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Article

On a conjecture of the harmonic index with given minimum degree of graphs and short proof of Liu's result on Randić index

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Abstract: The harmonic index $H(G)$ of a graph G is defined as the sum of the weights $\frac{2}{d_u+d_v}$ of all edges uv of G, where d_u denotes the degree of a vertex u. Delorme et al. [1] (2002) put forward a conjecture concerning the minimum Randić index among all connected graphs with n vertices and the minimum degree at least k. Motivated by this paper, a conjecture related to the minimum harmonic index among all connected graphs with n vertices and the minimum degree at least k was posed in [2]. In this work, we show that the conjecture is true for a connected graph on n vertices with k vertices of degree $n-1$, and it is also true for a k-tree. Moreover, we give a shorter proof of Liu's result [3].

Keywords: Randić index, harmonic index, minimum degree, k-tree.
Mathematics Subject Classification: 05C07, 05C15.

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1. Introduction

Let $G = (V, E)$ denote a simple connected graph. The Randić (or connectivity) index $R(G)$ is defined in [4] by $R(G) = \sum\limits_{uv\in E(G)} \frac{1}{\sqrt{d_u d_v}}$, where d_u denotes the degree of a vertex u in G, and $\frac{1}{\sqrt{d_u d_v}}$ is called the weight of the edge uv in the Randić index. This index was extensively studied in mathematical chemistry. The harmonic index $H(G)$ is defined in [5] as $H(G) = \sum\limits_{uv\in E(G)} \frac{2}{d_u+d_v}$, where $\frac{2}{d_u+d_v}$ is called the weight of the edge uv in the harmonic index. In [6], the authors considered the relation between the harmonic index and the eigenvalues of graphs. In [7], the author presented the minimum and maximum values of harmonic index on simple connected graphs and trees, and characterized the corresponding extremal graphs. In [8], the authors gave a best possible lower bound for the harmonic index of a graph (a triangle-free graph, respectively) with minimum degree at least two and characterized the corresponding extremal graphs.

In 1968, Harary and Palmer [9] defined an n-plex as an n-dimensional complex in which every k-simplex with $k < n$ is contained in an n-complex. For convenience, 0-simplexes, 1-simplexes, and 2-simplexes are called points, lines, and cells respectively. The two-dimensional trees, also called 2-trees can now be defined inductively. The 2-plex with three points is a 2-tree, and a 2-tree with

$p+1$ points is obtained from a 2-tree with p points by adjoining a new point w adjoint to each of two adjacent points u and v together with the accompanying cell $\{u, v, w\}$. The definition of a k-tree for $k > 2$ is similar. In graph theory, a k-tree is a chordal graph all of whose maximal cliques are the same size $k+1$ and all of whose minimal clique separators are also all the same size k. Every k-tree may be formed by starting with a $(k+1)$-vertex complete graph and then repeatedly adding vertices in such a way that each added vertex has exactly k neighbors that form a clique (see [10]). The minimum degree of a k-tree is k. And a 1-tree is a tree in traditional graph theory.

Delorme et al. (2002) [1] put forward a conjecture concerning the minimum Randić index among all connected graphs with n vertices and the minimum degree at least k.

Conjecture 1. *([1]) For any connected graph with n vertices and the minimum degree at least k, $R(G) \geq \frac{k(n-k)}{\sqrt{k(n-1)}} + \frac{k(k-1)}{2(n-1)}$, the equality holds if and only if $G \cong K^*_{k,n-k}$, which arises from complete bipartite graph $K_{n,n-k}$ by joining each pair of vertices in the partite set with k vertices by a new edge.*

Using linear programming, Pavlovic [11] proved that Conjecture 1 holds when $k = \frac{n-1}{2}$ or $k = \frac{n}{2}$ (see also [12] for further results proved by quadratic programming). Liu [3] showed that Conjecture 1 is true given the graph contains k vertices of degree $n-1$ and it is true among k-trees. But, Aouchiche and Hansen [13] refuted the conjecture by using the AutoGraphiX 2 system, and modified the conjecture. Motivated by this paper, a conjecture related to the minimum harmonic index among all connected graphs with n vertices and the minimum degree at least k was also posed in [2].

Conjecture 2. *([2]) Let G be a graph with $n \geq 4$ vertices and the minimum degree $\delta(G) \geq k$, where $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$. Then $H(G) \geq H(K^*_{k,n-k})$ with equality if and only if $G = K^*_{k,n-k}$.*

Deng et al posed the following conjecture in [14].

Conjecture 3. *([14]) For any simple and connected graph G with n vertices and the minimum degree at least k, $H(G) \geq \frac{k(k-1)}{2(n-1)} + \frac{2k(n-k)}{n+k-1}$ with equality if and only if $G \cong K^*_{k,n-k}$.*

A counter-example to Conjecture 3 is the graph obtained from K_7 by deleting two independent edges.

Motivated by the paper [3] and [15], here, we will show that Conjecture 2 and Conjecture 3 is true for a connected graph containing k vertices of degree $n-1$ with n vertices and the minimum degree at least k, and it is also true for a k-tree. For additional results on this index, see [16–34].

2. Main Results

In the following, we first determine the minimum value of $H(G)$ of a graph on n vertices with k vertices of degree $n-1$, and show that Conjecture 2 and Conjecture 3 is true for a graph containing k vertices of degree $n-1$.

Lemma 1. *([8]) If e is an edge with maximum weight in G, then $H(G-e) < H(G)$.*

Theorem 1. *Let G be a graph with n vertices and k vertices of degree $n-1$. Then*

$$H(G) \geq \frac{k(k-1)}{2(n-1)} + \frac{2k(n-k)}{n+k-1}$$

with equality if and only if $G \cong K^*_{k,n-k}$.

Proof. It is easy to compute that $H(K^*_{k,n-k}) = \frac{k(k-1)}{2(n-1)} + \frac{2k(n-k)}{n+k-1}$.

Let G be the graph with the minimum harmonic index among all graphs with n vertices and k vertices of degree $n-1$. Suppose X denotes the set of k vertices of degree $n-1$ in G and $Y = V(G)-X$. Then the subgraph induced by X is a clique. If $G \not\cong K^*_{k,n-k}$, then the subgraph induced by Y is not an empty graph. Since the degree of every vertex in Y is at most $n-2$, an edge e with the maximum weight belongs to the subgraph induced by Y. By Lemma 1, $H(G-e) < H(G)$ and $G-e$ is also a graph with n vertices and k vertices of degree $n-1$, a contradiction. So, the subgraph induced by Y is an empty graph, and $G \cong K^*_{k,n-k}$. □

Theorem 1 shows that Conjecture 2 and Conjecture 3 is true for a connected graph on n vertices with k vertices of degree $n-1$, certainly, the minimum degree at least k.

Using linear programming, Liu [3] showed that Conjecture 1 is true for a connected graph with n vertices and k vertices of degree $n-1$. In the following, we give a shorter proof of this result.

Lemma 2. *([35]) Let e be an edge with maximum weight in a graph G. Then* $R(G-e) < R(G)$.

Theorem 2. *([3]) Let G be a graph with n vertices and k vertices of degree* $n-1$. *Then*

$$R(G) \geq \frac{k(n-k)}{\sqrt{k(n-1)}} + \frac{k(k-1)}{2(n-1)}$$

with equality if and only if $G \cong K^*_{k,n-k}$.

Proof. It is easy to compute that $R(K^*_{k,n-k}) = \frac{k(n-k)}{\sqrt{k(n-1)}} + \frac{k(k-1)}{2(n-1)}$.

Let G be the graph with the minimum Randić index among all graphs with n vertices and k vertices of degree $n-1$. X denotes the set of k vertices of degree $n-1$ in G and $Y = V(G) - X$. Then the subgraph induced by X is a clique. If $G \ncong K^*_{k,n-k}$, then the subgraph induced by Y is not an empty graph. Since the degree of every vertex in Y is at most $n-2$, an edge e with the maximum weight belongs to the subgraph induced by Y. By Lemma 2, $R(G-e) < R(G)$ and $G-e$ is also a graph with n vertices and k vertices of degree $n-1$, a contradiction. So, the subgraph induced by Y is an empty graph, and $G \cong K^*_{k,n-k}$. □

Now, we consider the harmonic index of a k-tree. The following lemma can be proved easily and it will be used in the back.

Lemma 3. *If* $x \geq k$, *then* $\frac{1}{x+d} - \frac{1}{x+d-1} \geq \frac{1}{k+d} - \frac{1}{k+d-1}$ *and* $\frac{1}{(x+d)^2} - \frac{1}{(x+d-2)^2} \geq \frac{1}{(k+d)^2} - \frac{1}{(k+d-2)^2}$.

Lemma 4. *Let G be a k-tree with n* ($n \geq k+1$) *vertices.* v_0 *is a vertex of degree k in G and its neighbors* $N(v_0) = \{v_1, v_2, \cdots, v_k\}$ *and* $d_{v_i} = d_i$ $(i = 1, 2, \ldots, k)$, *then*

$$H(G) - H(G - v_0) \geq f(d_1, d_2, \cdots, d_k).$$

where $f(d_1, d_2, \cdots, d_k) = \sum_{i=1}^{k} \left(\frac{2(d_i-k+1)}{k+d_i} - \frac{2(d_i-k)}{k+d_i-1}\right) + \sum_{1\leq i<j\leq k} \left(\frac{2}{d_i+d_j} - \frac{2}{d_i+d_j-2}\right)$. *Moreover,* $f(d_1, d_2, \cdots, d_k) \geq f(n-1, n-1, \cdots, n-1)$, *and* $H(G) - H(G-v_0) = f(n-1, n-1, \cdots, n-1)$ *if and only if* $G = K^*_{k,n-k}$.

Proof. By the definition of a k-tree, the subgraph induced by $N(v_0)$ is a clique. We have

$$\begin{aligned} H(G) - H(G-v_0) &= \sum_{i=1}^{k} \frac{2}{k+d_i} + \sum_{1\leq i<j\leq k} \left(\frac{2}{d_i+d_j} - \frac{2}{d_i+d_j-2}\right) \\ &+ \sum_{i=1}^{k} \sum_{x\in N(v_i)-\{v_0,v_1,\cdots,v_k\}} \left(\frac{2}{d_x+d_i} - \frac{2}{d_x+d_i-1}\right) \\ &\geq \sum_{i=1}^{k} \frac{2}{k+d_i} + \sum_{1\leq i<j\leq k} \left(\frac{2}{d_i+d_j} - \frac{2}{d_i+d_j-2}\right) \\ &+ \sum_{i=1}^{k} \left(\frac{2(d_i-k)}{k+d_i} - \frac{2(d_i-k)}{k+d_i-1}\right) \text{ (by Lemma 3 and } d_x \geq k) \\ &= \sum_{i=1}^{k} \left(\frac{2(d_i-k+1)}{k+d_i} - \frac{2(d_i-k)}{k+d_i-1}\right) + \sum_{1\leq i<j\leq k} \left(\frac{2}{d_i+d_j} - \frac{2}{d_i+d_j-2}\right) \end{aligned}$$

$$= f(d_1, d_2, \cdots, d_k).$$

Note that the partial derivatives

$$\begin{aligned}
&\frac{\partial f(d_1, d_2, \cdots, d_k)}{\partial d_i} \\
=&\frac{2k-1}{(k+d_i)^2} - \frac{(2k-1)}{(k+d_i-1)^2} - \sum_{1\le j\le k, j\ne i} \frac{1}{(d_i+d_j)^2} + \sum_{1\le j\le k, j\ne i} \frac{1}{(d_i+d_j-2)^2} \\
\le&\frac{2k-1}{(k+d_i)^2} - \frac{(2k-1)}{(k+d_i-1)^2} - \frac{k-1}{(d_i+k)^2} + \frac{k-1}{(d_i+k-2)^2} \text{ (by Lemma 3 and } d_j \ge k) \\
<&0
\end{aligned}$$

for $i = 1, 2, \cdots, k$, and $k \le d_i \le n-1$ ($i = 1, 2, \cdots, k$), we have $f(d_1, d_2, \cdots, d_k) \ge f(n-1, n-1, \cdots, n-1)$ if and only if equalities hold throughout the above inequalities, i.e., $d_i = n-1$ and $d_x = k$ for all $x \in N(v_i) - \{v_0, v_1, \cdots, v_k\}$ ($i = 1, 2, \cdots, k$), and G is $K^*_{k,n-k}$. □

Theorem 3. *Let G be a k-tree with n ($n \ge k+1$) vertices. Then $H(G) \ge \frac{k(k-1)}{2(n-1)} + \frac{2k(n-k)}{n+k-1}$ with equality if and only if $G \cong K^*_{k,n-k}$.*

Proof. We prove the result by induction on n. The result is true for $n = k+1$, since a k-tree G with $k+1$ is a complete graph K_{k+1} and $H(G) = \frac{k+1}{2}$. Assume that the result is true for any k-tree with $n = n_0$ ($n_0 \ge k+1$) vertices. Let G be a k-tree with $n_0 + 1$ vertices. From the structure of a k-tree, there is a vertex v_0 of degree k such that $G - v_0$ is a k-tree with n_0 vertices. Denote $N(v_0) = \{v_1, v_2, \cdots, v_k\}$ and $d_{v_i} = d_i$ ($i = 1, 2, \cdots, k$). By Lemma 4 and the inductive assumption, we have

$$\begin{aligned}
H(G) &\ge H(G - v_0) + f(n-1, n-1, \cdots, n-1) \\
&\ge H(K^*_{k,n-1-k}) + f(n-1, n-1, \cdots, n-1) \\
&= \frac{k(k-1)}{2(n-2)} + \frac{2k(n-1-k)}{n+k-2} + \sum_{i=1}^{k} \left(\frac{2(n-k)}{k+n-1} - \frac{2(n-1-k)}{k+n-2} \right) \\
&+ \sum_{1\le i<j\le k} \left(\frac{2}{2n-2} - \frac{2}{2n-4} \right) \\
&= \frac{k(k-1)}{2(n-1)} + \frac{2k(n-k)}{n+k-1}
\end{aligned}$$

with equality if and only if $G = K^*_{k,n-k}$. □

Theorem 3 shows that Conjecture 2 and Conjecture 3 is true for k-trees. Note that a 1-tree is a tree, we get the following result immediately.

Corollary 1. *([7]) Let T be a tree with n vertices. Then $H(T) \ge \frac{2(n-1)}{n}$ with equality if and only if $T \cong S_n$.*

Conflict of Interest

The authors declare no conflict of interests.

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