Combinatorial Press

*Article*

# Prefixes of Stanley's Catalan Paths with Odd Returns to the x-axis–Standard Version and Skew Catalan-Stanley Paths

**Helmut Prodinger**[1,2,*]

[1] Department of Mathematics, University of Stellenbosch 7602, Stellenbosch, South Africa

[2] NITheCS (National Institute for Theoretical and Computational Sciences), South Africa

**\* Correspondence:** hproding@sun.ac.za

**Abstract:** For $r = 1, 2, \ldots, 6$, we obtain generating functions $F_k^{(r)}(y)$ for words over the alphabet $[k]$, where $y$ tracks the number of parts and $[y^n]$ is the total number of distinct adjacent $r$-tuples in words with $n$ parts. In order to develop these generating functions for $1 \le r \le 3$, we make use of intuitive decompositions but for larger values of $r$, we switch to the cluster analysis method for decorated texts that was introduced by Bassino et al. Finally, we account for the coefficients of these generating functions in terms of Stirling set numbers. This is done by putting forward the full triangle of coefficients for all the sub-cases where $r = 5$ and $6$. This latter is shown to depend on both periodicity and number of letters used in the $r$-tuples.

**Keywords:** Catalan (Dyck) paths, Odd returns to x-axis, Skew paths, Prefix, Kernel method
**2010 Mathematics Subject Classification:** 05A15

---

## 1. Introduction

Dyck paths consist of up-steps $(1, 1)$ and down-steps $(1, -1)$, are not allowed to go below the x-axis and end at the x-axis. The enumeration (path consisting of $2n$ steps) is via the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$. Sometimes, paths that do not necessarily end at the x-axis are also called Dyck (Catalan) paths. The ultimate reference is Stanley's book [1]. This book offers in Exercise 26 an interesting alternative: Maximal sequences of down-steps, when the end on the x-axis must always have odd length. This restriction leads to smaller numbers of paths, but interestingly restricted paths consisting of $2n + 2$ steps are also enumerated by Catalan numbers $C_n$. The 5 allowed paths of length 8 are depicted in Figure 1:
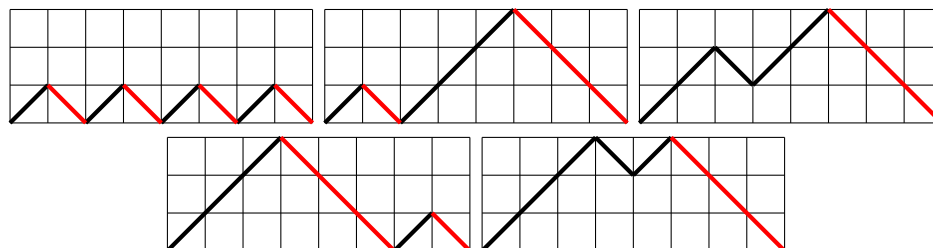


**Figure 1.** Restricted Paths à la Stanley of Length 8. The Maximal Runs of Down-steps to the x-axis are Depicted in Red

In the first of this paper we will enumerate prefixes of restricted paths à la Stanley: they can be completed to allowed paths. An alternative name of these combinatorial objects is *partial* Stanley-Dyck paths. We might also say that a maximal down-run the $x$-axis of *even* length has never occured in a partial Dyck path. The method of choice is generating functions (in $z$ marking the length) but also in $u$, marking the final level. The details will appear in the next section.

In the final section an additional restriction (skew Dyck paths) will be combined with the Stanley restriction: They allow an additional type of south-west step, but no overlap is allowed to occur. As in a previous publication [2] we prefer to draw a down-step $(1, -1)$ but depict it with the red color.

As in various examples discussed in the past [2–4] we will use the kernel method to set up appropriate generating functions. There are three bivariate generating functions $F(u), G(u), H(u)$ (also depending on $z$) related to the nature of the last step of the prefix of a Stanley-Dyck path. To solve the system, a 'bad' factor $u - r_2$ must be divided out from numerator and denominator, after which one can plug in $u = 0$ and identify the unknown quantities $f_1 = F(0)$, $h_0 = H(0)$. The function $r_2(z)$ generates Catalan numbers.

In the final section, instead of 3 functions, we have to deal with 5 functions, but the strategy is similar. The function that plays the role of $r_2$ now has coefficients 1, 3, 10, 36, 137, 543, 2219, 9285, 39587, 171369, 751236, which is sequence A2212 in [5]. The paper [4] contains a fair amount of combinatorics around this sequence.

## 2. Prefixes of Dyck Paths with Only Maximal Down-Runs to the $X$-axis of Odd Length

We present a graph that allows to identify all prefixes of Stanley-Dyck path. It has 3 layers of states, and maximal sequences of down-steps are depicted in blue. Parity plays a role: if such a sequence would lead to the $x$-axis after an even number of states, this is prohibited as such a state in the first layer is missing. Otherwise we introduce sequences $f_i, i \geq 1$, $g_i, h_i, i \geq 0$. These quantities all depend on the variable $z$ and describe generating functions of paths leading to a particular state. The state $h_0$ is special, it is related to Stanley-Dyck paths. We read off recursions and introduce bivariate
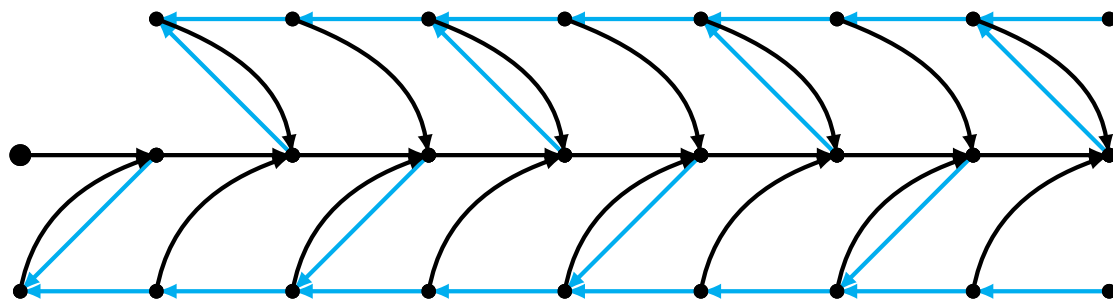


**Figure 2.** Three Layers of States, Labelled $F, g, h$, in that Order. The State $H_0$ is Responsible for Stanley-Dyck Paths, and all Others to Prefixes of them

generating functions

$$F(u) = \sum_{i \geq 1} f_i u^{i-1}, \quad G(u) = \sum_{i \geq 0} g_i u^i, \quad H(u) = \sum_{i \geq 0} h_i u^i.$$

The recursions are according to the last symbol read. Firstly,

$$f_i = z f_{i+1} + [i \text{ odd}] z g_{i+1}, \ i \geq 1, \quad \text{and}$$

$$\sum_{i \geq 1} f_i u^i = \sum_{i \geq 1} z f_{i+1} u^i + \sum_{i \geq 1} [i \text{ odd}] z g_{i+1} u^i,$$

$$u F(u) = z F(u) - z f_1 + \frac{z}{u} \sum_{i \geq 0} [i \text{ even}] g_i u^i - \frac{z}{u},$$

$$uF(u) = zF(u) - zf_1 + \frac{z}{2u}G(u) + \frac{z}{2u}G(-u) - \frac{z}{u}.$$

Secondly,

$$h_i = zh_{i+1} + [i \text{ even}]zg_{i+1}, \ i \geq 0, \quad \text{and}$$

$$H(u) = \frac{z}{u}H(u) - \frac{z}{u}h_0 + \frac{z}{u}\sum_{i\geq 1}[i \text{ odd}]g_i u^i,$$

$$H(u) = \frac{z}{u}H(u) - \frac{z}{u}h_0 + \frac{z}{2u}G(u) - \frac{z}{2u}G(-u).$$

Thirdly

$$g_{i+1} = zf_i + zg_i + zh_i, \ i \geq 1, \ g_1 = z + zh_0, \ g_0 = 1, \quad \text{and}$$

$$\sum_{i\geq 1}g_{i+1}u^i = \sum_{i\geq 1}zf_i u^i + \sum_{i\geq 1}zg_i u^i + \sum_{i\geq 1}zh_i u^i, \ g_1 = z + zh_0, \ g_0 = 1,$$

$$\frac{1}{u}G(u) - \frac{1}{u} = zuF(u) + zG(u) + zH(u).$$

The 3 equations can be solved: We start with the simplest function

$$G(u) = \frac{z^2u^2f_1 + z^2uh_0 - u + z^2u + z}{z - u + zu^2};$$

to get a form that leads to further developments, we factorize the denominator:

$$z - u + zu^2 = z(u - r_1)(u - r_2), \quad r_1 = \frac{1 + \sqrt{1 - 4z^2}}{2z}, \quad r_2 = \frac{1 - \sqrt{1 - 4z^2}}{2z}.$$

Dividing out the factor $u - r_2$ from numerator and denominator, we find[*]

$$G(u) = \frac{r_2z^2f_1 + z^2h_0 - 1 + z^2 + z^2uf_1}{r_2z - 1 + zu}, \quad \text{and} \quad G(0) = 1 = \frac{r_2z^2f_1 + z^2h_0 - 1 + z^2}{r_2z - 1}.$$

Using this for $G(u)$ and $G(-u)$, we get

$$F(u) = \frac{\Xi}{2u(z - u + zu^2)(u - z)}$$

with $\Xi = z(-2zf_1u^3 + z^2u^2f_1 + zu^2G(-u) - 2zu^2 + 2u^2f_1 + z^2uh_0 + z^2u - 2zf_1u - uG(-u) + u + zG(-u) - z)$.
Dividing out the factor $u(u - z)(u - r_2)$ we find an attractive answer

$$F(u) = \frac{z^3(1 + uf_1)}{1 - z^2 - zr_2 - z^2u^2}, \quad F(0) = f_1 = \frac{z^3}{1 - z^2 - zr_2} = \frac{1 - 2z^2 - \sqrt{1 - 4z^2}}{2z} = r_2 - z.$$

From the equation $G(0) = 1 = \dfrac{r_2z^2f_1 + z^2h_0 - 1 + z^2}{r_2z - 1}$, with the already known $f_1$, we conclude

$$h_0 = \frac{1 - \sqrt{1 - 4z^2}}{2} = zr_2 = z^2\sum_{n\geq 0}C_n z^{2n} = \sum_{n\geq 0}C_n z^{2n+2},$$

as predicted by Stanley [1]. Further simplifications include

$$F(u) = \frac{z(\frac{r_2}{z} - 1)(1 + zu(\frac{r_2}{z} - 1))}{1 - (\frac{r_2}{z} - 1)u^2} = \frac{(1 + uzr_2^2)r_2^2}{1 - r_2^2u^2},$$

---

[*]This is in accordance with the kernel method.

$$H(u) = \frac{z^2(1 - z^2 + uzh_0 - zf_1)}{1 - z^2 - r_2z - z^2u^2}$$

$$= \frac{z^2(1 - z^2 + uz^2r_2 - z(r_2 - z))}{1 - z^2 - r_2z} \frac{1}{1 - (\frac{r_2}{z} - 1)u^2} = \frac{(1 + uzr_2^2)zr_2}{1 - r_2^2u^2}$$

and

$$G(u) = \frac{1 - uz^2r_2^3}{1 - r_2u}.$$

This allows the following expansions

$$F(u) = \frac{(1 + uzr_2^2)r_2^2}{1 - r_2^2u^2} = \sum_{k \geq 0} r_2^{2k+2}u^{2k} + z\sum_{k \geq 0} r_2^{2k+4}u^{2k+1},$$

$$G(u) = \frac{1 - uz^2r_2^3}{1 - r_2u} = \sum_{k \geq 0} r_2^k u^k - z^2 \sum_{k \geq 0} r_2^{k+3}u^{k+1},$$

$$H(u) = \frac{(1 + uzr_2^2)zr_2}{1 - r_2^2u^2} = z\sum_{k \geq 0} r_2^{2k+1}u^{2k} + z^2 \sum_{k \geq 0} r_2^{2k+3}u^{2k+1}.$$

Classically, we know that

$$r_2^j = z^j \sum_{n \geq 0} \frac{j(2n + j - 1)!}{n!(n + j)!} z^{2n}.$$

Then

$$f_j = [u^{j-1}]F(u) = \begin{cases} r_2^{j+1} & j \text{ odd}, \\ zr_2^{j+2} & j \text{ even}. \end{cases}$$

Further, $g_0 = 1$ and for $j \geq 1$

$$g_j = r_2^j - z^2 r_2^{j+2}.$$

Finally,

$$h_j = [u^j]z \sum_{k \geq 0} r_2^{2k+1}u^{2k} + [u^j]z^2 \sum_{k \geq 0} r_2^{2k+3}u^{2k+1} = \begin{cases} z^2 r_2^{j+2} & j \text{ odd}, \\ zr_2^{j+1} & j \text{ even}. \end{cases}$$

An interesting quantity is $F(1) + G(1) + H(1)$, which is the generating function of *all* prefixes of Stanley-Dyck paths, regardless of where they end;

$$F(1) + G(1) + H(1) = \frac{1 - z}{2} + \frac{1}{2}\frac{1 + z}{1 - 2z}\sqrt{1 - 4z^2}$$

$$= 1 + z + 2z^2 + 3z^3 + 5z^4 + 9z^5 + 16z^6 + 30z^7 + 55z^8 + 105z^9 + 196z^{10} + 378z^{11} + \dots.$$

**Remark.** Stanley's [1] exercise 31 is a variation due to Callan [6]. The *last* run to the *x*-axis must have *even* length, all others (if any) odd length.

This can be found by $zf_1$, since the factor $z$ stands for an extra down-step, making the final number of down-steps even. But $zf_1 = zr_2 - z^2$ and $h_0 = zr_2$, so these series coincide except for a Dyck-path of length 2, as predicted and proved by Callan.

## 3. Skew Dyck Paths and the Stanley Restriction

Skew Dyck paths are augmented Dyck paths with additional south-west steps $(-1, -1)$, but no overlaps. In [2] we preferred to replace these extra steps by red down-steps; red-up resp. up-red are forbidden because of overlaps. The graph Figure 2 describes the allowed paths. If they end on level 0, they are skew Dyck paths, otherwise just prefixes of them.
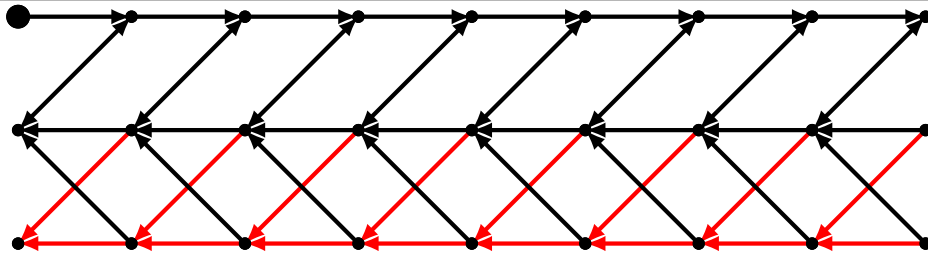
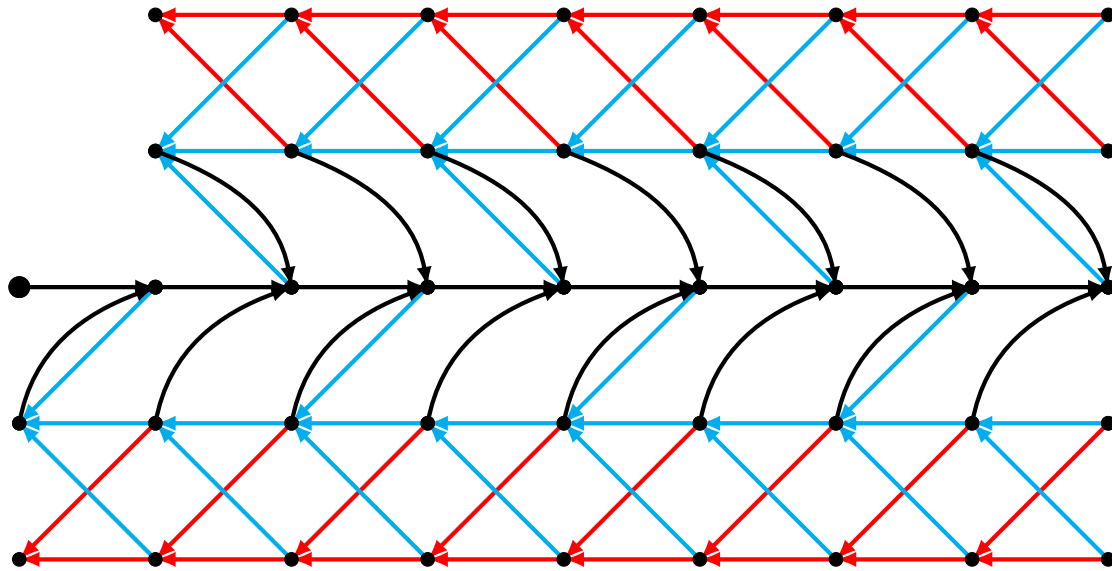**Figure 3.** Three Layers of States according to Skew Dyck Paths



**Figure 4.** Graph to Recognize Skew Stanley-Dyck Paths

The relevant recursions can be read off, again by considering a last step of a partial path.

$$e_i = ze_{i+1} + zf_{i+1}, i \geq 1$$
$$f_i = ze_{i+1} + zf_{i+1} + [i \text{ odd}]zg_{i+1}, i \geq 1$$
$$g_{i+1} = zf_i + zg_i + zh_i, \ i \geq 1, \ g_1 = z + zh_0, \ g_0 = 1$$
$$h_i = zh_{i+1} + [i \text{ even}]zg_{i+1} + zk_{i+1}, \ i \geq 0$$
$$k_i = zh_{i+1} + zk_{i+1}, \ i \geq 0.$$

Summing,

$$\sum_{i\geq 1} u^i e_i = \sum_{i\geq 1} u^i ze_{i+1} + \sum_{i\geq 1} u^i zf_{i+1}$$
$$\frac{u}{z}E(u) = E(u) - e_1 + F(u) - f_1;$$

$$\sum_{i\geq 1} u^i f_i = \sum_{i\geq 1} u^i ze_{i+1} + \sum_{i\geq 1} u^i zf_{i+1} + \sum_{i\geq 1} u^i [i \text{ odd}]zg_{i+1}$$
$$\frac{u}{z}F(u) = E(u) - e_1 + F(u) - f_1 + \frac{1}{u}\sum_{i\geq 2} u^i [i \text{ even }]g_i$$
$$\frac{u}{z}F(u) = E(u) - e_1 + F(u) - f_1 + \frac{1}{2u}G(u) + \frac{1}{2u}G(-u) - \frac{1}{u};$$

$$g_1 + \sum_{i\geq 1} u^i g_{i+1} = \sum_{i\geq 1} u^i zf_i + \sum_{i\geq 1} u^i zg_i + \sum_{i\geq 1} u^i zh_i + z + zh_0$$

$$\sum_{i\geq 0} u^i g_{i+1} = \sum_{i\geq 1} u^i z f_i + \sum_{i\geq 0} u^i z g_i + \sum_{i\geq 0} u^i z h_i$$

$$\frac{G(u)-1}{u} = uzF(u) + zG(u) + zH(u);$$

$$\sum_{i\geq 0} u^{i+1} h_i = \sum_{i\geq 0} u^{i+1} z h_{i+1} + \sum_{i\geq 0} u^{i+1} [i \text{ even}] z g_{i+1} + \sum_{i\geq 0} u^{i+1} z k_{i+1}$$

$$uH(u) = z(H(u) - h_0) + z \sum_{i\geq 1} u^i [i \text{ odd}] g_i + z(K(u) - k_0)$$

$$uH(u) = z(H(u) - h_0) + \frac{z}{2}G(u) - \frac{z}{2}G(-u) + z(K(u) - k_0);$$

$$\sum_{i\geq 0} u^{i+1} k_i = \sum_{i\geq 0} u^{i+1} z h_{i+1} + \sum_{i\geq 0} u^{i+1} z k_{i+1}$$

$$uK(u) = z(H(u) - h_0) + z(K(u) - k_0).$$

For convenience, we repeat the system that needs to be solved:

$$\frac{u}{z}E(u) = E(u) - e_1 + F(u) - f_1,$$

$$\frac{u}{z}F(u) = E(u) - e_1 + F(u) - f_1 + \frac{1}{2u}G(u) + \frac{1}{2u}G(-u) - \frac{1}{u},$$

$$\frac{G(u)-1}{u} = uzF(u) + zG(u) + zH(u),$$

$$uH(u) = z(H(u) - h_0) + \frac{z}{2}G(u) - \frac{z}{2}G(-u) + z(K(u) - k_0),$$

$$uK(u) = z(H(u) - h_0) + z(K(u) - k_0).$$

Again we start with the solution of the middle (simplest) function:

$$G(u) = \frac{z^2 u^2 f_1 + z^2 u^2 e_1 + z^2 u - u + z^2 u h_0 + z^2 u k_0 + 2z - z^3}{u^2 z - u - z^2 u + 2z - z^3}.$$

The denominator is of interest, as one of the factor is 'bad':

$$u^2 z - u - z^2 u + 2z - z^3 = z(u - r_1)(u - r_2),$$

with

$$r_1 = \frac{1 + z^2 + \sqrt{1 - 6z^2 + 5z^4}}{2z}, \quad r_2 = \frac{1 + z^2 - \sqrt{1 - 6z^2 + 5z^4}}{2z}.$$

Dividing out the factor $u - r_2$,

$$G(u) = \frac{r_2 z^2 f_1 + r_2 z^2 e_1 + z^2 - 1 + z^2 h_0 + z^2 k_0 + uz^2 f_0 + uz^2 e_1}{r_2 z - 1 - z^2 + zu},$$

which is what we need. Dividing out $(u - r_2)(u - 2z)u^2$ from the first two functions,

$$E(u) = -\frac{(f_1 + e_1)z^3(u + 2z)}{r_2 z + z^3 r_2 + z^2 u^2 - 1 - 2z^4}$$

and

$$F(u) = -\frac{z^3(u f_1 + u e_1 + 1 + f_1 z + e_1 z)}{r_2 z + z^3 r_2 + z^2 u^2 - 1 - 2z^4}.$$

Dividing out $(u - r_2)(u - 2z)u$ from the last two functions,

$$H(u) = \frac{z^2(-uzk_0 - uzh_0 + z^3e_1 + z^3f_1 - z^2h_0 - z^2k_0 + z^2 - 1 + f_1z + e_1z)}{r_2z + z^3r_2 + z^2u^2 - 1 - 2z^4}$$

and

$$K(u) = -\frac{z^3(k_0 + h_0)(u + 2z)}{r_2z + z^3r_2 + z^2u^2 - 1 - 2z^4}.$$

Now we can insert $u = 0$ and solve:

$$e_1 = -2\frac{z^5 - 3z^3 + 2r_2z^2 + 2z - r_2}{(1 + z^2)(-2 + z^2)^2},$$

$$f_1 = \frac{-4z + 2r_2 + r_2z^4 + 2z^3}{(1 + z^2)(-2 + z^2)^2},$$

$$h_0 = -\frac{(z^3 - 2z + 2r_2)z}{(-2 + z^2)(1 + z^2)},$$

$$k_0 = 2\frac{(-z^3 + r_2z^2 + 2z - r_2)z}{(-2 + z^2)(1 + z^2)}.$$

This leads eventually to

$$E(u) = \frac{(r_2 - 2z)z^3(u + 2z)}{(1 + z^2)(1 + 2z^4 - r_2z - z^3r_2 - z^2u^2)},$$

$$F(u) = -\frac{(r_2 - 2z)(r_2z + z^3rw - z^3u - 1)}{(1 + z^2)(1 + 2z^4 - r_2z - z^3r_2 - z^2u^2)},$$

$$G(u) = -\frac{(r_2 - 2z)(r_2z^5 - 2z^4 - z^5u - r_2z - z^2 + 1)}{z^3(1 + z^2)(r_2z - 1 - z^2 + zu)},$$

$$H(u) = -\frac{(r_2z + z^3r_2 - z^3u - 1)z(-3z + 2r_2)}{(1 + z^2)(1 + 2z^4 - r_2z - z^3r_2 - z^2u^2)},$$

$$K(u) = \frac{(-3z + 2r_2)(u + 2z)z^4}{(1 + z^2)(1 + 2z^4 - r_2z - z^3r_2 - z^2u^2)}$$

and, which is useful for expansion,

$$E(u) = \frac{1}{(2 - z^2)^2}\frac{(r_2 - 2z)z(u + 2z)}{(1 + z^2)(1 - \frac{z^2r_2u^2}{(2-z^2)^2})},$$

$$F(u) = \frac{1}{z^2(2 - z^2)^2}\frac{(r_2 - 2z)(1 - r_2z - z^3r_2 + z^3u)}{(1 + z^2)(1 - \frac{z^2r_2u^2}{(2-z^2)^2})},$$

$$G(u) = \frac{(r_2 - 2z)(r_2z^5 - 2z^4 - z^5u - r_2z - z^2 + 1)}{z^3(1 + z^2)(1 - r_2z + z^2 - zu)},$$

$$H(u) = \frac{1}{z(2 - z^2)^2}\frac{(1 - r_2z - z^3r_2 + z^3u)(-3z + 2r_2)}{(1 + z^2)(1 - \frac{z^2r_2u^2}{(2-z^2)^2})},$$

$$K(u) = \frac{1}{(2 - z^2)^2}\frac{(-3z + 2r_2)(u + 2z)z^2}{(1 + z^2)(1 - \frac{z^2r_2u^2}{(2-z^2)^2})}.$$

For the simplest function $G(u)$ we show the decomposition in detail:

$$G(u) = 1 + \frac{zu(1 + 3z^2 - zr_2)}{(1 + z^2)(1 + z^2 - zr_2 - zu)}$$

$$= 1 + \frac{u(1 + 3z^2 - zr_2)r_2}{(2 - z^2)(1 + z^2)\left(1 - \frac{r_2}{2-z^2}u\right)}$$

$$= 1 + \sum_{k \geq 1} \left(\frac{r_2}{2 - z^2}\right)^k u^k \frac{(1 + 3z^2 - zr_2)}{(1 + z^2)}.$$

Again we can compute the total generating function of paths, regardless where they end:

$$E(1) + F(1) + G(1) + H(1) + K(1)$$

$$= \frac{z(2z^3 + 9z^2 + 4z - 7) + (z + 2)(2z + 1)\sqrt{1 - 6z^2 + 5z^4}}{2(1 + z^2)(1 - 2z - z^2)}$$

$$= 1 + z + 2z^2 + 3z^3 + 5z^4 + 10z^5 + 20z^6 + 38z^7 + 75z^8 + 150z^9 + \dots.$$

*The Coefficients of $r_2$*

We report here briefly how to compute the coefficients of

$$r_2^\ell = z^\ell \left(\frac{r_2}{z}\right)^\ell = z^\ell \left(\frac{1 + Z - \sqrt{1 - 6Z + 5Z^2}}{2Z}\right)^\ell,$$

with $Z = z^2$. As in [4], we set $Z = \frac{v}{1 + 3v + v^2}$, so that $\frac{r_2}{z} = 2 + v$. Then

$$[Z^n]r_2^\ell = \frac{1}{2\pi i} \oint \frac{dZ}{Z^{n+1}}(2 + v)^\ell$$

$$= \frac{1}{2\pi i} \oint \frac{dv(1 - v^2)}{(1 + 3v + v^2)^2} \frac{(1 + 3v + v^2)^{n+1}}{v^{n+1}}(2 + v)^\ell$$

$$= [v^n](1 - v^2)(1 + 3v + v^2)^{N-1} \sum_{k=0}^{\ell} \binom{\ell}{k} 2^{\ell-k} v^k$$

$$= \sum_{k=0}^{\ell} \binom{\ell}{k} 2^{\ell-k} [v^{n-k}](1 - v^2)(1 + 3v + v^2)^{n-1}$$

$$= \sum_{k=0}^{\ell} \binom{\ell}{k} 2^{\ell-k} \left[ \binom{n - 1; 1, 3, 1}{n - k} - \binom{n - 1; 1, 3, 1}{n - k - 2} \right],$$

with weighted *trinomial coefficients* $\binom{n; 1, 3, 1}{k} := [t^k](1 + 3t + t^2)^n$.

## Conflict of Interest

The author declares that they have no conflicts of interest.

## References

1. Stanley, R. P., 2015. *Catalan numbers*. Cambridge University Press.

2. Prodinger, H., 2022. Partial skew Dyck paths: a kernel method approach. *Graphs and Combinatorics, 38*(5), pp.135-144.

3. Prodinger, H., 2004. The kernel method: a collection of examples. *Séminaire Lotharingien de Combinatoire, 50*, p.B50f.

4. Prodinger, H., 2023. A walk in my lattice path garden. *Séminaire Lotharingien de Combinatoire, 87b*, 49 p.

5. Sloane, N.J.A., 2007. The On-Line Encyclopedia of Integer Sequences. In: Kauers, M., Kerber, M., Miner, R., Windsteiger, W. (eds) Towards Mechanized Mathematical Assistants. MKM Calculemus 2007 2007. *Lecture Notes in Computer Science*, vol 4573. Springer, Berlin, Heidelberg. `https://doi.org/10.1007/978-3-540-73086-6_12`

6. Callan, D., 2005. The 136th manifestation of $C_n$. *arXiv*, math/0511010.