Ars Combinatoria, 158: 27–33
DOI:10.61091/ars158-04
http://www.combinatorialpress.com/ars
Received 02 January 2020, Accepted 11 March 2020, Published 31 March 2024

Article

Near Hexagons of Order $(3, t)$ having a Big Quad

Bart De Bruyn[1,*] **and Mou Gao**[1]

[1] Department of Mathematics: Algebra and Geometry, Ghent. University, Krijgslaan 281 (S25), B-9000 Gent, Belgium

* **Correspondence:** bart.debruyn@ugent.be

Abstract: We classify all near hexagons of order $(3, t)$ that contain a big quad. We show that, up to isomorphism, there are ten such near hexagons.

Keywords: Near hexagon, Big quad
Mathematics Subject Classification: 05B25, 51E12

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1. Introduction

A *generalised quadrangle* is a point-line geometry that satisfies the following properties:

(GQ1) There exist two disjoint lines.

(GQ2) For every non-incident point-line pair (x, L), there exists a unique point on L collinear with x.

An *ovoid* of a generalised quadrangle is a set of points that meets each line in a singleton. A *rosette of ovoids* is a set of ovoids through a given point x that partitions the set of points noncollinear with x. The point x is then called the *centre* of the rosette.

A *near hexagon* is a point-line geometry that satisfies the following properties:

(NH1) Every two distinct points are incident with at most one line.

(NH2) For every point x and every line L, there exists a unique point $\pi_L(x)$ on L which is nearest to x with respect to the distance function $d(\cdot, \cdot)$ of the collinearity graph Γ.

(NH3) The diameter of Γ is equal to 3.

A *quad* of a near hexagon $\mathcal{S}$ is a set Q of points satisfying the following properties:

(Q1) The diameter of Q is equal to 2.

(Q2) If x and y are two distinct points of Q, then any point collinear with x and y also belongs to Q.

(Q3) The point-line geometry formed by the points of Q and those lines of $\mathcal{S}$ that have all their points in Q is a generalised quadrangle.

Such a quad is called *big* if every point x of $\mathcal{S}$ outside Q is collinear with a (necessarily unique) point $\pi_Q(x)$ of Q.

A point-line geometry is said to have *order* (s, t) if every line is incident with precisely $s+1$ points and if every point is incident with exactly $t+1$ lines. A generalised quadrangle Q with at least three points on each line has a constant number $t_Q + 1$ of lines through each point.

A near hexagon is called *dense* if every line is incident with at least three points and if every two points at distance 2 have at least two common neighbours. In [1], we classified finite dense near hexagons with four points per line. There are ten known examples of such near hexagons, namely the near hexagons* $\mathbb{L}_4 \times \mathbb{L}_4 \times \mathbb{L}_4$, $W(3) \times \mathbb{L}_4$, $Q(4,3) \times \mathbb{L}_4$, $\mathrm{GQ}(3,5) \times \mathbb{L}_4$, $Q^-(5,3) \times \mathbb{L}_4$, $DW(5,3)$, $DQ(6,3)$, $DH(5,9)$, $\mathrm{GQ}(3,5) \otimes \mathrm{GQ}(3,5)$ and $Q^-(5,3) \otimes Q^-(5,3)$. Each of these has a big quad. In [1], we showed that the other (unknown) finite dense near hexagons with four points per line fall into four classes and that none of them has a big quad. This note arose from investigations whether these ten known examples are the only near hexagons of order $(3,t)$ with a big quad. Previously, also all near hexagons of order $(2,t)$ with a big quad had been classified and also there all examples turned out to be dense, see [2, §6] and [3, §10.7].

After successfully classifying all near hexagons of order $(3,t)$ containing a big quad, we noticed that the used arguments could be generalised to cover a more general result. We will therefore prove our results in this more general setting. Specifically, we show the following;

Theorem 1. *Suppose $\mathcal{S}$ is a finite near hexagon of order (s,t), where $s \geq 2$. Let $\mathcal{Q}$ denote the set of all quads of $\mathcal{S}$ and denote by (s, t_Q) the order of a quad $Q \in \mathcal{Q}$. Suppose that at least one quad of $\mathcal{S}$ is big and that all numbers t_Q, $Q \in \mathcal{Q}$, are odd. Suppose also that every quad $Q \in \mathcal{Q}$ that is not a grid has no partition in ovoids or no rosette of ovoids. Then $\mathcal{S}$ must be dense.*

Suppose now that $\mathcal{S}$ is a near hexagon of order $(3,t)$ with a big quad. We will show in Lemma 4 that $\mathcal{S}$ is finite. Every quad of $\mathcal{S}$ only has lines of size four. By [4] (see also [5, §6.2]), there are only five generalised quadrangles with four points per line, namely the (4×4)-grid of order $(3,1)$, the symplectic generalised quadrangle $W(3)$ of order $(3,3)$, the parabolic generalised quadrangle $Q(4,3)$ of order $(3,3)$, the unique generalised quadrangle $\mathrm{GQ}(3,5)$ of order $(3,5)$ and the elliptic generalised quadrangle $Q^-(5,3)$ of order $(3,9)$. Only one of these five generalised quadrangles has both partitions in ovoids and rosettes of ovoids, namely the (4×4)-grid, see Lemma 3. Theorem 1 then implies.

Corollary 1. *Suppose $\mathcal{S}$ is a (possibly infinite) near hexagon of order $(3,t)$ having a big quad. Then $\mathcal{S}$ is finite and dense, and therefore isomorphic to either $\mathbb{L}_4 \times \mathbb{L}_4 \times \mathbb{L}_4$, $W(3) \times \mathbb{L}_4$, $Q(4,3) \times \mathbb{L}_4$, $GQ(3,5) \times \mathbb{L}_4$, $Q^-(5,3) \times \mathbb{L}_4$, $DW(5,3)$, $DQ(6,3)$, $DH(5,9)$, $GQ(3,5) \otimes GQ(3,5)$ or $Q^-(5,3) \otimes Q^-(5,3)$.*

In a general near hexagon with four points per line, we show in Corollary 3 that any quad isomorphic to $W(3)$, $\mathrm{GQ}(3,5)$ or $Q^-(5,3)$ has to be big. Combining this with Corollary 1, we thus find the following;

Corollary 2. *Suppose $\mathcal{S}$ is a (possibly infinite) near hexagon of order $(3,t)$ that is not isomorphic to one of the ten examples mentioned in Corollary 1. Then every quad of $\mathcal{S}$ is isomorphic to the (4×4)-grid or to the generalised quadrangle $Q(4,3)$.*

2. Preliminaries

Suppose $\mathcal{S}$ is a near hexagon. For every point x, $\Gamma_i(x)$ with $i \in \mathbb{N}$ denotes the set of points at distance i from x. For every nonempty set X of points, $\Gamma_i(X)$ with $i \in \mathbb{N}$ denotes the set of points at distance i from X, i.e. the set of points x for which the minimal distance from x to a point of X is equal to i. The following result will be useful in our discussion;

Proposition 1 ([6, Proposition 2.5]). *Suppose x and y are two points of a near hexagon $\mathcal{S}$ at distance 2 from each other such that x and y have two common neighbours a and b such that at least one of the*

*For a definition of these near hexagons, see e.g. [1].

four lines xa, xb, ay, by contains at least three points. Then x and y are contained in a unique quad $Q(x, y)$.

For every point x of $\mathcal{S}$, the *local space* $\mathcal{L}_x$ at x is the point-line geometry whose points and lines are the lines and quads through x, respectively, with incidence being containment. This local space is always a *partial linear space*. Proposition 1 implies the following.

- In a dense near hexagon, every local space is linear.
- In a nondense near hexagon with more than two points on each line, at least one local space is not linear, i.e. there exist two intersecting lines that are not contained in a quad, or equivalently, there exist two points at distance 2 that have a unique common neighbour.

We now collect a number of known properties of quads in a near hexagon $\mathcal{S}$. We refer to the literature (e.g. [3, 7]) for proofs.

If (x, Q) is a point-quad pair of $\mathcal{S}$, then $d(x, Q) \leq 2$. If $d(x, Q) = 1$, then Q contains a unique point $\pi_Q(x)$ collinear with x and $d(x, y) = 1 + d(\pi_Q(x), y)$ for every point y of Q. If $d(x, Q) = 2$, then the set $O_x := \Gamma_2(x) \cap Q$ is an ovoid of Q.

If (x, Q) is a point-quad pair with $(x, Q) = 2$, then there are two possibilities for a line L through x:

(a) We have $L \subseteq \Gamma_2(Q)$. In this case, the ovoids O_y, $y \in L$, form a partition of ovoids of Q.
(b) The line L contains a unique point $u \in \Gamma_1(Q)$ and $L \setminus \{u\} \subseteq \Gamma_2(Q)$. In this case, the ovoids O_y, $y \in L \setminus \{u\}$, form a rosette of ovoids of Q with centre $\pi_Q(u)$.

If Q is a quad and L a line contained in $\Gamma_1(Q)$, then $\pi_Q(L) := \{\pi_Q(x) \mid x \in L\}$ is a line of Q, and the map $x \mapsto \pi_Q(x)$ defines a bijection between L and $\pi_Q(L)$. From this it can easily be deduced that if Q_1 and Q_2 are two disjoint quads with Q_1 big, then $\pi_{Q_1}(Q_2) := \{\pi_{Q_1}(x) \mid x \in Q_2\}$ is a subquadrangle of Q_1 isomorphic to Q_2. On the other hand, if Q_1 and Q_2 are two distinct intersecting quads of $\mathcal{S}$ with Q_1 big, then $Q_1 \cap Q_2$ must be a line.

Also the following lemmas will be useful in our treatment;

Lemma 1. *Suppose $\mathcal{S}$ is a finite near hexagon of order (s, t) having v points. Then* $|\Gamma_2(x)| = n_2 := \frac{v}{s+1} - 1 + s^2t - st$ *and* $|\Gamma_3(x)| = n_3 := \frac{sv}{s+1} - s - s^2t$ *for every point x of $\mathcal{S}$.*

Proof. For every point x of $\mathcal{S}$, we have $|\Gamma_0(x)| + |\Gamma_1(x)| + |\Gamma_2(x)| + |\Gamma_3(x)| = v$, where $|\Gamma_0(x)| = 1$ and $|\Gamma_1(x)| = s(t + 1)$. By [3, Theorem 1.2], we also know that $|\Gamma_0(x)| - \frac{1}{s}|\Gamma_1(x)| + \frac{1}{s^2}|\Gamma_2(x)| - \frac{1}{s^3}|\Gamma_3(x)| = 0$. The values of $|\Gamma_2(x)|$ and $|\Gamma_3(x)|$ are now easily computed. They are respectively equal to n_2 and n_3. □

A generalized quadrangle Q of order (s, t) contains $(s + 1)(st + 1)$ points, and for any point x of Q there are $(s + 1)(st + 1) - 1 - |\Gamma_1(x)| = (s + 1)(st + 1) - 1 - s(t + 1) = s^2t$ points in Q noncollinear with x.

Lemma 2. *Suppose $\mathcal{S}$ is a finite near hexagon of order (s, t), $s \geq 2$, having v points. Let x be a point of $\mathcal{S}$ and denote by $\mathcal{R}$ the set of all quads through x. For every $R \in \mathcal{R}$, denote by (s, t_R) the order of the quad R. Then*

$$\sum_{R \in \mathcal{R}} t_R^2 = t(t + 1) - \frac{1}{s^2}\left(\frac{v}{s + 1} - 1 + s^2t - st\right).$$

Proof. A quad $R \in \mathcal{R}$ contains s^2t_R points of $\Gamma_2(x)$. So, the number of points of $\Gamma_2(x)$ not contained in a quad together with x equals $n_2 - \sum_{R\in\mathcal{R}} s^2t_R$, where n_2 is the number defined in Lemma 1. Now, counting pairs (u, v) of collinear points in $\Gamma_1(x) \times \Gamma_2(x)$ gives $s(t+1) \cdot st = \sum_{R\in\mathcal{R}} s^2t_R \cdot (t_R + 1) + (n_2 - \sum_{R\in\mathcal{R}} s^2t_R) \cdot 1$, from which the mentioned equality readily follows. □

Regarding ovoids of generalized quadrangles of order $(3, t)$, the following can be said.

Lemma 3. (*a*) *The generalized quadrangles* $W(3)$ *and* $Q^-(5,3)$ *do not have ovoids.*

(*b*) *The generalized quadrangle* $GQ(3,5)$ *does not have rosettes of ovoids.*

(*c*) *The generalized quadrangle* $Q(4,3)$ *does not have partitions in ovoids.*

Proof. This is precisely Lemma 1 of [1]. Proofs for the nonexistence of ovoids in the generalized quadrangles $W(3)$ and $Q^-(5,3)$ can be found in [5, 3.4.1]. Payne [8, VI.1] classified all ovoids of GQ(3,5), and from this classification it readily follows that there cannot be rosettes of ovoids. The classification of the ovoids of $Q(4,3)$ can be found in [7, p. 160], and from this classification, it follows that there cannot be partitions in ovoids. □

Corollary 3. *In a near hexagon with four points per line, every quad isomorphic to either* $W(3)$*,* $GQ(3,5)$ *or* $Q^-(5,3)$ *is big.*

Proof. We must show that $\Gamma_2(Q) = \emptyset$ for every such quad Q. Suppose to the contrary that $\Gamma_2(Q) \neq \emptyset$ and let $x \in \Gamma_2(Q)$. Then $\Gamma_2(x) \cap Q$ is an ovoid of Q. As $x \in \Gamma_2(Q)$, there exists a line through x meeting $\Gamma_1(Q)$. This line would induce a rosette of ovoids of Q. A contradiction follows from Lemma 3. □

Regarding near hexagons of order $(3,t)$, the following can be said.

Lemma 4. *If* $\mathcal{S}$ *is a near hexagon of order* $(3,t)$ *having a big quad* Q*, then* $\mathcal{S}$ *is finite.*

Proof. It suffices to prove that t is finite. Any quad R of $\mathcal{S}$ also has four points on each line and hence must be finite by the main result of [9]. So, $t_R \in \mathbb{N}$ if $(3,t_R)$ denotes the order of R. In particular, $t_Q \in \mathbb{N}$. Let x be an arbitrary point outside Q and let L be the unique line through x meeting Q in a point (namely $\pi_Q(x)$). Let K be a line through x distinct from L, let y be a point of K distinct from x. Now, the points x and $\pi_Q(y)$ have at least two common neighbours (namely $\pi_Q(x)$ and y) and so are contained in a unique quad by Proposition 1. This quad contains the lines K, L and meets Q in a line. This implies that $t = \sum_{R\in\mathcal{R}'} t_R$, where $\mathcal{R}'$ is the set of quads through the line L. Recall that each number t_R, $R \in \mathcal{R}'$, is finite. Since any quad through L meets Q in a line, we have $|\mathcal{R}'| \leq t_Q + 1$ and so also $\mathcal{R}'$ is finite. We conclude that $t = \sum_{R\in\mathcal{R}'} t_R$ is finite as well. □

3. Proof of Theorem 1

Suppose $\mathcal{S}$ is a finite near hexagon of order (s,t), $s \geq 2$, containing a big quad of order (s,α).

Lemma 5. *The total number* v *of points of* $\mathcal{S}$ *is equal to* $(s+1)(s\alpha+1)(s(t-\alpha)+1) = (s+1)(s^2\alpha(t-\alpha)+st+1)$.

Proof. Let Q be a big quad of order (s,α). The total number of points of Q is equal to $(s+1)(s\alpha+1)$. Every point x of Q is collinear with $s(t-\alpha)$ points outside Q, namely s on each of the $t-\alpha$ lines through x not contained in Q. On the other hand, each point y outside Q is collinear with a unique point of Q, namely $\pi_Q(y)$. So, $|\mathcal{P} \setminus Q| = |Q| \cdot s(t-\alpha)$ and $|\mathcal{P}| = |Q| + |Q| \cdot s(t-\alpha) = (s+1)(s\alpha+1)(1+s(t-\alpha))$. □

Lemma 6. *Every quad of order* (s,α) *or* $(s,t-\alpha)$ *is big. There are no quads of order* (s,β) *with* $\min(\alpha,t-\alpha) < \beta < \max(\alpha,t-\alpha)$.

Proof. Suppose R is a quad of order (s,β) with $\min(\alpha,t-\alpha) \leq \beta \leq \max(\alpha,t-\alpha)$. With a completely similar reasoning as in Lemma 5, we find that the number of points at distance at most 1 from R is equal to $(s+1)(s\beta+1)(1+s(t-\beta)) = (s+1)(s^2\beta(t-\beta)+st+1)$. This number is at least $(s+1)(s^2\alpha(t-\alpha)+st+1) = v$ with equality if and only if β is equal to either α or $t-\alpha$. The conclusions of the lemma follow. □

Lemma 7. *If* x *is a point of a big quad* Q*, then the local space at the point* x *is linear.*

Proof. We need to prove that any two distinct lines K and L through x are contained in a (necessarily unique) quad. This is certainly the case if K and L are contained in Q. So, without loss of generality we may suppose that L is not contained in Q. Let $\mathcal{R}'$ denote the set of quads through L and take a point y on L distinct from x. The number of lines through x distinct from L and contained in a quad together with L is equal to $\sum_{R\in\mathcal{R}'} t_R$. This number also equals the number of lines through y distinct from L and contained in a quad together with L. So, it suffices to prove that if L' is a line through y distinct from L, then there is a (necessarily unique) quad through L and L'. Put $L'' := \pi_Q(L')$. Take $z \in L' \setminus \{y\}$ and put $u := \pi_Q(z)$. The points u and y have at least two common neighbours, namely z and x. As $s \geq 2$, Proposition 1 implies that u and y are contained in a unique quad. This unique quad also contains the lines L and L'. □

Using Lemmas 5, 6 and 7, we can now prove Theorem 1. So, suppose that $\mathcal{S}$ satisfies the conditions of that theorem. By way of contradiction, suppose that $\mathcal{S}$ is not dense.

Let Q be a big quad of order (s, α). Since $\mathcal{S}$ is not dense, there exists a point x whose local space is not linear. By Lemmas 6 and 7, $x \notin Q$ and x is not contained in quads of order (s, α). Let L denote the unique line through x meeting Q in the point $y := \pi_Q(x)$. By Lemma 7, the local space at y is linear and so there exists a quad through the line L. Since the local space at $x \in L$ is not linear, this quad is not big by Lemma 7. If (s,β) is the order of this quad, then from Lemma 6 we know that β cannot be contained in the interval $[\min(\alpha, t-\alpha), \max(\alpha, t-\alpha)]$. This implies that $\alpha \neq 1$.

Let $\mathcal{R}$ denote the set of all quads through x and let $\mathcal{R}'$ denote the set of all quads through the line L. Since the local space at y is linear, every two distinct lines through y are contained in a unique quad. Since Q is big, every quad through L meets Q in a line and so there are precisely $t_Q + 1 = \alpha + 1$ of them. We thus have $\sum_{R\in\mathcal{R}'} t_R = t$ with $|\mathcal{R}'| = t_Q + 1$. Since t_R, $R \in \mathcal{R}'$, is odd and $|\mathcal{R}'| = t_Q + 1$ is even, we thus have that $\sum_{R\in\mathcal{R}'} t_R = t$ is even.

Suppose R is a quad of order (s,β), $\beta \geq 2$, through x that is disjoint from Q, i.e. R does not contain the line L. Since x is not contained in quads of order (s, α), we have $\beta \neq \alpha$. Since $\pi_Q(R)$ is a subquadrangle of Q isomorphic to R, we thus have that $\beta < \alpha$ and $\pi_Q(R)$ is a proper subquadrangle of Q. There thus exists a point $z \in Q \setminus \pi_Q(R)$. This point lies at distance 2 from R. Any line through z containing a point of $\Gamma_1(R)$ defines a rosette of ovoids of R. So, we see that R cannot have a partition in ovoids. This implies that there are no lines through z contained in $\Gamma_2(R)$, i.e. every line through z contains a unique point of $\Gamma_1(R)$. Now, $\Gamma_2(z) \cap R$ is an ovoid of R and since the local space in $z \in Q$ is linear (see Lemma 7), there exists a unique quad $Q(z, u)$ through z and any point u of $\Gamma_2(z) \cap R$. We denote by $\mathcal{U}$ the set of all quads through z meeting R in a point of $\Gamma_2(z) \cap R$. The quads of $\mathcal{U}$ determine a partition of the set of all lines through z and so we have $t + 1 = \sum_{U\in\mathcal{U}}(t_U + 1)$. As $t_U + 1$ is even for every $U \in \mathcal{U}$, we have that also $t + 1$ is even. But this is impossible since we have already shown above that t is even.

So, every quad through x not containing L is a grid. The total number of such grid-quads is at most

$$\frac{t(t-1)}{2} - \sum_{R\in\mathcal{R}'} \frac{t_R(t_R-1)}{2}.$$

Indeed, there is at most one such quad through every two lines through x distinct from L. Moreover, for each $R \in \mathcal{R}'$, there are $\frac{t_R(t_R-1)}{2}$ pairs $\{L_1, L_2\}$ of distinct lines through x such that $L \neq L_1, L_2 \subseteq R$, and for none of these pairs $\{L_1, L_2\}$, the unique quad containing L_1 and L_2 (namely R) is a grid. So,

$$\sum_{R\in\mathcal{R}} t_R^2 \leq \frac{t(t-1)}{2} - \sum_{R\in\mathcal{R}'} \frac{t_R(t_R-1)}{2} + \sum_{R\in\mathcal{R}'} t_R^2.$$

Since $\sum_{R\in\mathcal{R}'} t_R = t$, we have

$$\sum_{R\in\mathcal{R}} t_R^2 \leq \frac{1}{2}\Big(t^2 + \sum_{R\in\mathcal{R}'} t_R^2\Big). \tag{1}$$

Since

- $t_R \in \mathbb{N} \setminus \{0\}$ for every $R \in \mathcal{R}'$,
- $\sum_{R\in\mathcal{R}'} t_R = t$ and $|\mathcal{R}'| = \alpha + 1$,
- $a^2 + b^2 < (a-1)^2 + (b+1)^2$ for all $a, b \in \mathbb{N}$ with $2 \leq a \leq b$,

we have

$$\sum_{R\in\mathcal{R}'} t_R^2 \leq \alpha \cdot 1^2 + 1 \cdot (t-\alpha)^2. \tag{2}$$

By (1), (2) and the fact that $\alpha > 1$, we have that

$$\sum_{R\in\mathcal{R}} t_R^2 \leq \frac{\alpha + \alpha^2 - 2\alpha t + t^2 + t^2}{2} < \alpha^2 - \alpha t + t^2. \tag{3}$$

On the other hand, by Lemmas 1 and 2, we know that $\sum_{R\in\mathcal{R}} t_R^2 = t(t+1) - \frac{n_2}{s^2}$, where $n_2 = \frac{v}{s+1} - 1 + s^2 t - st$. Since $v = (s+1)(s^2\alpha(t-\alpha) + st + 1)$, we have $n_2 = s^2(t + \alpha t - \alpha^2)$ and

$$\sum_{R\in\mathcal{R}} t_R^2 = t(t+1) - \frac{n_2}{s^2} = t^2 - \alpha t + \alpha^2. \tag{4}$$

A contradiction follows from (3) and (4).

Acknowledgment

The second author, Mou Gao, is supported by the State Scholarship Fund (File No. 201806065052) and the National Natural Science Foundation of China (grant number 71771035).

Conflict of Interest

The authors declare no conflict of interest.

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