rightarrow K_n$ ($m, n \geq 2$ and $r \geq 1$). If $G = P_3$, then by Theorem 6(i), $cg_{if}(G) = 1 = p - 2$. If $G = K_m \leftarrow P_r \rightarrow K_n$, then every vertex in $K_m \cup K_n$ is a simplicial vertex of G. It is clear that any independent set S of G contains atmost one vertex from each K_m and K_n. Also, every S-fixed connected geodetic set of G contains atleast $m - 1$ vertices from K_m and atleast $n - 1$ vertices from K_n. Since $m \geq 2$, $n \geq 2$ and $r \geq 1$, every vertex of P_r is an element of any S-fixed connected geodetic set of G. Hence $S' = V(G) - S$ is an S-fixed connected geodetic set of G and so $cg_{if}(G) \geq p - 2$. Let $S_1 = \{x, y\}$, where $x \in V(K_m)$ and $y \in V(K_n)$, and let $S_1' = V(G) - S_1$. It is clear that S_1 is an independent set of G and S_1' is an S_1-fixed connected geodetic set of G and so $cg_{if}(G) = p - 2$. □

Based on Theorem 4, we have the following realization result.

Theorem 9. *For any three positive integers a, b and p with $2 \leq a \leq b \leq p - 3$, it is possible to identify a connected graph G of order p with $g_{if}(G) = a$ and $cg_{if}(G) = b$.*

Proof. We consider two cases.
Case (1) $2 \leq a = b \leq p - 3$. Let $P_{p-a-1} : v_1, v_2, \ldots, v_{p-a-1}$ be a path of order $p - a - 1$ and let K_{a+1} be the complete graph of order $a + 1$. Let G be the graph obtained from P_{p-a-1} and K_{a+1} by joining an end vertex v_{p-a-1} of P_{p-a-1} with every vertex of K_{a+1}. The resultant graph G is shown in Figure 4 and its order is p.

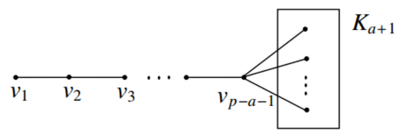

Figure 4. G.

It is clear that any independent set of G contains atmost one vertex from the complete graph K_{a+1}. Also, atleast a vertices in K_{a+1} lie in every S-fixed geodetic set of G, where S is an independent set of G. Hence $g_{if}(G) \geq a$. Let $S = \{v_1, x\}$ and $S' = V(K_{a+1}) - \{x\}$, where $x \in V(K_{a+1})$. Since $a \leq p - 3$, S is an independent set of G and it is clear that every vertex of $V(G) - S$ is on a $v_1 - y$ geodesic for any $y \in S'$. Hence S' is an S-fixed geodesic set of G and so $g_{if}(G) = |S'| = a$. Also, since $\langle S'\rangle$ is connected, we have $cg_{if}(G) = g_{if}(G) = a$.
Case (2) $2 \leq a < b \leq p - 3$. Let $P_{p-a-2} : v_1, v_2, \ldots, v_{p-b-2}, \ldots, v_{p-a-2}$ be a path of order $p - a - 2$. Let K_2 and K_a be two complete graphs of orders 2 and a, respectively. Let G be the graph obtained from P_{p-a-2}, K_2 and K_a, by joining the vertex v_{p-b-1} of P_{p-a-2} with every vertex of K_2 and joining the end vertex v_{p-a-2} of P_{p-a-2} with every vertex of K_a. The resultant graph G is shown in Figure 5 and its order is p.

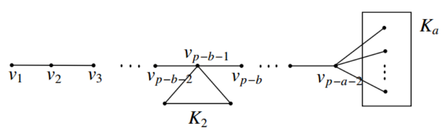

Figure 5. G.

It is clear that any independent set of G contains atmost one vertex from each complete graphs K_2 and K_a. Also, atleast one vertex in K_2 and atleast $a - 1$ vertices in K_a lie in every S-fixed geodetic set of G, where S is any independent set of G. Hence $g_{if}(G) \geq a$. Let $S = \{v_1, x, y\}$ and let $S' = (V(K_2) - \{x\}) \cup (V(K_a) - \{y\})$, where $x \in V(K_2)$ and $y \in V(K_a)$. Since $a < b \leq p - 3$, S is an independent set of G and it is clear that every vertex of $V(G) - S$ lies on a $u - v$ geodesic for some u in S and v in S'. Hence S' is an S-fixed geodetic set of G and so $g_{if}(G) = |S'| = a$. But $\langle S' \rangle$ is not connected and so $cg_{if}(G) > a$. Since every S-fixed geodetic set of G contains atleast one vertex from each complete graphs K_2 and K_a, every S-fixed connected geodetic set of G contains the cut vertices $\{v_{p-b-1}, v_{p-b}, \ldots, v_{p-a-2}\}$. Let $S'' = S' \cup \{v_{p-b-1}, v_{p-b}, \ldots, v_{p-a-2}\}$. It is clear that S'' is a minimum S-fixed connected geodetic set of G and so $cg_{if}(G) = |S''| = b$. □

We know that the diameter of any connected graph lies between its radius and two times of its radius. For that Ostrand [17] has given a realization result. Ostrand's theorem can be extended so that $cg_{if}(G)$ can also be prescribed.

Theorem 10. *For any three positive integers r, d and n with $r \leq d \leq 2r$, a connected graph G can be identified with radius r, diameter d and the independent fixed connected geodetic number n.*

Proof. If $r = 1$, then $d = 1$ or 2. If $d = 1$, then by Theorem 7, $G = K_{n+1}$ has the desired property. Now, let $d = 2$. Let G be the graph obtained from the complete graphs K_2 and K_{n+2} by merging a vertex of K_2, say y, and a vertex of K_{n+2}. Then G has radius 1, diameter 2 and is shown in Figure 6.

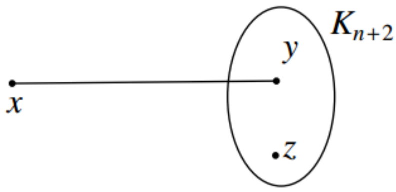

Figure 6. G.

Let $T = V(K_{n+2}) - \{y\}$. If S is any independent set of G, then atleast n vertices in T lie in every S-fixed connected geodetic set of G and so $cg_{if}(G) \geq n$. Let $S = \{x, z\}$, where $z \neq y$ in K_{n+2}, and let $S' = T - \{z\}$. Clearly, S is an independent set of G and S' is the minimum S-fixed connected geodetic set of G and so $cg_{if}(G) = n$.

Now, let $r \geq 2$. We construct a graph G which meets our requirement.

Case (1) $r = d$. Let K_{n+2} be the complete graph with vertex set $V(K_{n+2}) = \{u_1, u_2, \ldots, u_{n+2}\}$ and let C_{2r} be the even cycle with vertex set $V(C_{2r}) = \{v_1, v_2, \ldots, v_{2r}\}$. Let G be the graph obtained from K_{n+2} and C_{2r} by merging the edge u_1u_2 in K_{n+2} and v_rv_{r+1} in C_{2r}. The resultant graph G is shown in Figure 7.

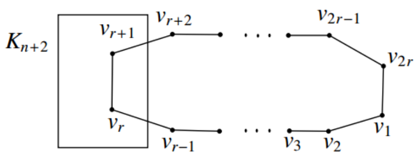

Figure 7. G.

It can be easily verified that $e(v) = r$ for any vertex $v \in G$ and so $rad\ G = diam\ G = r$. Also, $T = V(K_{n+2}) - \{u_1, u_2\}$ is the set of all simplicial vertices of G and any independent set of G contains atmost one element in T. If S is any independent set of G, then atleast $n - 1$ vertices in T lie in every S-fixed connected geodetic set of G and so $cg_{if}(G) \geq n - 1$. Also, it is clear that all the vertices of C_{2r} are not on any $x - y$ geodesic for some $x \in V(C_{2r})$ and $y \in T$. Hence $cg_{if}(G) > n - 1$. Let $S = \{v_1, u_3\}$ and let $S' = (T - \{u_3\}) \cup \{v_{r+1}\}$. Clearly, every vertex of C_{2r} is on a $v_1 - v_{r+1}$ geodesic and so S' is an S-fixed geodetic set of G. Also, $\langle S' \rangle$ is connected and so $cg_{if}(G) = n$.

Case (2) $r < d \leq 2r$. Let K_{n+1} be the complete graph with the vertex set $V(K_{n+1}) = \{u_1, u_2, \cdots, u_{n+1}\}$, let P_{d-r} be a path with the vertex set $V(P_{d-r}) = \{v_1, v_2, \ldots, v_{d-r}\}$ and let C_{2r} be the cycle with the vertex set $V(C_{2r}) = \{w_1, w_2, \ldots, w_{2r}\}$. Let G be the graph obtained from K_{n+1}, P_{d-r} and C_{2r} by joining every vertex of K_{n+1} with the vertex v_1 in P_{d-r}, and joining the vertex v_{d-r} of P_{d-r} with the vertex w_1 in C_{2r}. The graph G is shown in Figure 8.

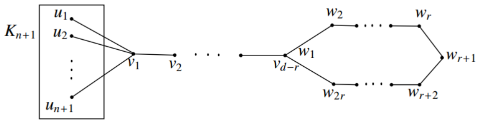

Figure 8. G.

It can be easily verified that $r \leq e(x) \leq d$, $e(w_1) = r$ and $e(w_{r+1}) = d$. Hence $rad\ G = r$ and $diam\ G = d$. It is clear that $T = V(K_{n+1})$ is the set of all simplicial vertices of G. Then by an argument similar to Case (1) of Theorem 2.10, we have $S = \{u_1, w_{r+1}\}$ is an independent set of G and $S' = T - \{u_1\}$ is a minimum S-fixed connected geodetic set of G. Hence $cg_{if}(G) = n$. □

3. Connected Geo-independent Number

Definition 4. *The minimum (maximum) independent set required to form a cg_{if}-set of G is called a connected geo-independent set (upper connected geo-independent set) of G. The cardinality of a connected geo-independent set (upper connected geo-independent set) of G is called the connected geo-independent number (upper connected geo-independent number) of G and is denoted by $cgin(G)$ ($cgin^+(G)$).*

Example 2. *For the graph G given in Figure 2.1, we have $cg_{if}(G) = 3$. From Table 1, it can be easily seen that $S_1 = \{x_1, x_4\}, S_2 = \{x_1, x_6\}$, $S_3 = \{x_2, x_4\}$ and $S_4 = \{x_2, x_6\}$ are the connected geo-independent sets of G, $S_5 = \{x_1, x_4, x_6\}$ and $S_6 = \{x_2, x_4, x_6\}$ are the upper connected geo-independent sets of G. Hence $cgin(G) = 2$ and $cgin^+(G) = 3$.*

The following result gives the connected geo-independent numbers and the upper connected geo-independent numbers of certain special classes of graphs.

Result 11. (*i*) *If $G = P_p$, then $cgin(G) = 1$ and $cgin^+(G) = \left\lceil \frac{p}{2} \right\rceil$.*

(*ii*) *If $G = K_{1,p-1} (p \geq 3)$, then $cgin(G) = p - 2$ and $cgin^+(G) = p - 1$.*

(*iii*) *If $G = K_p$, then $cgin(G) = cgin^+(G) = 1$.*

(*iv*) *If $G = C_p$, then $cgin(G) = 1$ and $cgin^+(G) = \left\lfloor \frac{p}{2} \right\rfloor$ or $\left\lfloor \frac{p}{2} \right\rfloor - 1$ according as $\left\lfloor \frac{p}{2} \right\rfloor$ is odd or even.*

(*v*) *If $G = K_{m,n}$ $(2 \leq m \leq n)$, then $cgin(G) = m - 1$ and $cgin^+(G) = n - 1$.*

(*vi*) *If $G = Q_n$ $(n \geq 3)$, then $cgin(G) = 1$ and $cgin^+(G) = 2^{n-1}$.*

The following observation is an easy consequence of some of the previous results.

Observation 12. *For any connected graph G of order $p \geq 2$,*

(i) $1 \leq cgin(G) \leq cgin^+(G) \leq \beta(G) \leq p - 1$, where $\beta(G)$ is the independence number of G.

(ii) $2 \leq cgin(G) + cg_{if}(G) \leq p$ and $2 \leq cgin^+(G) + cg_{if}(G) \leq p$.

Theorem 13. *Let G be a connected graph of order $p \geq 2$. Then*

(i) $cgin(G) = 1$ if and only if $cg_x(G) = cg_{if}(G)$ for some vertex x in G.

(ii) $cgin(G) = p - 1$ if and only if $G = K_2$.

(iii) $cgin^+(G) = p - 1$ if and only if $G = K_{1,p-1}$.

Proof. (i) Let $cgin(G) = 1$. Then there exists a vertex, say x, in G with $S = \{x\}$ and S', a minimum S-fixed connected geodetic set of G. Hence every vertex of G is on an $x - y$ geodesic for some $y \in S'$ and $\langle S' \rangle$ is connected, and so S' is a minimum connected x-geodominating set of G. Hence $cg_x(G) = |S'| = cg_{if}(G)$. Converse is clear from the respective definitions.
(ii) Let $cgin(G) = p - 1$. If $p = 2$, then we get the required result. So, let $p \geq 3$. Since the connected graph G has $p - 1$ independent vertices, all the independent vertices are end vertices of G. Hence G is a star. Then by Result 11(ii), $cgin(G) = p - 2$, but this leads to a contradiction. Conversely, let $G = K_2$. Then by Result 11(iii), $cgin(G) = 1 = p - 1$.
(iii) The result follows from the proof of (ii) and Result 11(ii). □

Problem 14. *Characterize graphs G for which $cgin^+(G) = 1$.*

The following theorem gives a realization result.

Theorem 15. *For any four positive integers a, b, c and n with $2 \leq a \leq b \leq c$, a connected graph G can be identified with $cgin(G) = a$, $cgin^+(G) = b$, $\beta(G) = c$ and $cg_{if}(G) = n$.*

Proof. **Case (1)** $a = b$. Let H_1, H_2, H_3 and H_4 be the complete graphs with vertex sets $V(H_1) = \{x, y\}$, $V(H_2) = \{u_1, u_2, \ldots, u_{c-a+1}\}$, $V(H_3) = \{v_1, v_2, \ldots, v_{a-2}\}$ and $V(H_4) = \{w_1, w_2, \ldots, w_{n+1}\}$ respectively. Let the graph G be constructed using $\bar{H}_1, \bar{H}_2, \bar{H}_3$ and H_4 by (i) joining the vertices x and y in $\bar{H}_1$ with every vertex of $\bar{H}_2$ and (ii) joining the vertex y in $\bar{H}_1$ with every vertex of $\bar{H}_3 \cup H_4$. The resultant graph G is shown in Figure 9.

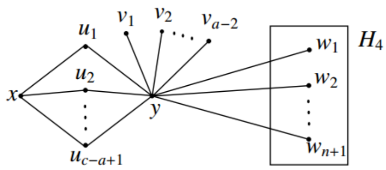

Figure 9. G.

Every independent set of G contains atmost one vertex from H_4. Then atleast n vertices in the complete graph H_4 lie on every S-fixed geodetic set of G, where S is an independent set of G. Hence $cg_{if}(G) \geq n$. Let $S = \{x, v_1, v_2, \ldots, v_{a-2}, w_i\}$ and let $S' = V(H_4) - \{w_i\}$ for any $i(1 \leq i \leq n+1)$. Clearly S is an independent set of G and every vertex of $V(G) - S$ is on an $x - s$ geodesic for any $s \in S'$ and $\langle S' \rangle$ is connected. Hence S' is an S-fixed connected geodetic set of G and so $cg_{if}(G) = |S'| = n$. Also, it is clear that, for any independent set S of G, every minimum S-fixed connected geodetic set contains n vertices from H_4.

Let T be any independent set of G with T', a T-fixed connected geodetic set of G and $|T'| = cg_{if}(G) = n$. Hence $T' \subset V(H_4)$. Now claim that $V(\bar{H}_3) \subseteq T$. If not, let $z \in V(\bar{H}_3)$ and $z \notin T$. Since z is an end vertex of G, z is not an internal vertex of any geodesic and so $z \in T'$, which is a contradiction to $T' \subset V(H_4)$. Hence $V(\bar{H}_3) \subseteq T$. Also, since $cg_{if}(G) = n$ and $T' \subset V(H_4)$, exactly one vertex in H_4, say w_1, belongs to T. Since w_1 is adjacent to y, $y \notin T$. Then either x or $u_i(1 \leq i \leq c-a+1)$ but not both belongs to T.
Subcase (1) $x \in T$. Then every vertex of $G - \bar{H}_3$ is on an $x - w$ geodesic for any $w \in T'$ and so $T = V(\bar{H}_3) \cup \{x, w_1\}$ is an independent set of G with T', a T-fixed connected geodetic set of G and $|T'| = n$.
Subcase (2) $u_i \in T(1 \leq i \leq c-a+1)$. If all the vertices $u_i \in T$, then x is not an internal vertex of any $s - t$ geodesic for any $s \in T$ and $t \in T'$. Hence $x \in T'$, which is a contradiction to $T' \subset V(H_4)$. If atleast one vertex $u_i \notin T$, then u_i is not an internal vertex of any $s - t$ geodesic for any $s \in T$ and $t \in T'$. Hence $u_i \in T'$, which is a contradiction to $T' \subset V(H_4)$. Thus $T_i = V(\bar{H}_3) \cup \{x, w_i\}$, where $w_i \in V(H_4)$ $(1 \leq i \leq n+1)$, is the independent set of G with $T'_i = V(H_4) - \{w_i\}$, the T_i-fixed connected geodetic set of G and $\left|T'_i\right| = n$. Hence $cgin(G) = cgin^+(G) = a$.

Claim $\beta(G) = c$. It can be easily verified that $W_i = V(\bar{H}_2 \cup \bar{H}_3) \cup \{w_i\}$, where $1 \le i \le n+1$, is a maximum independent set of G and so $\beta(G) = c$.

Case (2) $a < b$. Let H_1, H_2, H_3, H_4 and H_5 be the complete graphs with vertex sets $V(H_1) = \{x, y, z\}, V(H_2) = \{u_1, u_2, \ldots, u_{c-b+1}\}, V(H_3) = \{v_1, v_2, \ldots, v_{a-2}\}, V(H_4) = \{w_1, w_2, \ldots, w_{b-a}\}$ and $V(H_5) = \{t_1, t_2, \ldots, t_{n+1}\}$, respectively. Let G be the graph obtained from $\bar{H}_1, \bar{H}_2, \bar{H}_3, \bar{H}_4$ and H_5 by (i) joining the vertices x and y in $\bar{H}_1$ with every vertex of $\bar{H}_2$, (ii) joining the vertex y in $\bar{H}_1$ with every vertex of $\bar{H}_3$, (iii) joining the vertices y and z in $\bar{H}_1$ with every vertex of $\bar{H}_4$, and (iv) joining the vertex z in $\bar{H}_1$ with every vertex of H_5. The resultant graph G is shown in Figure 10. By an argument similar

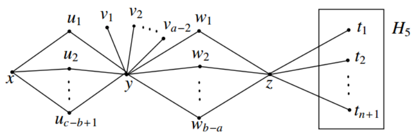

Figure 10. G.

to Case (1), for any independent set S of G, every minimum S-fixed connected geodetic set contains atleast n vertices from H_5. Let $S = V(\bar{H}_3) \cup \{x, t_1\}$ and let $S' = V(H_5) - \{t_1\}$. Then clearly S is an independent set of G and S' is an S-fixed connected geodetic set of G and so $cg_{if}(G) = |S'| = n$.

Next, prove that $cgin(G) = a$. By an argument similar to Case (1), to form an S-fixed connected geodetic set S' with $|S'| = n$, we need all the vertices in $\bar{H}_3$, the vertex x in $\bar{H}_1$ and exactly one vertex in H_5 for S. Hence $cgin(G) \ge a$. As in the above paragraph, $S = V(\bar{H}_3) \cup \{x, t_1\}$ is an independent set of G and $S' = V(H_5) - \{t_1\}$ is the S-fixed connected geodetic set with $|S'| = n$. Hence S is a connected geo-independent set of G and so $cgin(G) = |S| = a$.

Claim that $cgin^+(G) = b$. Let T be any independent set of G with T', a T-fixed connected geodetic set of G and $|T'| = cg_{if}(G) = n$. Then by an argument similar to Case (1), $V(\bar{H}_3) \cup \{x, t_i\} \subseteq T_i, V(H_5) - \{t_i\} = T'_i$ for any i $(1 \le i \le n+1)$, and $s \notin T_i$, where $s \in V(\bar{H}_2) \cup \{y, z\}$. Let $S_i = V(\bar{H}_3 \cup \bar{H}_4) \cup \{x, t_i\}$. Then clearly S_i is a maximum independent set of G with $S'_i = V(H_5) - \{t_i\}$, the S_i-fixed connected geodetic set of G and $\left|S'_i\right| = n$. Hence $cgin^+(G) = |S_i| = b$.

It can be easily verified that $W_i = V(\bar{H}_2 \cup \bar{H}_3 \cup \bar{H}_4) \cup \{t_i\}$, where $1 \le i \le n+1$, is a maximum independent set of G and so $\beta(G) = c$. □

Conflict of Interest

The authors declare no conflict of interests.

References

1. Buckley, F. and Harary, F., 1990. *Distance in Graphs*. Addison-Wesley.
2. Chartrand, G.and Zhang, P., 2005. *Introduction to Graph Theory*. McGraw Hill Education (India) Private Limited.
3. Harary, F., 1969. *Graph Theory*. Addison-Wesley.
4. Buckley, F., Harary, F. and Quintas, L. V., 1988. Extremal results on the geodetic number of a graph. *Scientia A2, 17*(1988), pp.17-26.
5. Kim, B. K., 2004. The geodetic number of a graph. *Journal of Computational and Applied Mathematics, 16*(1-2), pp.525-532.
6. Chartrand, G., Harary, F., Swart, H.C. and Zhang, P., 2001. Geodomination in graphs. *Bulletin of the ICA, 31*, pp.51-59.
7. Chartrand, G., Harary, F.and Zhang, P., 2002. On the geodetic number of a graph. *Networks*, 39, pp.1-6.

8. Harary, F., Loukakis, E.and Tsouros, C., 1993. The geodetic number of a graph. *Mathematical and Computer Modelling, 17*(11), pp.87-95.

9. Santhakumaran, A. P., Titus, P. and John, J., 2009. On the connected geodetic number of a graph. *Journal of Combinatorial Mathematics and Combinatorial Computing, 69*, pp.219-229.

10. Santhakumaran, A. P., Titus, P. and John, J., 2009. The upper connected geodetic number and forcing connected geodetic number of a graph. *Discrete Applied Mathematics, 157*, pp.1571-1580.

11. Santhakumaran, A. P.and Titus, P., 2005. Vertex geodomination in graphs. *Bulletin of Kerala Mathematics Association, 2*(2), pp.45-57.

12. Santhakumaran, A. P. and Titus, P., 2011. On the vertex geodomination number of a graph. *Ars Combinatoria, 101*, pp.137-151.

13. Santhakumaran, A. P. and Titus, P., 2012. The geo-number of a graph. *Ars Combinatoria, 106*, pp.65-78.

14. Santhakumaran, A. P. and Titus, P., 2009. The connected vertex geodomination number of a graph. *Journal of Prime Research in Mathematics, 5*, pp.101-114.

15. Santhakumaran, A. P. and Titus, P., 2009. The edge fixed geodomination number of a graph. *Analele Stiintifice ale Universitatii Ovidius Constanta, 17*(1), pp.187-200.

16. Titus, P.and Antin Mary, S., Communicated. Computation of independent fixed geodetic number of a graph.

17. Ostrand, P. A., 1973. Graphs with specified radius and diameter. *Discrete Mathematics, 4,* pp.71-75.