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Article

On Tades of Disjoint Union of Some Graphs

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Abstract: Consider a total labeling ξ of a graph G. For every two different edges e and f of G, let $wt(e) \neq wt(f)$ where weight of $e = xy$ is defined as $wt(e) = |\xi(e) - \xi(x) - \xi(y)|$. Then ξ is called edge irregular total absolute difference k-labeling of G. Let k be the minimum integer for which there is a graph G with edge irregular total absolute difference labeling. This k is called the total absolute difference edge irregularity strength of the graph G, denoted $tades(G)$. We compute $tades$ of SC_n, disjoint union of grid and zigzag graph.

Keywords: total absolute difference edge irregularity strength, edge irregularity strength, zigzag graph, grid graph, staircase graph
Mathematics Subject Classification: 05C78

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1. Introduction and Motivation

Chartrand et al. [1] introduced irregularity strength and irregular assignments of a graph in 1988. The other kind of total labeling the total edge irregularity strength of a graph was studied by Baca et al. [2]: Consider a graph $G = (V, E)$. A labeling $\xi : V \bigcup E \rightarrow \{1, 2, \cdots, k\}$ is called an edge irregular total k-labeling if for every pair of distinct edges uv and xy, $\xi(u) + \xi(v) + \xi(uv) \neq \xi(x) + \xi(y) + \xi(xy)$. If a graph G admits an edge irregular total k-labeling and k is minimum then G is said to have a total edge irrgularity strength denoted by $tes(G)$. The results about the $tes(G)$ can be found in [3–10].

Ramalakshmi and Kathiresan introduced the total absolute difference edge irregularity strength of graphs to lower edge weights, using $tes(G)$ and graceful labeling. Consider a total labeling ξ of a graph G. For every two different edges e and f of G, let $wt(e) \neq wt(f)$ where weight of $e = xy$ is defined as $wt(e) = |\xi(e) - \xi(x) - \xi(y)|$. Then ξ is called edge irregular total absolute difference k-labeling of G. Let k be the minimum integer for which there is a graph G with edge irregular total absolute difference labeling. This k is called the total absolute difference edge irregularity strength of the graph G, denoted $tades(G)$.

Lourdusamy et al. [11] determined the total absolute difference edge irregular strength for snake related graphs, wheel related graphs, lotus inside the circle and double fan graph. Also, they obtained the $tades$ of T_p-tree related graphs [12]. Lourdusamy et al. [13] discussed the $tades$ of super subdivision of certain families of graphs and corona graphs. Also, they obtained the $tades$ of transformed tree and path related graphs [14]. Here, we discuss the $tades$ of staircase graph, disjoint union of zigzag and grid graphs.

Theorem 1. *[?] For a graph $G = (V, E)$, we have* $\left\lceil \frac{|E|}{2} \right\rceil \leq tades(G) \leq |E| + 1$.

2. Main Results

In this section, we compute the exact value of total absolute difference edge irregularity strength of staircase graph.

Theorem 2. *For SC_n, the total absolute difference edge irregularity strength is* $tades(SC_n) = \left\lceil \frac{n(n+3)}{2} \right\rceil$.

Proof. Let $k = \left\lceil \frac{n(n+3)}{2} \right\rceil$. Let $V(SC_n) = \{a_{r,s} : r = 0, 1, 0 \leq s \leq n\} \cup \{a_{r,s} : 2 \leq r \leq n, r-1 \leq s \leq n\}$ and $E(SC_n) = \{a_{r,s}a_{r+1,s} : r = 0, 0 \leq s \leq n\} \cup \{a_{r,s}a_{r+1,s} : 1 \leq r \leq n-1, r \leq s \leq n\} \cup \{a_{r,s}a_{r,s+1} : r = 0, 1, 0 \leq s \leq n-1\} \cup \{a_{r,s}a_{r,s+1} : 2 \leq r \leq n, r-1 \leq s \leq n-1\}$. Note that $|V(SC_n)| = \frac{1}{2}(n+1)(n+2) + n$ and $|E(SC_n)| = n(n+3)$. From Theorem 1.1, $tades(SC_n) \geq k$. To complete the proof we show that

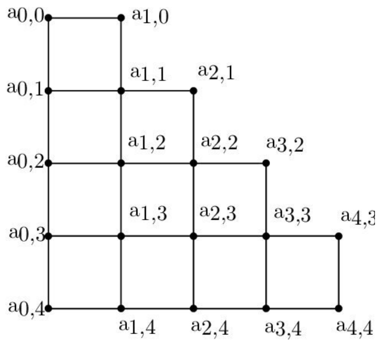

Figure 1. SC_4

$tades(SC_n) \leq k$. We define a, k-labeling $\xi : V(SC_n) \cup E(SC_n) \to \{1, 2, \cdots k\}$ as follows:

$\xi(a_{r,0}) = 1, r = 0, 1;$

For $1 \leq s \leq n$

$\xi(a_{0,s}) = \left\lceil \frac{s^2+3s}{2} \right\rceil - \left\lfloor \frac{s}{2} \right\rfloor;$

Case 1. s is odd

Let $1 \leq s \leq n$ and s is odd .

Fix $\xi(a_{1,s}) = \left\lceil \frac{s^2+3s}{2} \right\rceil - \left\lfloor \frac{s}{2} \right\rfloor$.

Let $2 \leq r \leq n,\ r-1 \leq s \leq n$ and s is odd.

$$\text{Fix } \xi(a_{r,s}) = \begin{cases} \left\lceil \frac{s^2+3s}{2} \right\rceil - \left\lfloor \frac{s}{2} \right\rfloor + \frac{r-1}{2} & \text{if } r \text{ is odd} \\ \left\lceil \frac{s^2+3s}{2} \right\rceil - \left\lfloor \frac{s}{2} \right\rfloor + \frac{r}{2} & \text{if } r \text{ is even ;} \end{cases}$$

Case 2. s is even

Let $1 \leq s \leq n$ and s is even.

Fix $\xi(a_{1,s}) = \left\lceil \frac{s^2+3s}{2} \right\rceil - \left\lfloor \frac{s}{2} \right\rfloor + 1$.

Let $2 \leq r \leq n,\ r-1 \leq s \leq n$ and s is even.

$$\text{Fix } \xi(a_{r,s}) = \begin{cases} \left\lceil \frac{s^2+3s}{2} \right\rceil - \left\lfloor \frac{s}{2} \right\rfloor + \frac{r+1}{2} & \text{if } r \text{ is odd} \\ \left\lceil \frac{s^2+3s}{2} \right\rceil - \left\lfloor \frac{s}{2} \right\rfloor + \frac{r}{2} & \text{if } r \text{ is even .} \end{cases}$$

We fix the edge labels as follows:

$\xi(a_{0,0}a_{1,0}) = 2;$

$\xi(a_{0,0}a_{0,1}) = 2;$

$\xi(a_{1,0}a_{1,1}) = 1;$

$\xi(a_{0,s}a_{1,s}) = 1$, for $1 \leq s \leq n$;

$\xi(a_{r,s}a_{r+1,s}) = 1$, for $1 \leq r \leq n-1$ and $r \leq s \leq n$;

$\xi(a_{r,s}a_{r,s+1}) = 1$, for $r = 0, 1$ and $1 \leq s \leq n-1$;

$\xi(a_{r,s}a_{r,s+1}) = 1$, for $2 \leq r \leq n$ and $r-1 \leq s \leq n-1$.

We then have the weight of the edges as follows:

$wt(a_{0,0}a_{1,0}) = 0;$
$wt(a_{0,0}a_{0,1}) = 1;$
$wt(a_{1,0}a_{1,1}) = 2;$
$wt(a_{0,s}a_{1,s}) = s^2 + 2s$ for $1 \leq s \leq n$;
$wt(a_{r,s}a_{r+1,s}) = s^2 + 2s + r$, for $1 \leq r \leq n-1$ and $r \leq s \leq n$;
$wt(a_{r,s}a_{r,s+1}) = s^2 + 3s + 1$, for $r = 0, 1$ and $1 \leq s \leq n-1$;
$wt(a_{r,s}a_{r,s+1}) = s^2 + 3s + r + 1$, for $2 \leq r \leq n$ and $r-1 \leq s \leq n-1$.

Hence ξ is total absolute difference edge irregular k-labeling with $k = \left\lceil \frac{n(n+3)}{2} \right\rceil$ as the weights for the edges are different. □

3. Disjoint Union of Zigzag Graph

In this section, we compute the exact value of total absolute difference edge irregularity strength of disjoint union of zigzag graphs $\bigcup_{j=1}^{p} Z_{n_j}^{m_j}$ with $n_j \geq 2$ and $m_j \geq 2$.

Theorem 3. *For any integer $n_j \geq 2$, $m_j \geq 2$, $tades(\bigcup_{j=1}^{p} Z_{n_j}^{m_j}) = \left\lceil \frac{\sum_{j=1}^{p}(n_j-1)(2m_j-1)}{2} \right\rceil$.*

Proof. Let $k = \left\lceil \frac{\sum_{j=1}^{p}(n_j-1)(2m_j-1)}{2} \right\rceil$. The disjoint union $\bigcup_{j=1}^{p} Z_{n_j}^{m_j}$ of zigzag graphs Z_n^m is defined to be a graph with vertex set $V(\bigcup_{j=1}^{p} Z_{n_j}^{m_j}) = \{a_{i,s}^{j} : 1 \leq i \leq n_j, 1 \leq s \leq m_j, 1 \leq j \leq p\}$ and the edge set $E(\bigcup_{j=1}^{p} Z_{n_j}^{m_j}) = \{a_{i,s}^{j}a_{i+1,s}^{j} : 1 \leq i \leq n_j - 1, 1 \leq s \leq m_j, 1 \leq j \leq p\} \bigcup \{a_{i,s}^{j}a_{i-1,s+1}^{j} : 1 \leq i \leq n_j, 1 \leq s \leq m_j - 1, 1 \leq j \leq p\}$. The disjoint union of zigzag graphs $\bigcup_{j=1}^{p} Z_{n_j}^{m_j}$ has $\sum_{j=1}^{p} n_j m_j$ vertices and $\sum_{j=1}^{p}(n_j-1)(2m_j-1)$ edges. Based on Theorem 1.1, we have $tades(\bigcup_{j=1}^{p} Z_{n_j}^{m_j}) \geq \left\lceil \frac{\sum_{j=1}^{p}(n_j-1)(2m_j-1)}{2} \right\rceil$.

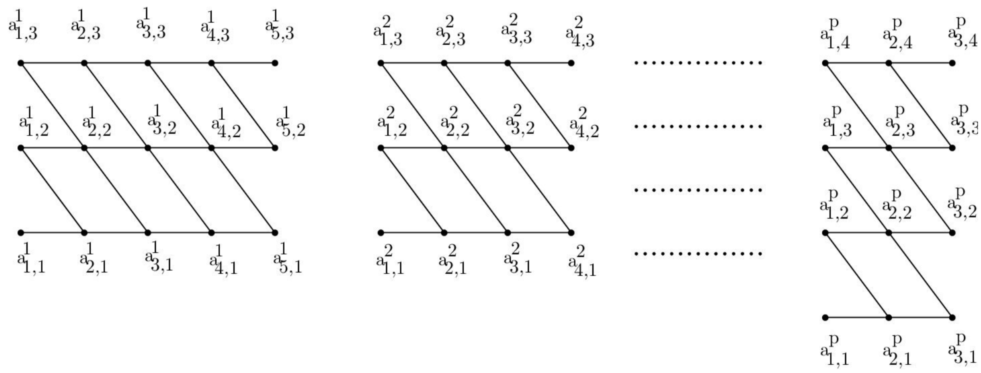

Figure 2. $Z_5^3 \bigcup Z_4^3 \bigcup \cdots \bigcup Z_3^4$

We define ξ as follows:

for $1 \leq i \leq n_j$, $1 \leq s \leq m_j$ and $1 \leq j \leq p-1$,

$\xi(a_{i,s}^{j}) = \left\lceil \frac{r}{2} \right\rceil + (s-1)(n_j-1) + \left\lfloor \frac{i+\frac{1}{2}((-1)^r+1)}{2} \right\rfloor$ where $r = \sum_{q=1}^{j-1}(n_q-1)(2m_q-1)$;

for $1 \leq i \leq n_p$, $1 \leq s \leq m_p - 1$ and $r = \sum_{q=1}^{p-1}(n_q-1)(2m_q-1)$,

$\xi(a_{i,s}^{p}) = \left\lceil \frac{r}{2} \right\rceil + (s-1)(n_p-1) + \left\lfloor \frac{i+\frac{1}{2}((-1)^r+1)}{2} \right\rfloor$;

$$\xi(a_{i,m_p}^{p}) = \begin{cases} \left\lceil \frac{r}{2} \right\rceil + (m_p-1)(n_p-1) + \left\lfloor \frac{i+\frac{1}{2}((-1)^r+1)}{2} \right\rfloor & \text{if } 1 \leq i \leq n_p - 1 \\ k & \text{if } i = n_p \text{ ;} \end{cases}$$

$\xi(a_{i,s}^{j}a_{i+1,s}^{j}) = 2$, for $1 \leq i \leq n_j - 1$, $1 \leq s \leq m_j$ and $1 \leq j \leq p-1$;
$\xi(a_{i,s}^{p}a_{i+1,s}^{p} = 2$, for $1 \leq i \leq n_p - 1$, $1 \leq s \leq m_p - 1$;
$\xi(a_{i,m_p}^{p}a_{i+1,m_p}^{p}) = 2$, for $1 \leq i \leq n_p - 2$;

$$\xi(a^p_{n_{p-1},m_p}a^p_{n_p,m_p}) = \begin{cases} 1 & \text{if } |E(\bigcup_{j=1}^{p} Z_{n_j}^{m_j})| \text{ is even} \\ 2 & \text{if } |E(\bigcup_{j=1}^{p} Z_{n_j}^{m_j})| \text{ is odd} \end{cases};$$

$\xi(a^j_{i-1,s+1}a^j_{i,s}) = 2,$ for $2 \leq i \leq n_j$, $1 \leq s \leq m_j - 1$ and $1 \leq j \leq p$.

We now arrive at the weight of the edges:

for $1 \leq i \leq n_j - 1$, $1 \leq s \leq m_j$, $1 \leq j \leq p$ and $r = \sum_{q=1}^{j-1}(n_q - 1)(2m_q - 1)$,

$$wt(a^j_{i,s}a^j_{i+1,s}) = 2\left\lceil \tfrac{r}{2} \right\rceil + 2(s-1)(n_j - 1) + i + \tfrac{1}{2}((-1)^r + 1) - 2;$$

for $1 \leq i \leq n_j$, $1 \leq s \leq m_j - 1$, $1 \leq j \leq p$ and $r = \sum_{q=1}^{j-1}(n_q - 1)(2m_q - 1)$,

$$wt(a^j_{i,s}a^j_{i-1,s+1}) = 2\left\lceil \tfrac{r}{2} \right\rceil + (2s-1)(n_j - 1) + i + \tfrac{1}{2}((-1)^r - 1) - 3.$$

It is clear that, the labels for vertices and edges receive values are not more than k. Also we see that the weights for the edges are all distinct. Hence $tades(\bigcup_{j=1}^{p} Z_{n_j}^{m_j}) = \left\lceil \frac{\sum_{j=1}^{p}(n_j-1)(2m_j-1)}{2} \right\rceil$. □

Illustration for *tades* of $Z_5^4 \bigcup Z_6^5 \bigcup Z_4^6 \bigcup Z_3^7$ is shown in Figure 3.

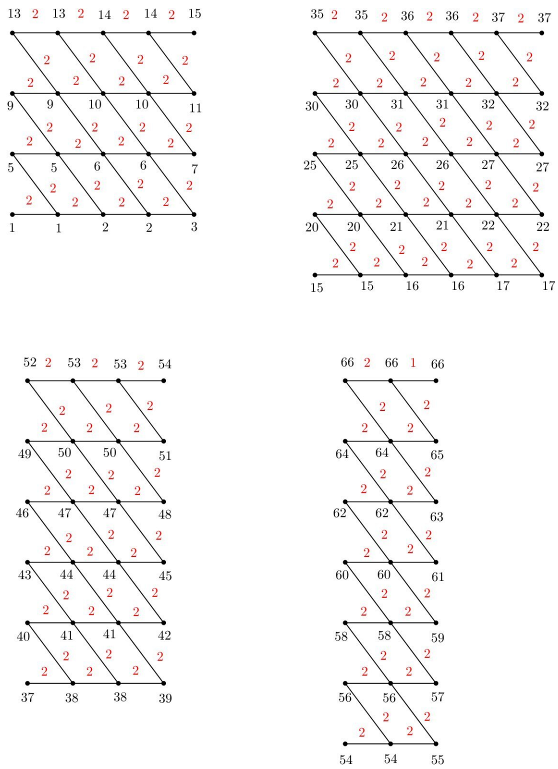

Figure 3. $Z_5^4 \bigcup Z_6^5 \bigcup Z_4^6 \bigcup Z_3^7$

4. Disjoint Union of Grid Graph

In this section, we compute the exact value of total absolute difference edge irregularity strength of disjoint union of grid graphs $\bigcup_{j=1}^{p} Z_{n_j}^{m_j}$ with $n_j, m_j \geq 2$.

Theorem 4. *For any integer $n_j, m_j \geq 2$ and $1 \leq j \leq p$, $tades(\bigcup_{j=1}^{p} G_{n_j,m_j}) = \left\lceil \frac{\sum_{j=1}^{p}(2n_jm_j-m_j-n_j)}{2} \right\rceil$.*

Proof. Let $k = \left\lceil \frac{\sum_{j=1}^{p}(2n_jm_j-m_j-n_j)}{2} \right\rceil$. We define disjoint union $\bigcup_{j=1}^{p} G_{n_j,m_j}$ of grid graphs $G_{n,m}$ as follows: Let $V(\bigcup_{j=1}^{p} G_{n_j,m_j}) = \{a^j_{i,s} : 1 \leq i \leq n_j, 1 \leq s \leq m_j, 1 \leq j \leq p\}$. Let $E(\bigcup_{j=1}^{p} G_{n_j,m_j}) = \{a^j_{i,s}, a^j_{i+1,s} : 1 \leq i \leq n_j - 1, 1 \leq s \leq m_j, 1 \leq j \leq p\} \bigcup \{a^j_{i,s}, a^j_{i,s+1} : 1 \leq i \leq n_j, 1 \leq s \leq m_{j-1}, 1 \leq j \leq p\}$. From Theorem 1.1, $tades(\bigcup_{j=1}^{p} G_{n_j,m_j}) \geq \left\lceil \sum_{j=1}^{p} \frac{(2n_jm_j-m_j-n_j)}{2} \right\rceil$. Now we prove the converse part.

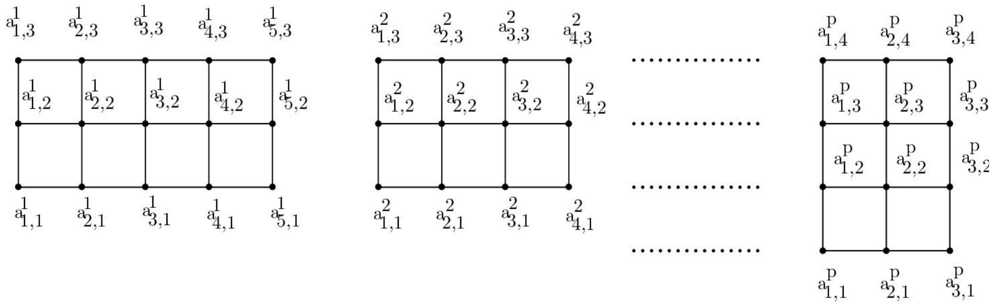

Figure 4. $G_{5,3} \bigcup G_{4,3} \bigcup \cdots \bigcup G_{3,4}$

Let us define $\xi : V(\bigcup_{j=1}^{p} G_{n_j,m_j}) \cup E(\bigcup_{j=1}^{p} G_{n_j,m_j}) \to \{1, 2, \cdots, \left\lceil \frac{\sum_{j=1}^{p}(2n_jm_j - m_j - n_j)}{2} \right\rceil\}$ as follows:

for $1 \le i \le n_j$, $1 \le s \le m_j$ and $1 \le j \le p-1$,

$$\xi(a^j_{i,s}) = \begin{cases} \left\lceil \frac{t}{2} \right\rceil + \frac{s-1}{2}(2n_j - 1) + \left\lfloor \frac{i + \frac{1}{2}((-1)^t + 1)}{2} \right\rfloor & \text{if } s \text{ is odd} \\ \left\lceil \frac{t}{2} \right\rceil + n_j(s-1) - \frac{s}{2} + \left\lceil \frac{i + \frac{1}{2}((-1)^t + 1)}{2} \right\rceil & \text{if } s \text{ is even ;} \end{cases}$$

where $t = \sum_{q=1}^{j-1} 2n_qm_q - m_q - n_q$

for $1 \le i \le n_p$, $1 \le s \le m_p - 1$ and $t = \sum_{q=1}^{p-1} 2n_qm_q - m_q - n_q$

$$\xi(a^p_{i,s}) = \begin{cases} \left\lceil \frac{t}{2} \right\rceil + \frac{s-1}{2}(2n_p - 1) + \left\lfloor \frac{i + \frac{1}{2}((-1)^t + 1)}{2} \right\rfloor & \text{if } s \text{ is odd} \\ \left\lceil \frac{t}{2} \right\rceil + n_p(s-1) - \frac{s}{2} + \left\lceil \frac{i + \frac{1}{2}((-1)^t + 1)}{2} \right\rceil & \text{if } s \text{ is even ;} \end{cases}$$

for $1 \le i \le n_p - 1$,

$$\xi(a^p_{i,m_p}) = \begin{cases} \left\lceil \frac{t}{2} \right\rceil + \frac{m_p-1}{2}(2n_p - 1) + \left\lfloor \frac{i + \frac{1}{2}((-1)^t + 1)}{2} \right\rfloor & \text{if } s \text{ is odd} \\ \left\lceil \frac{t}{2} \right\rceil + n_p(m_p-1) - \frac{m_p}{2} + \left\lceil \frac{i + \frac{1}{2}((-1)^t + 1)}{2} \right\rceil & \text{if } s \text{ is even ;} \end{cases}$$

$$\xi(a^p_{n_p,m_p}) = k;$$
$$\xi(a^j_{i,s}a^j_{i+1,s}) = 2, \text{ for } 1 \le i \le n_j - 1,\ 1 \le s \le m_j \text{ and } 1 \le j \le p-1;$$
$$\xi(a^p_{i,s}a^p_{i+1,s}) = 2, \text{ for } 1 \le i \le n_p - 1,\ 1 \le s \le m_p - 1;$$
$$\xi(a^p_{i,m_p}a^p_{i+1,m_p}) = 2, \text{ for } 1 \le i \le n_p - 2;$$
$$\xi(a^p_{n_p-1,a_p}a^p_{n_p,m_p}) = \begin{cases} 1 & \text{if } |E(\bigcup_{j=1}^{p} G_{n_j,m_j})| \text{ is even} \\ 2 & \text{if } |E(\bigcup_{j=1}^{p} G_{n_j,m_j})| \text{ is odd} \end{cases};$$
$$\xi(a^j_{i,s}a^j_{i,s+1}) = 2, \text{ for } 2 \le i \le n_j,\ 1 \le s \le m_j - 1 \text{ and } 1 \le j \le p.$$

Below we arrive at the weight of the edges. for $1 \le i \le n_j - 1$, $1 \le s \le m_j$, $1 \le j \le p$ and $t = \sum_{q=1}^{j-1} 2n_qm_q - m_q - n_q$,

$$wt(a^j_{i,s}a^j_{i+1,s}) = \begin{cases} 2\left\lceil \frac{t}{2} \right\rceil + (s-1)(2n_j - 1) + i + \frac{1}{2}((-1)^t + 1) - 2 & s \text{ is odd} \\ 2\left\lceil \frac{t}{2} \right\rceil + (s-1)2n_j - s + i + 1 + \frac{1}{2}((-1)^t + 1) - 2 & s \text{ is even;} \end{cases}$$

for $1 \le i \le n_j$, $1 \le s \le m_j - 1$, $1 \le j \le p$ and $t = \sum_{k=1}^{j-1} 2n_jm_j - m_j - n_j$,

$$wt(a^j_{i,s}a^j_{i,s+1}) = 2\left\lceil \frac{t}{2} \right\rceil + (2s-1)n_j - s + i + \frac{1}{2}((-1)^t + 1) - 2.$$

It is clear that, the labels for vertices and edges receive values are not more than k. Also we see that the weights for the edges are all distinct. Hence $tades(\bigcup_{j=1}^{p} G_{n_j,m_j}) = \left\lceil \frac{\sum_{j=1}^{p}(2n_jm_j - m_j - n_j)}{2} \right\rceil$. □

Illustration for *tades* of $G_{6,7} \bigcup G_{5,6} \bigcup G_{4,5}$ is shown in Figure 5.

Conflict of Interest

The author declares no conflict of interests.

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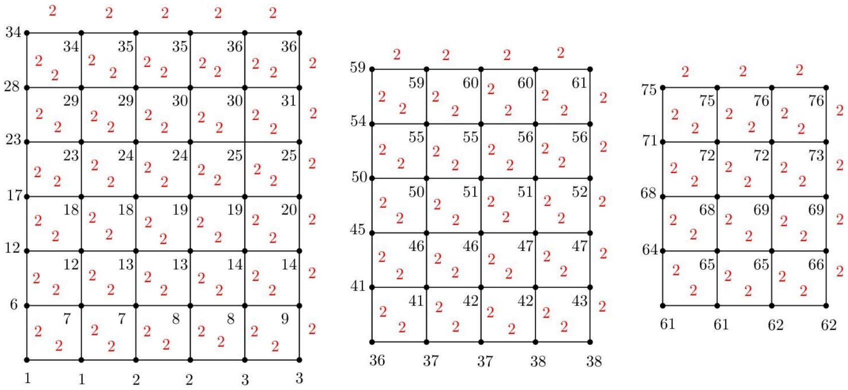

Figure 5. $G_{6,7} \bigcup G_{5,6} \bigcup G_{4,5}$

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