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Article

On Lie Derivations, Generalized Lie Derivations and Lie Centralizers of Octonion Algebras

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Abstract: Let L be a unital ring with characteristic different from 2 and $\mathcal{O}(L)$ be an algebra of Octonion over L. In the present article, our attempt is to present the characterization as well as the matrix representation of some variants of derivations on $\mathcal{O}(L)$. The matrix representation of Lie derivation of $\mathcal{O}(L)$ and its decomposition in terms of Lie derivation and Jordan derivation of L and inner derivation of $\mathcal{O}$ is presented. The result about the decomposition of Lie centralizer of $\mathcal{O}$ in terms of Lie centralizer and Jordan centralizer of L is given. Moreover, the matrix representation of generalized Lie derivation (also known as D-Lie derivation) of $\mathcal{O}(L)$ is computed.

Keywords: The Octonion algebra, Lie Derivations, D-Lie Derivations, Lie Centralizer
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1. Introduction

It is well known that derivation is a function of an algebra which generalizes certain features of the derivative operator. It gives an interesting insight to understand the structure and local properties of an algebra. The concept to study the structure of ring theory and derivations though established long back, but got stimulated after Posner in [1] described some important results on the derivations of prime rings. A fundamental problem in the theory of derivations is to determine all the derivations on an algebra. Over the years, many important variants of derivation have been presented. Among these variants, Lie derivation and its generic extension is currently more interesting and attracting. Lie centralizer is not so common but very meaningful variant. These variants are being widely discussed now a days.

Let $\mathcal{O}$ be a unital algebra and the center of $\mathcal{O}$ is denoted by $Z(\mathcal{O})$. We denote the commutator (Lie product) and Jordan product of $x_1,\ x_2$ by $[x_1, x_2] = x_1x_2 - x_2x_1$ and $x_1 \circ x_2 = x_1x_2 + x_2x_1$ respectively for all $x_1, x_2 \in \mathcal{O}$. We say that a ring $\mathcal{O}$ is an F-algebra (F is a field) if $\mathcal{O}$ is an F-vector space equipped with a bilinear product. Let $D : \mathcal{O} \to \mathcal{O}$ be an additive map. We say D is a derivation (respectively Jordan derivation) if $D[x_1, x_2] = [D(x_1), x_2] + [x_1, D(x_2)]$ (respectively $D(x^2) = D(x)x + xD(x)$), for all $x, x_1, x_2 \in \mathcal{O}$. For an element $\alpha \in \mathcal{O}$, the mapping $I_\alpha : \mathcal{O} \to \mathcal{O}$ given by $I_\alpha(x) = x\alpha - \alpha x$ for all $x \in \mathcal{O}$ is called an inner derivation of $\mathcal{O}$ induced by α. Let $\xi : \mathcal{O} \to \mathcal{O}$ be an additive map. The map ξ is said to be right (left) centralizer if $\xi(x_1x_2) = x_1\xi(x_2)$ ($\xi(x_1x_2) = \xi(x_1)x_2$) for all $x_1, x_2 \in \mathcal{O}$. We say that φ is an Jordan centralizer if $\varphi(x_1 \circ x_2) = \varphi(x_1) \circ x_2$ for all $x_1, x_2 \in \mathcal{O}$. An additive map $\xi : \mathcal{O} \to \mathcal{O}$ is called a Lie centralizer if $\xi[x_1, x_2] = [\xi(x_1), x_2]$ for all $x_1, x_2 \in \mathcal{O}$.

The characterization of Lie centralizer of quaternion algebra is given in [2]. The characterization of Lie centralizers and their generalizations is now widely studied on different kinds of algebras by many authors in ([3–7]). Fosner et al. characterised Lie centralizers of triangular tings and nest algebras in [3]. Ashrafi et al. computed multiplicative generalized Lie n-derivations of unital rings with idempotents and σ-centralizers generalized matrix algebras in [4] and [5] respectively. Fadaee et al. characterized Lie centralizers at the zero product of generalized matrix algebras in [6] and Lie triple centralizers of generalized matrix algebras in [7]. In [8], Martindale described the standard form of Lie derivation on certain primitive rings. Similar results have been discussed on von Neumann algebras by Miers [9]. Mokhtari *et al*. computed Lie derivations on trivial extension algebras in [10]. It is obvious that every Lie derivation is a generalized Lie derivation. Besides there are two different definitions of generalized Lie derivations in literature. One is introduced by Atsushi Nakajima [11] which is stated as: for an additive map $F : \mathcal{O} \to \mathcal{O}$, we say that F is generalized Lie derivation (also known as D-Lie derivation) if there exist a Lie derivation $D : \mathcal{O} \to \mathcal{O}$ such that $F[x_1, x_2] = [F(x_1), x_2] + [x_1, D(x_2)]$ for all $x_1, x_2 \in \mathcal{D}$ and the other is by Bojan Hvala [12] which is stated as: for an additive map $F : \mathcal{O} \to \mathcal{O}$, we say that F is generalized Lie derivation if there exist a linear map $D : \mathcal{O} \to \mathcal{O}$ such that $F[x_1, x_2] = F(x_1)x_2 - F(x_2)x_1 + x_1D(x_2) - x_2D(x_1)$ for all $x_1, x_2 \in \mathcal{D}$. Both definitions are equally being discussed by mathematicians. Hvala discussed the generalized Lie derivations on rings and proved that every generalized Lie derivation on a prime ring can be written as the sum of a generalized derivation and a central map. Benkovič [13] proved that every generalized Lie derivation from a unital algebra onto a unitary bimodule can be written in the sum of a generalized derivation and a central map that vanishes on the commutators of the algebra. The description of generalized Lie derivation of Lie ideals of prime algebras and nonlinear generalized Lie derivation of some classical triangular algebras are respectively given in [14, 15]. The generalized Lie derivations of prime rings are discussed in [16] by using both definitions. Hvala's definition of generalized Lie derivation covers both generalized derivations and D-Lie derivations. On the other hand, Nakajima's definition is more favourable which unifies the notions of Lie derivation and Lie centralizer. We will, in particular focus on Nakajima's definition to compute the matrix representation of generalized Lie derivations of algebra of Octonion. The discussion about local and 2-local derivation of Octonion algebra have been discussed in [17]. Later, this discussion was extended to Cayley algebras in [18].

Mainly, our focus in this article is to describe the matrix representation as well as the characterization of Lie derivation of Octonion algebras equipped with commutator product. Authors characterized in [19], the Lie triple derivations of algebra of tensor product of some algebra T and quaternion algebra. Ghahramani *et al*. in [20] proved results on the characterization of generalized derivation and generalized Jordan derivation of ring of quaternion and in [21] discussed the characterization of Lie derivation and its natural generic extension of quaternion ring.

This article is arranged in the following order: Section 2 contains some minor details of Octonion algebras equipped with commutator product denoted by $\mathcal{O}$. In Sections 3 and 4, matrix representation of Lie derivation as well as decomposition of Lie derivation of octonion algebra in terms of Lie derivation and Jordan derivation of L and inner derivation of $\mathcal{O}$ is presented. Section 5 contains the characterization of Lie centralizer of Octonion algebras. In Section 6, the matrix representation of generalized Lie derivation is computed.

2. The Octonion Algebra $\mathcal{O}$

Let L be an arbitrary 2-torsion free unital ring. The octonion algebra (denoted by $\mathcal{O}$) over L is a class of non-associative algebra. It is a unital nonassociative algebra of dimension 8 with the basis $\mathcal{B} = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ and the product defined in the following table.

	e_j								
	$e_i.e_j$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
	e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
	e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
	e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_i	e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
	e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
	e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
	e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
	e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

The table can be summarized as follows:

$$e_i.e_j = \begin{cases} e_j, & \text{if } i = 0; \\ e_i, & \text{if } j = 0; \\ -\delta_{ij}e_0 + \epsilon_{ijk}e_k, & \text{otherwise,} \end{cases}$$

where δ_{ij} is the Kronecker delta and ϵ_{ijk} is a completely antisymmetric tensor with value $+1$ when $ijk = 123,\ 145,\ 176,\ 246,\ 257,\ 347,\ 365$.

An octonion x is of the form $x = x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7$ with real coefficients x_i. By using the product defined in the table given above, we can have the following relations;

$$[e_1, e_2] = 2e_3, [e_1, e_3] = -2e_2, [e_1, e_4] = 2e_5, [e_1, e_5] = -2e_4, [e_1, e_6] = -2e_7, [e_1, e_7] = 2e_6, [e_2, e_3] = 2e_1,$$
$$[e_2, e_4] = 2e_6, [e_2, e_5] = 2e_7, [e_2, e_6] = -2e_4, [e_2, e_7] = -2e_5, [e_3, e_4] = 2e_7, [e_3, e_5] = -2e_6, [e_3, e_6] = 2e_5,$$
$$[e_3, e_7] = -2e_4, [e_4, e_5] = 2e_1, [e_4, e_6] = 2e_2, [e_4, e_7] = 2e_3, [e_5, e_6] = -2e_3, [e_5, e_7] = 2e_2, [e_6, e_7] = -2e_1.$$

Using the above product on the basis as Lie product and extend it by linearity, we can equip this product on O.

3. Lie Derivation of Octonion Algebra O

In this section, we compute matrix representation of Lie derivation of the octonion Algebra. Let $D : O \to O$ be a Lie derivation. D admits a matrix representation with respect to the basis, which is an 8×8 matrix $[D] = (\beta_{ij})^T$ whose entries are defined by

$$D(e_{i-1}) = \sum_{j=1}^{8} \beta_{ij} e_{j-1}, \qquad 1 \le i \le 8. \tag{1}$$

Each column of $[D]$ is an element of O.

Theorem 1. *The algebra of Lie derivations of Octonions is generated by the following matrices:*

$$[D] = \begin{pmatrix} \beta_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta_{23} & -\beta_{24} & -\beta_{25} & -\beta_{26} & -\beta_{27} & -\beta_{28} \\ 0 & \beta_{23} & 0 & -\beta_{34} & -\beta_{35} & -\beta_{36} & -\beta_{37} & -\beta_{38} \\ 0 & \beta_{24} & \beta_{34} & 0 & -\beta_{27}+\beta_{36} & -\beta_{28}-\beta_{35} & \beta_{25}-\beta_{38} & \beta_{26}+\beta_{37} \\ 0 & \beta_{25} & \beta_{35} & \beta_{27}-\beta_{36} & 0 & -\beta_{56} & -\beta_{57} & -\beta_{58} \\ 0 & \beta_{26} & \beta_{36} & \beta_{28}+\beta_{35} & \beta_{56} & 0 & -\beta_{23}-\beta_{58} & -\beta_{24}+\beta_{57} \\ 0 & \beta_{27} & \beta_{37} & -\beta_{25}+\beta_{38} & \beta_{57} & \beta_{23}+\beta_{58} & 0 & -\beta_{34}-\beta_{56} \\ 0 & \beta_{28} & \beta_{38} & -\beta_{26}-\beta_{37} & \beta_{58} & \beta_{24}-\beta_{57} & \beta_{34}+\beta_{56} & 0 \end{pmatrix}$$

Proof. Let D be a Lie derivation of O. Then we write

$$D(e_{i-1}) = \sum_{j=1}^{8} \beta_{ij} e_{j-1}, \qquad 1 \le i \le 8$$

for some arbitrary $\beta'_{ij}s \in L$. Applying D on the identity $[e_0, e_j] = 0$ for $1 \le j \le 7$. So, we get $\beta_{1j} = 0$ for $2 \le j \le 7$.

Applying D on identities $e_3 = \frac{1}{2}[e_1, e_2]$, $e_3 = \frac{1}{2}[e_4, e_7]$ and $e_3 = \frac{1}{2}[e_6, e_5]$, we get

$$\begin{aligned} D(e_3) =& \frac{1}{2}(2\beta_{22}e_3 - 2\beta_{24}e_1 - 2\beta_{25}e_6 - 2\beta_{26}e_7 + 2\beta_{27}e_4 + 2\beta_{28}e_5 + 2\beta_{33}e_3 - 2\beta_{34}e_2 + 2\beta_{35}e_5 \\ & - 2\beta_{36}e_4 - 2\beta_{37}e_7 + 2\beta_{38}e_6) \\ =& (2\beta_{52}e_6 - 2\beta_{53}e_5 - 2\beta_{54}e_4 + 2\beta_{55}e_3 + 2\beta_{56}e_2 - 2\beta_{57}e_1 - 2\beta_{82}e_5 - 2\beta_{83}e_6 - 2\beta_{84}e_7 \\ & + 2\beta_{86}e_1 + 2\beta_{87}e_2 + 2\beta_{88}e_3) \\ =& (-2\beta_{62}e_7 - 2\beta_{63}e_4 + 2\beta_{64}e_5 + 2\beta_{65}e_2 - 2\beta_{66}e_3 + 2\beta_{68}e_1 + 2\beta_{72}e_4 - 2\beta_{73}e_7 + 2\beta_{74}e_6 \\ & - 2\beta_{75}e_1 - 2\beta_{77}e_3 + 2\beta_{78}e_2). \end{aligned}$$

Now by applying D on $e_1 = \frac{1}{2}[e_2, e_3]$, $e_1 = \frac{1}{2}[e_4, e_5]$ and $e_1 = -\frac{1}{2}[e_6, e_7]$, we get

$$\begin{aligned} D(e_1) =& \frac{1}{2}(-2\beta_{32}e_2 + 2\beta_{33}e_1 - 2\beta_{35}e_7 + 2\beta_{36}e_6 - 2\beta_{37}e_5 + 2\beta_{38}e_4 - 2\beta_{42}e_3 + 2\beta_{44}e_1 + 2\beta_{45}e_6 \\ & + 2\beta_{46}e_7 - 2\beta_{47}e_4 - 2\beta_{48}e_5) \\ =& \frac{1}{2}(-2\beta_{52}e_4 + 2\beta_{53}e_7 - 2\beta_{54}e_6 + 2\beta_{55}e_1 + 2\beta_{57}e_3 - 2\beta_{58}e_2 - 2\beta_{62}e_5 - 2\beta_{63}e_6 - 2\beta_{64}e_7 \\ & + 2\beta_{66}e_1 + 2\beta_{67}e_2 + 2\beta_{68}e_3) \\ =& -\frac{1}{2}(2\beta_{72}e_6 - 2\beta_{73}e_5 - 2\beta_{74}e_4 + 2\beta_{75}e_3 + 2\beta_{76}e_2 - 2\beta_{77}e_1 + 2\beta_{82}e_7 + 2\beta_{83}e_4 - 2\beta_{84}e_5 \\ & - 2\beta_{85}e_2 + 2\beta_{86}e_3 - 2\beta_{88}e_1). \end{aligned}$$

Now by applying D on $[e_1, e_3]$, $[e_4, e_6]$ and $[e_5, e_7]$, we get

$$\begin{aligned} D(e_2) =& -\frac{1}{2}(-2\beta_{22}e_2 + 2\beta_{23}e_1 - 2\beta_{25}e_7 + 2\beta_{26}e_6 - 2\beta_{27}e_5 + 2\beta_{28}e_4 + 2\beta_{43}e_3 - 2\beta_{44}e_2 + 2\beta_{45}e_5 \\ & - 2\beta_{46}e_4 - 2\beta_{47}e_7 + 2\beta_{48}e_6) \\ =& \frac{1}{2}(-2\beta_{52}e_7 - 2\beta_{53}e_4 + 2\beta_{54}e_5 + 2\beta_{55}e_2 - 2\beta_{56}e_3 + 2\beta_{58}e_1 - 2\beta_{72}e_5 - 2\beta_{73}e_6 - 2\beta_{74}e_7 \\ & + 2\beta_{76}e_1 + 2\beta_{77}e_2 + 2\beta_{78}e_3) \\ =& \frac{1}{2}(2\beta_{62}e_6 - 2\beta_{63}e_5 - 2\beta_{64}e_4 + 2\beta_{65}e_3 + 2\beta_{66}e_2 - 2\beta_{67}e_1 + 2\beta_{82}e_4 - 2\beta_{83}e_7 + 2\beta_{84}e_6 \\ & - 2\beta_{85}e_1 - 2\beta_{87}e_3 + 2\beta_{88}e_2). \end{aligned}$$

Similarly by applying D on all the remaining identities, we get for $D(e_4)$

$$\begin{aligned} D(e_4) =& -\frac{1}{2}(-2\beta_{22}e_4 + 2\beta_{23}e_7 - 2\beta_{24}e_6 + 2\beta_{25}e_1 + 2\beta_{27}e_3 - 2\beta_{28}e_2 + 2\beta_{63}e_3 - 2\beta_{64}e_2 + 2\beta_{65}e_5 \\ & - 2\beta_{66}e_4 - 2\beta_{67}e_7 + 2\beta_{68}e_6) \\ =& -\frac{1}{2}(-2\beta_{32}e_7 - 2\beta_{33}e_4 + 2\beta_{34}e_5 + 2\beta_{35}e_2 - 2\beta_{36}e_3 + 2\beta_{38}e_1 - 2\beta_{72}e_3 + 2\beta_{74}e_1 + 2\beta_{75}e_6 \\ & + 2\beta_{76}e_7 - 2\beta_{77}e_4 - 2\beta_{78}e_5) \end{aligned}$$

$$= -\frac{1}{2}(2\beta_{42}e_6 - 2\beta_{43}e_5 - 2\beta_{44}e_4 + 2\beta_{45}e_3 + 2\beta_{46}e_2 - 2\beta_{47}e_1 + 2\beta_{82}e_2 - 2\beta_{83}e_1 + 2\beta_{85}e_7 - 2\beta_{86}e_6 + 2\beta_{87}e_5 - 2\beta_{88}e_4).$$

For $D(e_5)$,

$$\begin{aligned}D(e_5) =&\frac{1}{2}(2\beta_{22}e_5 + 2\beta_{23}e_6 + 2\beta_{24}e_7 - 2\beta_{26}e_1 - 2\beta_{27}e_2 - 2\beta_{28}e_3 + 2\beta_{53}e_3 - 2\beta_{54}e_2 + 2\beta_{55}e_5\\ &- 2\beta_{56}e_4 - 2\beta_{57}e_7 + 2\beta_{58}e_6)\\ =& -\frac{1}{2}(2\beta_{32}e_6 - 2\beta_{33}e_5 - 2\beta_{34}e_4 + 2\beta_{35}e_3 + 2\beta_{36}e_2 - 2\beta_{37}e_1 - 2\beta_{82}e_3 + 2\beta_{84}e_1 + 2\beta_{85}e_6\\ &+ 2\beta_{86}e_7 - 2\beta_{87}e_4 - 2\beta_{88}e_5)\\ =&\frac{1}{2}(-2\beta_{42}e_7 - 2\beta_{43}e_4 + 2\beta_{44}e_5 + 2\beta_{45}e_2 - 2\beta_{46}e_3 + 2\beta_{48}e_1 + 2\beta_{72}e_2 - 2\beta_{73}e_1 + 2\beta_{75}e_7\\ &- 2\beta_{76}e_6 + 2\beta_{77}e_5 - 2\beta_{78}e_4).\end{aligned}$$

For $D(e_6)$,

$$\begin{aligned}D(e_6) =&\frac{1}{2}(2\beta_{22}e_6 - 2\beta_{23}e_5 - 2\beta_{24}e_4 + 2\beta_{25}e_3 + 2\beta_{26}e_2 - 2\beta_{27}e_1 + 2\beta_{83}e_3 - 2\beta_{84}e_2 + 2\beta_{85}e_5\\ &- 2\beta_{86}e_4 - 2\beta_{87}e_7 + 2\beta_{88}e_6)\\ =&\frac{1}{2}(2\beta_{32}e_5 + 2\beta_{33}e_6 + 2\beta_{34}e_7 - 2\beta_{36}e_1 - 2\beta_{37}e_2 - 2\beta_{38}e_3 - 2\beta_{52}e_3 + 2\beta_{54}e_1 + 2\beta_{55}e_6\\ &+ 2\beta_{56}e_7 - 2\beta_{57}e_4 - 2\beta_{58}e_5)\\ =& -\frac{1}{2}(-2\beta_{42}e_4 + 2\beta_{43}e_7 - 2\beta_{44}e_6 + 2\beta_{45}e_1 + 2\beta_{47}e_3 - 2\beta_{48}e_2 + 2\beta_{62}e_2 - 2\beta_{63}e_1 + 2\beta_{65}e_7\\ &- 2\beta_{66}e_6 + 2\beta_{67}e_5 - 2\beta_{68}e_4).\end{aligned}$$

For $D(e_7)$,

$$\begin{aligned}D(e_7) =& -\frac{1}{2}(-2\beta_{22}e_7 - 2\beta_{23}e_4 + 2\beta_{24}e_5 + 2\beta_{25}e_2 - 2\beta_{26}e_3 + 2\beta_{28}e_1 + 2\beta_{73}e_3 - 2\beta_{74}e_2 + 2\beta_{75}e_5\\ &- 2\beta_{76}e_4 - 2\beta_{77}e_7 + 2\beta_{78}e_6)\\ =&\frac{1}{2}(-2\beta_{32}e_4 + 2\beta_{33}e_7 - 2\beta_{34}e_6 + 2\beta_{35}e_1 + 2\beta_{37}e_3 - 2\beta_{38}e_2 - 2\beta_{62}e_3 + 2\beta_{64}e_1 + 2\beta_{65}e_6\\ &+ 2\beta_{66}e_7 - 2\beta_{67}e_4 - 2\beta_{68}e_5)\\ =&\frac{1}{2}(2\beta_{42}e_5 + 2\beta_{43}e_6 + 2\beta_{44}e_7 - 2\beta_{46}e_1 - 2\beta_{47}e_2 - 2\beta_{48}e_3 + 2\beta_{52}e_2 - 2\beta_{53}e_1 + 2\beta_{55}e_7\\ &- 2\beta_{56}e_6 + 2\beta_{57}e_5 - 2\beta_{58}e_4).\end{aligned}$$

By comparing the coefficients, we get $\beta_{i1} = 0$ for $2 \le i \le 8$, $\beta_{ij} = 0$ for $2 \le i, j \le 8$ with $i = j$ and $\beta_{ji} = -\beta_{ij}$ for $2 \le i, j \le 8$ with $i \ne j$. Specifically, $\beta_{45} = \beta_{27} - \beta_{36}$, $\beta_{46} = \beta_{28} + \beta_{35}$, $\beta_{47} = -\beta_{25} + \beta_{38}$, $\beta_{48} = -\beta_{26} - \beta_{37}$, $\beta_{67} = \beta_{23} + \beta_{58}$, $\beta_{68} = \beta_{24} - \beta_{57}$ and $\beta_{78} = \beta_{34} + \beta_{56}$. □

Theorem 2. *Let $x = \sum_{i=1}^{8} x_i e_{i-1} \in \mathbb{O}$. Let $D : \mathbb{O} \to \mathbb{O}$ be a Lie derivation. Then $D(x)$ can be written as* $D(x) = x_1\beta_{11}e_0 + (-x_3\beta_{23} - x_4\beta_{24} - x_5\beta_{25} - x_6\beta_{26} - x_7\beta_{27} - x_8\beta_{28})e_1 + (x_2\beta_{23} - x_4\beta_{34} - x_5\beta_{35} - x_6\beta_{36} - x_7\beta_{37} - x_8\beta_{38})e_2 + (x_2\beta_{24} + x_3\beta_{34} + x_5(-\beta_{27} + \beta_{36}) + x_6(-\beta_{28} - \beta_{35}) + x_7(\beta_{25} - \beta_{38}) + x_8(\beta_{26} + \beta_{37}))e_3 + (x_2\beta_{25} + x_3\beta_{35} + x_4(\beta_{27} - \beta_{36}) - x_6\beta_{56} - x_7\beta_{57} - x_8\beta_{58})e_4 + (x_2\beta_{26} + x_3\beta_{36} + x_4(\beta_{28} + \beta_{35}) + x_5\beta_{56} + x_7(-\beta_{23} - \beta_{58}) + x_8(-\beta_{24} + \beta_{57}))e_5 + (x_2\beta_{27} + x_3\beta_{37} + x_4(-\beta_{25} + \beta_{38}) + x_5\beta_{57} + x_6(\beta_{23} + \beta_{58}) + x_8(-\beta_{34} - \beta_{56}))e_6 + (x_2\beta_{28} + x_3\beta_{38} + x_4(-\beta_{26} - \beta_{37}) + x_5\beta_{58} + x_6(\beta_{24} - \beta_{57}) + x_7(\beta_{34} + \beta_{56}))e_7.$

Proof. Applying D on x gives

$$D(x) = \sum_{i=1}^{8} x_i D(e_{i-1}).$$

Substituting the values of $D(e_i)'s$, which is computed in matrix representation of D in above theorem yields

$$\begin{aligned}D(x) =& x_1\beta_{11}e_0 + x_2(\beta_{23}e_2 + \beta_{24}e_3 + \beta_{25}e_4 + \beta_{26}e_5 + \beta_{27}e_6 + \beta_{28}e_7) + x_3(-\beta_{23}e_1 \\ &+ \beta_{34}e_3 + \beta_{35}e_4 + \beta_{36}e_5 + \beta_{37}e_6 + \beta_{38}e_7) + x_4(-\beta_{24}e_1 - \beta_{34}e_2 + (\beta_{27} - \beta_{36})e_4 \\ &+ (\beta_{28} + \beta_{35})e_5 + (-\beta_{25} + \beta_{38})e_6 + (-\beta_{26} - \beta_{37})e_7) + x_5(-\beta_{25}e_1 - \beta_{35}e_2 \\ &+ (-\beta_{27} + \beta_{36})e_3 + \beta_{56}e_5 + \beta_{57}e_6 + \beta_{58}e_7) + x_6(-\beta_{26}e_1 - \beta_{36}e_2 + (-\beta_{28} - \beta_{35})e_3 \\ &- \beta_{56}e_4 + (\beta_{23} + \beta_{58})e_6 + (\beta_{24} - \beta_{57})e_7) + x_7(-\beta_{27}e_1 - \beta_{37}e_2 + (\beta_{25} - \beta_{38})e_3 \\ &- \beta_{57}e_4 + (-\beta_{23} - \beta_{58})e_5 + (\beta_{34} + \beta_{56})e_7) + x_8(-\beta_{28}e_1 - \beta_{38}e_2 + (\beta_{26} + \beta_{37})e_3 \\ &- \beta_{58}e_4 + (-\beta_{24} + \beta_{57})e_5 + (-\beta_{34} - \beta_{56})e_6).\end{aligned}$$

Summarizing the above expression yields our required result. □

4. Characterizing Lie Derivation of Octonion Algebra $\mathbb{O}$

Our next task is to present characterization of Lie derivations of the algebra of Octonion. In Theorem 2.2 of [21], it is shown that if S be a 2-torsion free ring and $R = H(S)$ be quaternion ring, then every Lie derivation of R can be decomposed in terms of Jordan derivation and Lie derivation of S and an inner derivation of R, for every element $t \in R$. Here, we have:

Theorem 3. *Let $D : \mathbb{O} \to \mathbb{O}$ be a Lie derivation. Then there exist an element A in $\mathbb{O}$, a Lie derivation δ and a Jordan derivation ψ on L such that*

$$D(t) = \delta(x_1)e_0 + \sum_{i=2}^{8} \psi(x_i)e_{i-1} + I_A(t)$$

for every element $t = \sum_{i=1}^{8} x_i e_{i-1} \in \mathbb{O}$.

Proof. Since D is an additive map, we can write

$$D(e_{i-1}) = \sum_{j=1}^{8} \beta_{ij} e_{j-1}, \qquad 1 \le i \le 8. \tag{2}$$

for some $\beta'_{ij}s \in L$. It can be easily seen that $\beta_{11} \in Z(L)$. Next, we will fine $D(se_i)'s$ with $i = 0, 1, ..., 7$, for arbitrary $l \in L$. Set $D(le_1) = \sum_{i=1}^{8} x_i e_{i-1}$. Applying D on $[le_1, e_1]$, we get

$$\begin{aligned}0 = D[le_1, e_1] =& -2x_3e_3 + 2x_4e_2 - 2x_5e_5 + 2x_6e_4 + 2x_7e_7 - 2x_8e_6 + (l \circ \beta_{23})e_3 - (l \circ \beta_{24})e_2 \\ &+ (l \circ \beta_{25})e_5 - (l \circ \beta_{26})e_4 - (l \circ \beta_{27})e_7 + (l \circ \beta_{28})e_6.\end{aligned}$$

By comparing the coefficients, we get

$$\begin{aligned}x_3 =& \frac{1}{2}(l \circ \beta_{23}), \quad x_4 = \frac{1}{2}(l \circ \beta_{24}), \quad x_5 = \frac{1}{2}(l \circ \beta_{25}), \\ x_6 =& \frac{1}{2}(l \circ \beta_{26}), \quad x_7 = \frac{1}{2}(l \circ \beta_{27}), \quad x_8 = \frac{1}{2}(l \circ \beta_{28}),\end{aligned}$$

which implies

$$D(le_1) = x_1e_0 + x_2e_1 + \frac{1}{2}(l \circ \beta_{23})e_2 + \frac{1}{2}(l \circ \beta_{24}e_3 + \frac{1}{2}(l \circ \beta_{25})e_4 + \frac{1}{2}(l \circ \beta_{26})e_5 + \frac{1}{2}(l \circ \beta_{27})e_6 + \frac{1}{2}(l \circ \beta_{28})e_7.$$

Now, applying D on the identities $le_3 = \frac{1}{2}[le_1, e_2]$, $le_2 = -\frac{1}{2}[le_1, e_3]$, $le_1 = \frac{1}{2}[le_2, e_3]$, $le_4 = -\frac{1}{2}[le_1, e_5]$, $le_5 = \frac{1}{2}[le_1, e_4]$, $le_6 = \frac{1}{2}[le_1, e_7]$, $le_7 = -\frac{1}{2}[le_1, e_6]$ and putting $x_2 = \psi(l)$ where $\psi : L \to L$ is an additive map which is uniquely determined by D, we get

$$\begin{aligned}
D(le_1) =& \frac{1}{2}I_{\beta_{34}}(l)e_0 + \psi(l)e_1 + \frac{1}{2}(l \circ \beta_{23})e_2 + \frac{1}{2}(l \circ \beta_{24}e_3 + \frac{1}{2}(l \circ \beta_{25})e_4 \\
&+ \frac{1}{2}(l \circ \beta_{26})e_5 + \frac{1}{2}(l \circ \beta_{27})e_6 + \frac{1}{2}(l \circ \beta_{28})e_7. \\
D(le_2) =& -\frac{1}{2}I_{\beta_{24}}(l)e_0 - \frac{1}{2}(l \circ \beta_{23})e_1 + \psi(l)e_2 + \frac{1}{2}(l \circ \beta_{34})e_3 + \frac{1}{2}(l \circ \beta_{35})e_4 \\
&+ \frac{1}{2}(l \circ \beta_{36})e_5 + \frac{1}{2}(l \circ \beta_{37})e_6 + \frac{1}{2}(l \circ \beta_{38})e_7. \\
D(le_3) =& \frac{1}{2}I_{\beta_{23}}(l)e_0 - \frac{1}{2}(l \circ \beta_{24})e_1 - \frac{1}{2}(l \circ \beta_{34})e_2 + \psi(l)e_3 + \frac{1}{2}(l \circ (\beta_{27} - \beta_{36}))e_4 \\
&+ \frac{1}{2}(l \circ (\beta_{28} + \beta_{35}))e_5 - \frac{1}{2}(l \circ (\beta_{25} - \beta_{38}))e_6 - \frac{1}{2}(l \circ (\beta_{26} + \beta_{37}))e_7 \\
D(le_4) =& -\frac{1}{2}I_{\beta_{26}}(l)e_0 - \frac{1}{2}(l \circ \beta_{25})e_1 - \frac{1}{2}(l \circ \beta_{25})e_2 - \frac{1}{2}(l \circ (\beta_{27} - \beta_{36}))e_3 \\
&+ \psi(l)e_4 + \frac{1}{2}(l \circ \beta_{56})e_5 + \frac{1}{2}(l \circ \beta_{57})e_6 + \frac{1}{2}(l \circ \beta_{58})e_7 \\
D(le_5) =& \frac{1}{2}I_{\beta_{25}}(l)e_0 - \frac{1}{2}(l \circ \beta_{26})e_1 - \frac{1}{2}(l \circ \beta_{36})e_2 - \frac{1}{2}(l \circ (\beta_{28} + \beta_{35}))e_3 \\
&- \frac{1}{2}(l \circ \beta_{56})e_4 + \psi(l)e_5 + \frac{1}{2}(l \circ (\beta_{23} + \beta_{58}))e_6 + \frac{1}{2}(l \circ (\beta_{24} - \beta_{57})) \\
D(le_6) =& \frac{1}{2}I_{\beta_{28}}(l)e_0 - \frac{1}{2}(l \circ \beta_{27})e_1 - \frac{1}{2}(l \circ \beta_{37})e_2 + \frac{1}{2}(l \circ (\beta_{25} - \beta_{38}))e_3 \\
&- \frac{1}{2}(l \circ \beta_{57})e_4 - \frac{1}{2}(l \circ (\beta_{23} + \beta_{58}))e_5 + \psi(l)e_6 + \frac{1}{2}(l \circ (\beta_{34} + \beta_{56})) \\
D(le_7) =& -\frac{1}{2}I_{\beta_{27}}(l)e_0 - \frac{1}{2}(l \circ \beta_{28})e_1 - \frac{1}{2}(l \circ \beta_{38})e_2 + \frac{1}{2}(l \circ (\beta_{26} + \beta_{37}))e_3 \\
&- \frac{1}{2}(l \circ \beta_{58})e_4 - \frac{1}{2}(l \circ (\beta_{24} - \beta_{57}))e_5 - \frac{1}{2}(l \circ (\beta_{34} + \beta_{56}))e_6 + \psi(l)e_7.
\end{aligned} \tag{3}$$

Next, let $l \in L$ be arbitrary and put $D(le_0) = D(l) = x_1e_0 + x_2e_1 + x_3e_2 + x_4e_3 + x_5e_4 + x_6e_5 + x_7e_6 + x_8e_7$. Applying D on $[le_0, e_1] = 0$ and using (2), we obtain

$$0 = D[le_0, e_1] = -2x_3e_3 + 2x_4e_2 - 2x_5e_4 + 2x_6e_4 + 2x_7e_7 - 2x_8e_6 + I_{\beta_{23}}(l)e_2 + I_{\beta_{24}}(l)e_3 + I_{\beta_{25}}(l)e_4 + I_{\beta_{26}}(l)e_5 + I_{\beta_{27}}(l)e_6 + I_{\beta_{28}}(l)e_7.$$

By comparing the coefficients, we get

$$x_3 = \frac{1}{2}I_{\beta_{24}}(l), x_4 = -\frac{1}{2}I_{\beta_{23}}(l), x_5 = \frac{1}{2}I_{\beta_{26}}(l), x_6 = -\frac{1}{2}I_{\beta_{25}}(l), x_7 = -\frac{1}{2}I_{\beta_{28}}(l), x_8 = \frac{1}{2}I_{\beta_{27}}(l).$$

Applying D on the identities $[le_0, e_i]$ where $i = 2, ..., 7$ and taking $x_1 = \delta(l)$ for some additive map $\delta : L \to L$ uniquely determined by D, we get

$$D(le_0) = \delta(l)e_0 - \frac{1}{2}I_{\beta_{34}}(l)e_1 + \frac{1}{2}I_{\beta_{24}}(l)e_2 - \frac{1}{2}I_{\beta_{23}}(l)e_3 + \frac{1}{2}I_{\beta_{26}}(l)e_4 \tag{4}$$

$$-\frac{1}{2}I_{\beta_{25}}(l)e_5-\frac{1}{2}I_{\beta_{28}}(l)e_6+\frac{1}{2}I_{\beta_{27}}(l)e_7$$

and

$$\begin{aligned}-I_{\beta_{34}}&=-I_{\beta_{56}}=I_{\beta_{34}+\beta_{56}}\\ I_{\beta_{24}}&=-I_{\beta_{57}}=-I_{\beta_{24}-\beta_{57}}\\ -I_{\beta_{23}}&=-I_{\beta_{58}}=I_{\beta_{23}+\beta_{58}}\\ I_{\beta_{26}}&=I_{\beta_{37}}=-I_{\beta_{26}+\beta_{37}}\\ -I_{\beta_{25}}&=I_{\beta_{38}}=-I_{-\beta_{25}+\beta_{38}}\\ -I_{\beta_{28}}&=-I_{\beta_{35}}=I_{\beta_{28}+\beta_{35}}\\ I_{\beta_{27}}&=-I_{\beta_{36}}=-I_{\beta_{27}-\beta_{36}}.\end{aligned}\tag{5}$$

Replacing l by $[l_1,l_2]$ in (4), for some $l_1,l_2\in L$, we infer that δ is a Lie derivation of L. Moreover, applying D on the identity $[l_1e_1,l_2e_2]=(l_1\circ l_2)e_3$ and using the foregoing calculations, we can see that ψ is a Jordan derivation. Now let $t=\sum_{i=1}^{8}x_ie_{i-1}\in\mathbb{O}$ be an arbitrary element. Using (3), (4) and (5), we find that

$$D(t)=\delta(x_1)e_0+\sum_{i=2}^{8}\psi(x_i)e_{i-1}+h(t)$$

where

$$\begin{aligned}h(t)=&\frac{e_0}{2}(I_{\beta_{34}}(x_2)-I_{\beta_{24}}(x_3)+I_{\beta_{23}}(x_4)-I_{\beta_{26}}(x_5)+I_{\beta_{25}}(x_6)+I_{\beta_{28}}(x_7)-I_{\beta_{27}}(x_8))\\ &+\frac{e_1}{2}(-I_{\beta_{34}}(x_1)-(x_3\circ\beta_{23})-(x_4\circ\beta_{24})-(x_5\circ\beta_{25})-(x_6\circ\beta_{26})-(x_7\circ\beta_{27})\\ &-(x_8\circ\beta_{28}))\\ &+\frac{e_2}{2}(I_{\beta_{24}}x_1+(x_2\circ\beta_{23})-(x_4\circ\beta_{34})-(x_5\circ\beta_{35})-(x_6\circ\beta_{36})-(x_7\circ\beta_{37})\\ &-(x_8\circ\beta_{38}))\\ &+\frac{e_3}{2}(-I_{\beta_{23}}x_1+(x_2\circ\beta_{24})+(x_3\circ\beta_{34})-(x_5\circ(\beta_{27}-\beta_{36}))-(x_6\circ(\beta_{28}+\beta_{35}))\\ &+(x_7\circ(\beta_{25}-\beta_{38}))+(x_8\circ(\beta_{26}+\beta_{37})))\\ &+\frac{e_4}{2}(I_{\beta_{26}}x_1+(x_2\circ\beta_{25})+(x_3\circ\beta_{35})+(x_4\circ(\beta_{27}-\beta_{36}))-(x_6\circ\beta_{56})-(x_7\circ\beta_{57})\\ &-(x_8\circ\beta_{58}))\\ &+\frac{e_5}{2}(-I_{\beta_{25}}(x_1)+(x_2\circ\beta_{26})+(x_3\circ\beta_{36})+(x_4\circ(\beta_{28}+\beta_{35}))+(x_5\circ\beta_{56})\\ &-(x_7\circ(\beta_{23}+\beta_{58}))-(x_8\circ(\beta_{24}-\beta_{57})))\\ &+\frac{e_6}{2}(-I_{\beta_{28}}x_1+(x_2\circ\beta_{27})+(x_3\circ\beta_{37})-(x_4\circ(\beta_{25}-\beta_{38}))+(x_5\circ\beta_{57})\\ &+(x_6\circ(\beta_{23}+\beta_{58}))-(x_8\circ(\beta_{34}+\beta_{56})))\\ &+\frac{e_7}{2}(I_{\beta_{27}}x_1+(x_2\circ\beta_{28})+(x_3\circ\beta_{38})-(x_4\circ(\beta_{26}+\beta_{37}))+(x_5\circ\beta_{58})\\ &+(x_6\circ(\beta_{24}-\beta_{57}))+(x_7\circ(\beta_{34}\circ+\beta_{56}))).\end{aligned}$$

It can be easily verified that $h(t)=I_A(t)$ where $t=\sum_{i=1}^{8}x_ie_{i-1}$ and

$$A=\frac{1}{2}(-\beta_{34}e_1+\beta_{24}e_2-\beta_{23}e_3+\beta_{26}e_4-\beta_{25}e_5-\beta_{28}e_6+\beta_{27}e_7).$$

Consequently,

$$D(t) = \delta(x_1)e_0 + \sum_{i=2}^{8} \psi(x_i)e_{i-1} + I_A(t).$$

□

5. Lie Centralizer of Octonion Algebra O

This section contains the characterization of Lie centralizer of octonion algebra. In Theorem 2.1 of [2], it is shown that if S is a 2-torsion free unital ring and $R = H(S)$ is quaternion ring. Then every Lie centralizer of R can be represented in terms of a Lie centralizer and Jordan centralizer of S. Here, we have:

Theorem 4. *Let $\xi : O \to O$ be a Lie centralizer. Then there exists a Lie centralizer α and a Jordan centralizer φ on L such that*

$$\xi(t) = \alpha(x_1)e_0 + \sum_{i=2}^{8} \varphi(x_i)e_{i-1}$$

for every element $t = \sum_{i=1}^{8} x_i e_{i-1} \in O$.

Proof. We have already assumed the form

$$\xi(e_{i-1}) = \sum_{j=1}^{8} \beta_{ij}e_{j-1}, \qquad 1 \le i \le 8$$

for some $\beta'_{ij}s \in L$. Since ξ is a Lie centralizer, we have

$$\begin{aligned}\xi(e_3) = \frac{1}{2}\xi[e_1, e_2] = \frac{1}{2}[\xi(e_1), e_2] =& \frac{1}{2}(2\beta_{22}e_3 - 2\beta_{24}e_1 - 2\beta_{25}e_6 - 2\beta_{26}e_7 + 2\beta_{27}e_4 + 2\beta_{28}e_5)\\ =& -\beta_{24}e_1 + \beta_{22}e_3 + \beta_{27}e_4 + \beta_{28}e_5 - \beta_{25}e_6 - \beta_{26}e_7.\end{aligned}$$

Furthermore,

$$\xi(e_1) = \frac{1}{2}\xi[e_2, e_3] = \frac{1}{2}[\xi(e_2), e_3] = \beta_{33}e_1 - \beta_{32}e_2 + \beta_{38}e_4 - \beta_{37}e_5 + \beta_{36}e_6 - \beta_{35}e_7.$$

By comparing the coefficients, we have $\beta_{21} = \beta_{24} = \beta_{42} = \beta_{41} = \beta_{43} = 0$, $\beta_{22} = \beta_{33} = \beta_{44}$, $\beta_{25} = \beta_{38} = -\beta_{47}$, $\beta_{26} = -\beta_{37} = -\beta_{48}$, $\beta_{27} = \beta_{36} = \beta_{45}$, $\beta_{28} = -\beta_{35} = \beta_{46}$, which reduces $\xi(e_1)$ and $\xi(e_3)$ to

$$\begin{aligned}\xi(e_1) =& \beta_{22}e_1 + \beta_{23}e_2 + \beta_{25}e_4 + \beta_{26}e_5 + \beta_{27}e_6 + \beta_{28}e_7\\ \xi(e_3) =& \beta_{22}e_3 + \beta_{27}e_4 + \beta_{28}e_5 - \beta_{25}e_6 - \beta_{26}e_7.\end{aligned}$$

Applying ξ on the identities $e_2 = \frac{1}{2}[e_3, e_1]$, $e_4 = \frac{1}{2}[e_5, e_1]$, $e_5 = \frac{1}{2}[e_1, e_4]$, $e_6 = \frac{1}{2}[e_1, e_7]$, $e_7 = \frac{1}{2}[e_6, e_1]$, we get

$$\begin{aligned}\xi(e_2) =& \beta_{22}e_2 + \beta_{28}e_4 - \beta_{27}e_5 + \beta_{26}e_6 - \beta_{25}e_7\\ \xi(e_4) =& \beta_{28}e_2 + \beta_{27}e_3 + \beta_{22}e_4 + \beta_{23}e_7\\ \xi(e_5) =& -\beta_{26}e_1 - \beta_{27}e_2 + \beta_{28}e_3 + \beta_{22}e_5 + \beta_{23}e_6\\ \xi(e_6) =& -\beta_{27}e_1 + \beta_{26}e_2 + \beta_{25}e_3 - \beta_{23}e_5 + \beta_{22}e_6\\ \xi(e_7) =& \beta_{25}e_2 - \beta_{26}e_3 - \beta_{23}e_4 + \beta_{22}e_7.\end{aligned}$$

Now assume that, $\xi(e_0) = t = \sum_{i=1}^{8} x_i e_{i-1}$. We have $0 = \xi[e_0, t] = te_0 - e_0 t = -2x_3e_3 + 2x_4e_2 - 2x_5e_5 + 2x_6e_4 + 2x_7e_7 - 2x_8e_6$, which implies $x_3 = x_4 = x_5 = x_6 = x_7 = x_8 = 0$. Application of ξ on the identity $[e_0, e_2]$ gives $x_2 = 0$, which implies $\xi(e_0) = x_1e_0 = x_1 \in L$. Let $s \in L$ be an arbitrary then $0 = \xi[e_0, se_1] = [\xi(e_0), se_1] = (x_1s - sx_1)$. From this we get, $x_1 = \xi(e_0) \in Z(L)$.
Let $l \in L$ and set $\xi(le_1) = \sum_{i=1}^{8} x_i e_{i-1}$. Applying ξ on $[le_1, e_1] = 0$, we get $x_3 = x_4 = x_5 = x_6 = x_7 = x_8 = 0$, which reduces $\xi(le_1)$ to $\xi(le_1) = x_1e_0 + x_2e_1$. Now applying ξ on the identities $le_3 = \frac{1}{2}[le_1, e_2]$, $le_2 = \frac{1}{2}[le_3, e_1]$, $le_1 = \frac{1}{2}[le_2, e_3]$, $le_5 = \frac{1}{2}[le_1, e_4]$, $le_4 = \frac{1}{2}[le_5, e_1]$, $le_6 = \frac{1}{2}[le_1, e_7]$, $le_7 = \frac{1}{2}[le_6, e_1]$ and taking $x_2 = \varphi(l)$, where $\varphi : L \to L$ is an additive map which is uniquely determined by ξ, we get

$$\begin{aligned} &\xi(le_1) = \varphi(l)e_1, \quad \xi(le_2) = \varphi(l)e_2, \quad \xi(le_3) = \varphi(l)e_3, \\ &\xi(le_4) = \varphi(l)e_4, \quad \xi(le_5) = \varphi(l)e_5, \quad \xi(le_6) = \varphi(l)e_6, \\ &\xi(le_7) = \varphi(l)e_7 \end{aligned} \tag{6}$$

Our next goal is to calculate $\xi(le_0)$ for arbitrary $l \in L$. Set $\xi(l) = \sum_{i=1}^{8} x_i e_{i-1}$. Applying ξ on $[l, e_1] = 0$ and $[l, e_2] = 0$ and putting $x_1 = \alpha(l)$, where $\alpha : L \to L$ is an additive map which is uniquely determined by ξ, we get

$$\xi(l) = \alpha(l). \tag{7}$$

Since ξ is a Lie centralizer, (7) implies that α is a Lie centralizer on L. Let $l_1, l_2 \in L$. Applying ξ on the identity $[l_1e_1, l_2e_2] = (l_1 \circ l_2)e_3$ and using (6) and (7), we get $\varphi(l_1 \circ l_2) = \varphi(l_1) \circ l_2$ shows that φ is a Jordan centralizer on L. Now let $t = \sum_{i=1}^{8} x_i e_{i-1}$ be an arbitrary element in L. By (6) and (7), we get $\xi(t) = \alpha(x_1)e_0 + \sum_{i=2}^{8} \varphi(x_i)e_{i-1}$, which completes the proof. □

6. Generalized Lie Derivation of Octonion Algebra $\mathcal{O}$

Generalized derivation is an extension of natural derivation. It has many applications in the literature since it is quite helpful in the geometric classification of rings and algebras. In this section, we compute the matrix representation of generalized Lie derivation of the octonion algebra. Let $F : \mathcal{O} \to \mathcal{O}$ be a generalized Lie derivation. F admits a matrix representation with respect to the basis, which is an 8×8 matrix $[F] = (\gamma_{ij})^T$ whose entries are defined by

$$F(e_{i-1}) = \sum_{j=1}^{8} \gamma_{ij} e_{j-1}, \qquad 1 \le i \le 8. \tag{8}$$

Each column of $[F]$ is an element of $\mathcal{O}$.

Theorem 5. *Let $F : \mathcal{O} \to \mathcal{O}$ be a generalized Lie derivation of $\mathcal{O}$ and $\mathcal{B}$ be the basis of $\mathcal{O}$. Then the matrix representation of F is as follows*

$$[F] = \begin{pmatrix} \gamma_{11} - \beta_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_{22} \end{pmatrix}$$

$$+\begin{pmatrix} \beta_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta_{23} & -\beta_{24} & -\beta_{25} & -\beta_{26} & -\beta_{27} & -\beta_{28} \\ 0 & \beta_{23} & 0 & -\beta_{34} & -\beta_{35} & -\beta_{36} & -\beta_{37} & -\beta_{38} \\ 0 & \beta_{24} & \beta_{34} & 0 & -\beta_{27}+\beta_{36} & -\beta_{28}-\beta_{35} & \beta_{25}-\beta_{38} & \beta_{26}+\beta_{37} \\ 0 & \beta_{25} & \beta_{35} & \beta_{27}-\beta_{36} & 0 & -\beta_{56} & -\beta_{57} & -\beta_{58} \\ 0 & \beta_{26} & \beta_{36} & \beta_{28}+\beta_{35} & \beta_{56} & 0 & -\beta_{23}-\beta_{58} & -\beta_{24}+\beta_{57} \\ 0 & \beta_{27} & \beta_{37} & -\beta_{25}+\beta_{38} & \beta_{57} & \beta_{23}+\beta_{58} & 0 & -\beta_{34}-\beta_{56} \\ 0 & \beta_{28} & \beta_{38} & -\beta_{26}-\beta_{37} & \beta_{58} & \beta_{24}-\beta_{57} & \beta_{34}+\beta_{56} & 0 \end{pmatrix}.$$

Proof. Let F be a generalized Lie derivation of O, then

$$F[e_i, e_j] = [F(e_i), e_j] + [e_i, D(e_j)] \tag{9}$$

where D is the derivation of the octonion algebra.
Put $i = 0$, then $F[e_0, e_j] = [F(e_0), e_j] + [e_0, D(e_j)] = 0$ for $1 \leq j \leq 7$. So, by using the equation(8), we get $\gamma_{1j} = 0$ for $2 \leq j \leq 7$.
Unlike the procedure of finding the Lie derivation of O, we don't need to verify equation (9) for the products between the octonionic units to compute the matrix representation of F. Hence generalized Lie derivation is much easier to compute once a Lie derivation is obtained.
Suppose that

$$F(e_{i-1}) = \sum_{j=1}^{8} \gamma_{ij} e_{j-1}, \qquad 1 \leq i \leq 8.$$

Applying F on the identity $[e_i, e_j]$ for $i = j$ and comparing the coefficients, we get $\gamma_{ij} = \beta_{ij}$ for $2 \leq i, j \leq 8$ with $i \neq j$.

By using the same technique proposed in the previous theorem, we get, for $F(e_1)$

$$\begin{aligned} F(e_1) =& \frac{1}{2}(-2\gamma_{32}e_2 + 2\gamma_{33}e_1 - 2\gamma_{35}e_7 + 2\gamma_{36}e_6 - 2\gamma_{37}e_5 + 2\gamma_{38}e_4 - 2\beta_{42}e_3 + 2\beta_{44}e_1 + 2\beta_{45}e_6 \\ &+ 2\beta_{46}e_7 - 2\beta_{47}e_4 - 2\beta_{48}e_5) \\ =& \frac{1}{2}(-2\gamma_{52}e_4 + 2\gamma_{53}e_7 - 2\gamma_{54}e_6 + 2\gamma_{55}e_1 + 2\gamma_{57}e_3 - 2\gamma_{58}e_2 - 2\beta_{62}e_5 - 2\beta_{63}e_6 - 2\beta_{64}e_7 \\ &+ 2\beta_{66}e_1 + 2\beta_{67}e_2 + 2\beta_{68}e_3) \\ =& -\frac{1}{2}(2\gamma_{72}e_6 - 2\gamma_{73}e_5 - 2\gamma_{74}e_4 + 2\gamma_{75}e_3 + 2\gamma_{76}e_2 - 2\gamma_{77}e_1 + 2\beta_{82}e_7 + 2\beta_{83}e_4 - 2\beta_{84}e_5 \\ &- 2\beta_{85}e_2 + 2\beta_{86}e_3 - 2\beta_{88}e_1). \end{aligned}$$

For $F(e_2)$

$$\begin{aligned} F(e_2) =& -\frac{1}{2}(-2\gamma_{22}e_2 + 2\gamma_{23}e_1 - 2\gamma_{25}e_7 + 2\gamma_{26}e_6 - 2\gamma_{27}e_5 + 2\gamma_{28}e_4 + 2\beta_{43}e_3 - 2\beta_{44}e_2 + 2\beta_{45}e_5 \\ &- 2\beta_{46}e_4 - 2\beta_{47}e_7 + 2\beta_{48}e_6) \\ =& \frac{1}{2}(-2\gamma_{52}e_7 - 2\gamma_{53}e_4 + 2\gamma_{54}e_5 + 2\gamma_{55}e_2 - 2\gamma_{56}e_3 + 2\gamma_{58}e_1 - 2\beta_{72}e_5 - 2\beta_{73}e_6 - 2\beta_{74}e_7 \\ &+ 2\beta_{76}e_1 + 2\beta_{77}e_2 + 2\beta_{78}e_3) \\ =& \frac{1}{2}(2\gamma_{62}e_6 - 2\gamma_{63}e_5 - 2\gamma_{64}e_4 + 2\gamma_{65}e_3 + 2\gamma_{66}e_2 - 2\gamma_{67}e_1 + 2\beta_{82}e_4 - 2\beta_{83}e_7 + 2\beta_{84}e_6 \\ &- 2\beta_{85}e_1 - 2\beta_{87}e_3 + 2\beta_{88}e_2). \end{aligned}$$

For $F(e_3)$

$$F(e_3) = \frac{1}{2}(2\gamma_{22}e_3 - 2\gamma_{24}e_1 - 2\gamma_{25}e_6 - 2\gamma_{26}e_7 + 2\gamma_{27}e_4 + 2\gamma_{28}e_5 + 2\beta_{33}e_3 - 2\beta_{34}e_2 + 2\beta_{35}e_5$$

$$- 2\beta_{36}e_4 - 2\beta_{37}e_7 + 2\beta_{38}e_6)$$
$$= \frac{1}{2}(2\gamma_{52}e_6 - 2\gamma_{53}e_5 - 2\gamma_{54}e_4 + 2\gamma_{55}e_3 + 2\gamma_{56}e_2 - 2\gamma_{57}e_1 - 2\beta_{82}e_5 - 2\beta_{83}e_6 - 2\beta_{84}e_7 + 2\beta_{86}e_1 + 2\beta_{87}e_2 + 2\beta_{88}e_3)$$
$$= -\frac{1}{2}(-2\gamma_{62}e_7 - 2\gamma_{63}e_4 + 2\gamma_{64}e_5 + 2\gamma_{65}e_2 - 2\gamma_{66}e_3 + 2\gamma_{68}e_1 + 2\beta_{72}e_4 - 2\beta_{73}e_7 + 2\beta_{74}e_6 - 2\beta_{75}e_1 - 2\beta_{77}e_3 + 2\beta_{78}e_2).$$

For $F(e_4)$

$$F(e_4) = -\frac{1}{2}(-2\gamma_{22}e_4 + 2\gamma_{23}e_7 - 2\gamma_{24}e_6 + 2\gamma_{25}e_1 + 2\gamma_{27}e_3 - 2\gamma_{28}e_2 + 2\beta_{63}e_3 - 2\beta_{64}e_2 + 2\beta_{65}e_5 - 2\beta_{66}e_4 - 2\beta_{67}e_7 + 2\beta_{68}e_6)$$
$$= -\frac{1}{2}(-2\gamma_{32}e_7 - 2\gamma_{33}e_4 + 2\gamma_{34}e_5 + 2\gamma_{35}e_2 - 2\gamma_{36}e_3 + 2\gamma_{38}e_1 - 2\beta_{72}e_3 + 2\beta_{74}e_1 + 2\beta_{75}e_6 + 2\beta_{76}e_7 - 2\beta_{77}e_4 - 2\beta_{78}e_5)$$
$$= -\frac{1}{2}(2\gamma_{42}e_6 - 2\gamma_{43}e_5 - 2\gamma_{44}e_4 + 2\gamma_{45}e_3 + 2\gamma_{46}e_2 - 2\gamma_{47}e_1 + 2\beta_{82}e_2 - 2\beta_{83}e_1 + 2\beta_{85}e_7 - 2\beta_{86}e_6 + 2\beta_{87}e_5 - 2\beta_{88}e_4).$$

For $F(e_5)$

$$F(e_5) = \frac{1}{2}(2\gamma_{22}e_5 + 2\gamma_{23}e_6 + 2\gamma_{24}e_7 - 2\gamma_{26}e_1 - 2\gamma_{27}e_2 - 2\gamma_{28}e_3 + 2\beta_{53}e_3 - 2\beta_{54}e_2 + 2\beta_{55}e_5 - 2\beta_{56}e_4 - 2\beta_{57}e_7 + 2\beta_{58}e_6)$$
$$= -\frac{1}{2}(2\gamma_{32}e_6 - 2\gamma_{33}e_5 - 2\gamma_{34}e_4 + 2\gamma_{35}e_3 + 2\gamma_{36}e_2 - 2\gamma_{37}e_1 - 2\beta_{82}e_3 + 2\beta_{84}e_1 + 2\beta_{85}e_6 + 2\beta_{86}e_7 - 2\beta_{87}e_4 - 2\beta_{88}e_5)$$
$$= \frac{1}{2}(-2\gamma_{42}e_7 - 2\gamma_{43}e_4 + 2\gamma_{44}e_5 + 2\gamma_{45}e_2 - 2\gamma_{46}e_3 + 2\gamma_{48}e_1 + 2\beta_{72}e_2 - 2\beta_{73}e_1 + 2\beta_{75}e_7 - 2\beta_{76}e_6 + 2\beta_{77}e_5 - 2\beta_{78}e_4).$$

For $F(e_6)$

$$F(e_6) = \frac{1}{2}(2\gamma_{22}e_6 - 2\gamma_{23}e_5 - 2\gamma_{24}e_4 + 2\gamma_{25}e_3 + 2\gamma_{26}e_2 - 2\gamma_{27}e_1 + 2\beta_{83}e_3 - 2\beta_{84}e_2 + 2\beta_{85}e_5 - 2\beta_{86}e_4 - 2\beta_{87}e_7 + 2\beta_{88}e_6)$$
$$= \frac{1}{2}(2\gamma_{32}e_5 + 2\gamma_{33}e_6 + 2\gamma_{34}e_7 - 2\gamma_{36}e_1 - 2\gamma_{37}e_2 - 2\gamma_{38}e_3 - 2\beta_{52}e_3 + 2\beta_{54}e_1 + 2\beta_{55}e_6 + 2\beta_{56}e_7 - 2\beta_{57}e_4 - 2\beta_{58}e_5)$$
$$= -\frac{1}{2}(-2\gamma_{42}e_4 + 2\gamma_{43}e_7 - 2\gamma_{44}e_6 + 2\gamma_{45}e_1 + 2\gamma_{47}e_3 - 2\gamma_{48}e_2 + 2\beta_{62}e_2 - 2\beta_{63}e_1 + 2\beta_{65}e_7 - 2\beta_{66}e_6 + 2\beta_{67}e_5 - 2\beta_{68}e_4).$$

For $F(e_7)$

$$F(e_7) = -\frac{1}{2}(-2\gamma_{22}e_7 - 2\gamma_{23}e_4 + 2\gamma_{24}e_5 + 2\gamma_{25}e_2 - 2\gamma_{26}e_3 + 2\gamma_{28}e_1 + 2\beta_{73}e_3 - 2\beta_{74}e_2 + 2\beta_{75}e_5 - 2\beta_{76}e_4 - 2\beta_{77}e_7 + 2\beta_{78}e_6)$$
$$= \frac{1}{2}(-2\gamma_{32}e_4 + 2\gamma_{33}e_7 - 2\gamma_{34}e_6 + 2\gamma_{35}e_1 + 2\gamma_{37}e_3 - 2\gamma_{38}e_2 - 2\beta_{62}e_3 + 2\beta_{64}e_1 + 2\beta_{65}e_6$$

$$+ 2\beta_{66}e_7 - 2\beta_{67}e_4 - 2\beta_{68}e_5)$$
$$= \frac{1}{2}(2\gamma_{42}e_5 + 2\gamma_{43}e_6 + 2\gamma_{44}e_7 - 2\gamma_{46}e_1 - 2\gamma_{47}e_2 - 2\gamma_{48}e_3 + 2\beta_{52}e_2 - 2\beta_{53}e_1 + 2\beta_{55}e_7 - 2\beta_{56}e_6 + 2\beta_{57}e_5 - 2\beta_{58}e_4).$$

By comparing the coefficients, we get $\gamma_{i1} = 0$ for $2 \leq i \leq 8$ and $\gamma_{22} = \gamma_{33} = \gamma_{44} = \gamma_{55} = \gamma_{66} = \gamma_{77} = \gamma_{88}$. □

Theorem 6. *Let $x = x_1e_0 + x_2e_1 + x_3e_2 + x_4e_3 + x_5e_4 + x_6e_5 + x_7e_6 + x_8e_7 \in \mathbb{O}$. Let $F : \mathbb{O} \to \mathbb{O}$ be a generalized Lie derivation with respect to D then $F(x)$ can be written as* $F(x) = x_1\gamma_{11}e_0 + (x_2\gamma_{22} - x_3\beta_{23} - x_4\beta_{24} - x_5\beta_{25} - x_6\beta_{26} - x_7\beta_{27} - x_8\beta_{28})e_1 + (x_2\beta_{23} + x_3\gamma_{22} - x_4\beta_{34} - x_5\beta_{35} - x_6\beta_{36} - x_7\beta_{37} - x_8\beta_{38})e_2 + (x_2\beta_{24} + x_3\beta_{34} + x_4\gamma_{22} + x_5(-\beta_{27} + \beta_{36}) + x_6(-\beta_{28} - \beta_{35}) + x_7(\beta_{25} - \beta_{38}) + x_8(\beta_{26} + \beta_{37}))e_3 + (x_2\beta_{25} + x_3\beta_{35} + x_4(\beta_{27} - \beta_{36}) + x_5\gamma_{22} - x_6\beta_{56} - x_7\beta_{57} - x_8\beta_{58})e_4 + (x_2\beta_{26} + x_3\beta_{36} + x_4(\beta_{28} + \beta_{35}) + x_5\beta_{56} + x_6\gamma_{22} + x_7(-\beta_{23} - \beta_{58}) + x_8(-\beta_{24} + \beta_{57}))e_5 + (x_2\beta_{27} + x_3\beta_{37} + x_4(-\beta_{25} + \beta_{38}) + x_5\beta_{57} + x_6(\beta_{23} + \beta_{58}) + x_7\gamma_{22} + x_8(-\beta_{34} - \beta_{56}))e_6 + (x_2\beta_{28} + x_3\beta_{38} + x_4(-\beta_{26} - \beta_{37}) + x_5\beta_{58} + x_6(\beta_{24} - \beta_{57}) + x_7(\beta_{34} + \beta_{56}) + x_8\gamma_{22})e_7.$

Proof. Applying F on x gives

$$F(x) = x_1F(e_0) + x_2F(e_1) + x_3F(e_2) + x_4F(e_3) + x_5F(e_4) + x_6F(e_5) + x_7F(e_6) + x_8F(e_7).$$

Substituting the values of $F(e_i)'s$, which is computed in matrix representation of F in above theorem yields

$$F(x) = x_1\gamma_{11}e_0 + x_2(\gamma_{22}e_1 + \beta_{23}e_2 + \beta_{24}e_3 + \beta_{25}e_4 + \beta_{26}e_5 + \beta_{27}e_6 + \beta_{28}e_7) + x_3(-\beta_{23}e_1 + \gamma_{22}e_2 + \beta_{34}e_3 + \beta_{35}e_4 + \beta_{36}e_5 + \beta_{37}e_6 + \beta_{38}e_7) + x_4(-\beta_{24}e_1 - \beta_{34}e_2 + \gamma_{22}e_3 + (\beta_{27} - \beta_{36})e_4 + (\beta_{28} + \beta_{35})e_5 + (-\beta_{25} + \beta_{38})e_6 + (-\beta_{26} - \beta_{37})e_7) + x_5(-\beta_{25}e_1 - \beta_{35}e_2 + (-\beta_{27} + \beta_{36})e_3 + \gamma_{22}e_4 + \beta_{56}e_5 + \beta_{57}e_6 + \beta_{58}e_7) + x_6(-\beta_{26}e_1 - \beta_{36}e_2 + (-\beta_{28} - \beta_{35})e_3 - \beta_{56}e_4 + \gamma_{22}e_5 + (\beta_{23} + \beta_{58})e_6 + (\beta_{24} - \beta_{57})e_7) + x_7(-\beta_{27}e_1 - \beta_{37}e_2 + (\beta_{25} - \beta_{38})e_3 - \beta_{57}e_4 + (-\beta_{23} - \beta_{58})e_5 + \gamma_{22}e_6 + (\beta_{34} + \beta_{56})e_7) + x_8(-\beta_{28}e_1 - \beta_{38}e_2 + (\beta_{26} + \beta_{37})e_3 - \beta_{58}e_4 + (-\beta_{24} + \beta_{57})e_5 + (-\beta_{34} - \beta_{56})e_6 + \gamma_{22}e_7).$$

Summarizing the above expression yields our required result. □

Example 1. *Let an arbitrary element $x = \sum_{i=1}^{8} x_ie_{i-1} \in \mathbb{O}$. Let $\beta_{23} = 1,\ \beta_{26} = 1,\ \beta_{37} = 1,\ \beta_{58} = 1$ and $\beta_{ij} = 0$ otherwise in Lie derivation of $\mathbb{O}$ then D will be*

$$D(x) = (-x_3 - x_6)e_1 + (x_2 - x_7)e_2 + 2x_8e_3 - x_8e_4 + (x_2 - 2x_7)e_5 + (x_3 + 2x_6)e_6 + (-2x_4 + x_5)e_7. \tag{10}$$

Select $\gamma_{11} = 1$ and $\gamma_{22} = 1$ in the generalized Lie derivation, then

$$F(x) = x_1e_0 + (x_2 - x_3 - x_6)e_1 + (x_2 + x_3 - x_7)e_2 + (x_4 + 2x_8)e_3 + (x_5 - x_8)e_4 + (x_2 + x_6 - 2x_7)e_5 + (x_3 + 2x_6 + x_7)e_6 + (-2x_4 + x_5 + x_8)e_7. \tag{11}$$

Put $i = 1,\ j = 7$ in $F[e_i, e_j] = [F(e_i), e_j] + [e_i, D(e_j)]$, we get

$$F[e_1, e_7] = [F(e_1), e_7] + [e_1, D(e_7)].$$

The left hand side of the above equation will be

$$F[e_1, e_7] = 2F(e_6) = -2e_2 - 4e_5 + 2e_6.$$

By using (11), we get $F(e_1) = e_1 + e_2 + e_5$ and using (10), we get $D(e_7) = 2e_3 - e_4$.
Then by direct calculation the right hand side will be $2e_6 - 4e_5 - 2e_2$.

7. Conclusion

Lie algebras of derivations and its variants explore the nature of given algebras. In this research article, we have described of matrix representation as well as the characterization of Lie derivation of Octonion algebra. The characterization of Lie centralizer of Octonion algebra is also presented. We have also computed the matrix representation of generalized Lie derivations of Octonion algebras.

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Conflict of Interest

The authors declare no conflict of interests.

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