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Article

Rainbow Vertex Connection Numbers and Total Rainbow Connection Numbers of Middle and Total Graphs

Yingbin Ma[1] **and Kairui Nie**[1,*]

[1] College of Mathematics and Information Science Henan Normal University, Xinxiang 453007, P.R. China

* **Correspondence:** niekairui@163.com

Abstract: A vertex-colouring of a graph Γ is rainbow vertex connected if every pair of vertices (u, v) in Γ there is a $u-v$ path whose internal vertices have different colours. The rainbow vertex connection number of a graph Γ, is the minimum number of colours needed to make Γ rainbow vertex connected, denoted by rvc(Γ). Here, we study the rainbow vertex connection numbers of middle and total graphs. A total-colouring of a graph Γ is total rainbow connected if every pair of vertices (u, v) in Γ there is a $u - v$ path whose edges and internal vertices have different colours. The total rainbow connection number of Γ, is the minimum number of colours required to colour the edges and vertices of Γ in order to make Γ total rainbow connected, denoted by trc(Γ). In this paper, we also research the total rainbow connection numbers of middle and total graphs.

Keywords: Rainbow vertex connection number; total rainbow connection number; middle graph; total graph
Mathematics Subject Classification: 05C15, 05C35, 05C40.

1. Introduction

We consider finite and simple graphs only. That is, we do not allow the existence of loops and multiple edges. We follow the terminology and notation of [1] for those not described here. A graph Γ is an ordered pair $(V(\Gamma), E(\Gamma))$ consisting of a set $V(\Gamma)$ of vertices and a set $E(\Gamma)$ of edges. Let P_s, C_s, K_s and $K_{1,s-1}$, denote the path, cycle, complete graph, and star graph with s vertices, respectively.

An *edge-colouring* of a connected graph $\Gamma=(V(\Gamma), E(\Gamma))$ is a mapping $c : E \to S$, where S is a set of colours. Usually, the set S of colours is taken to be $\{1, 2, \ldots, t\}$, $t \in N$. A $u - v$ path P in an edge-coloured graph is defined as a *rainbow path* if it does not exist two edges on this path coloured the same. An edge-coloured graph Γ is *rainbow connected* if any two vertices of Γ are connected by a rainbow path. The *rainbow connection number* of a connected graph, denoted by rc(Γ), is the minimum t for which are needed to make the graph rainbow connected. The concept of rainbow connection number was first introduced and researched by Chartrand et al. in [2].

Likewise the concept of rainbow connection number, Krivelevich and Yuster [3] put forward the concept of rainbow vertex connection number. A *vertex-colouring* of a connected graph $\Gamma = (V(\Gamma), E(\Gamma))$ is a mapping $c : V \to S$, where S is a set of colours. Usually, the set S of colours is taken to be $\{1, 2, \ldots, t\}$, $t \in N$. A path in a vertex-coloured graph is defined as a *vertex rainbow* *path* if its internal vertices are assigned distinct colours. A vertex-coloured graph Γ is *rainbow vertex connected* if any two vertices of Γ are connected by a vertex rainbow path, while the colouring is defined as *rainbow vertex-colouring*. The *rainbow vertex connection number* of Γ, is the minimum t for which are needed to make Γ rainbow vertex connected, denoted by rvc(Γ). Let Γ be a connected graph. The graph Γ is complete, we have rvc(Γ) = 0, and rvc(Γ) ≥ diam(Γ) − 1 with equality if and only if the diameter of a graph is 1 or 2. Moreover, if Σ is a connected spanning subgraph of Γ(that is $V(\Gamma) = V(\Sigma)$), then rvc(Γ) ≤ rvc(Σ).

The research of rainbow connection has attracted tremendous attention in the literature, and many conclusions have been published, see [3–11] for example. For more results, The reader can refer to the survey [12] and a new monograph [13].

As a natural generalization, Uchizawa et al. [14] and Liu et al. [15] introdeced the concept of total rainbow connection number, respectively. A *total-colouring* of a graph Γ is a mapping $c : V \cup E \to S$, where S is a set of colours. Usually, the set S of colours is taken to be $\{1, 2, \ldots, t\}$, $t \in N$. A $u-v$ path P in a graph is defined as a *total rainbow path* if its internal vertices and edges are assigned distinct colours. A total-coloured graph Γ is *total rainbow connected* if any two vertices in Γ are connected by a total rainbow path, while the colouring is defined as *total rainbow colouring*. The *total rainbow connection number* of Γ, is the minimum t for which are needed to make Γ total rainbow connected, denoted by trc(Γ). A simple observation is that the graph Γ is complete if and only if trc(Γ) = 1, otherwise trc(Γ) ≥ 3. Moreover, trc(Γ) = 3 if diam(Γ) = 2 and rc(Γ) = 2. We also noticed some trivial fact that trc(Γ) ≤ trc(Σ), where Σ is a connected spanning subgraph of Γ. For more results about the function trc(Γ), the reader can see [16–22] for details.

Definition 1 ([23]). *The middle graph $M(\Gamma)$ of a connected graph Γ is defined as follows,*

(i) $V(M(\Gamma)) = V(\Gamma) \cup E(\Gamma)$.

(ii) *Joining edges with those pairs of these vertices ($x \in E(\Gamma), y \in E(\Gamma)$; $x \in E(\Gamma), y \in V(\Gamma)$) which is adjacent (incident) in Γ.*

In [24], Li investigated the rainbow connection numbers of middle graphs of Γ, where $\Gamma \cong P_s, C_s, K_{1,s}$ or K_s. Here, we research the total rainbow connection numbers (rainbow vertex connection numbers) for the above middle graphs. Our first main result is stated as follows.

Theorem 1. (i) *Let $M(P_s)$ be the middle graph of P_s. Then $rvc(M(P_s)) = s-1$ and $trc(M(P_s)) = 2s-1$.*

(ii) *Let $M(C_s)$ be the middle graph of C_s. Then*

$$rvc(M(C_s)) = \begin{cases} \frac{s}{2} & \text{if } s \text{ is even;} \\ \frac{s+1}{2} & \text{if } s \text{ is odd.} \end{cases}$$

and

$$trc(M(C_s)) = \begin{cases} s+1 & \text{if } s \text{ is even;} \\ s \text{ or } s+1 & \text{if } s \text{ is odd.} \end{cases}$$

(iii) *Let $M(K_{1,s})$ be the middle graph of $K_{1,s}$. Then $rvc(M(K_{1,s})) = s$ and $trc(M(K_{1,s})) = 2s$.*

(iv) *Let $M(K_s)$ be the middle graph of K_s. Then $rvc(M(K_s)) = 1$ and $3 \le trc(M(K_s)) \le s+1$.*

Definition 2 ([25]). *The total graph $T(\Gamma)$ of a graph Γ is defined as follows.*

(i) $V(T(\Gamma)) = V(\Gamma) \cup E(\Gamma)$.

(ii) *Joining edges with those pairs of these vertices ($x \in E(\Gamma), y \in E(\Gamma)$; $x \in E(\Gamma), y \in V(\Gamma)$; $x \in V(\Gamma), y \in V(\Gamma)$) which is adjacent (incident) in Γ.*

Li [24] considered the rainbow connection numbers of total graphs of Γ, where $\Gamma \cong P_s, C_s, K_{1,s}$ or K_s. Here, we also consider the total rainbow connection numbers (rainbow vertex connection numbers) for the above total graphs. Our second main result is stated as follows.

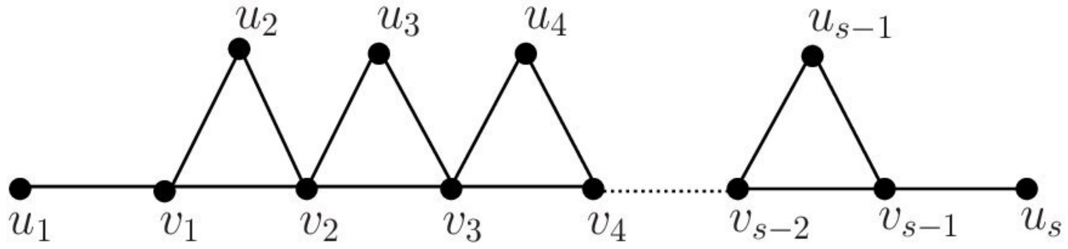

Figure 1. The middle graph of P_s.

Theorem 2. (i) *Let $T(P_s)$ be the total graph of P_s. Then $rvc(T(P_s)) = s-2$ and $trc(T(P_s)) = 2s-3$.*
(ii) *Let $T(C_s)$ be the total graph of C_s. Then*

$$rvc(T(C_s)) = \begin{cases} \frac{s-2}{2} \text{ or } \frac{s}{2} & \text{if } s \text{ is even;} \\ \frac{s-1}{2} \text{ or } \frac{s+1}{2} & \text{if } s \text{ is odd.} \end{cases}$$

and

$$trc(T(C_s)) = \begin{cases} s-1, s \text{ or } s+1 & \text{if } s \text{ is even;} \\ s \text{ or } s+1 & \text{if } s \text{ is odd.} \end{cases}$$

(iii) *Let $T(K_{1,s})$ be the total graph of $K_{1,s}$. Then $rvc(T(K_{1,s})) = 1, trc(T(K_{1,2})) = 3, trc(T(K_{1,3})) = 4$ and $trc(T(K_{1,s})) = 5$ with $s \geq 4$.*
(iv) *Let $T(K_s)$ be the total graph of K_s. Then $rvc(T(K_s)) = 1$ and $3 \leq trc(T(K_s)) \leq s+1$.*

2. Proof of Theorem 1

Proposition 1. *Let $M(P_s)$ be the middle graph of P_s. Then $rvc(M(P_s)) = s-1$ and $trc(M(P_s)) = 2s-1$.*

Proof. The graph $M(P_s)$ is depicted in Figure 1. First we prove that $\mathrm{rvc}(M(P_s)) = s-1$. Let c be a vertex-colouring of $M(P_s)$ defined as follows: $c(u_1) = 1, c(v_i) = c(u_{i+1}) = i$ for $1 \leq i \leq s-2$, $c(v_{s-1}) = c(u_s) = s-1$. We will see that $M(P_s)$ is rainbow vertex connected, and so $\mathrm{rvc}(M(P_s)) \leq s-1$. On the other hand, since $\mathrm{diam}(M(P_s)) = s$, this implies $\mathrm{rvc}(M(P_s)) \geq s-1$, and so $\mathrm{rvc}(M(P_s)) = s-1$.

Now we prove that $\mathrm{trc}(M(P_s)) = 2s-1$. Let c be a total-colouring of $M(P_s)$ defined as follows: $c(u_1v_1) = 1, c(v_iu_{i+1}) = c(v_iv_{i+1}) = c(u_{i+1}v_{i+1}) = i+1$ for $1 \leq i \leq s-2$, $c(v_{s-1}u_s) = s, c(u_1) = s+1$, $c(v_i) = c(u_{i+1}) = s+i$ for $1 \leq i \leq s-2$, $c(v_{s-1}) = c(u_s) = 2s-1$. We will see that $M(P_s)$ is total rainbow connected. Thus $\mathrm{trc}(M(P_s)) \leq 2s-1$. On the other hand, since $\mathrm{diam}(M(P_s)) = s$, this implies $\mathrm{trc}(M(P_s)) \geq 2s-1$, which follows that $\mathrm{trc}(M(P_s)) = 2s-1$. □

Proposition 2. *Let $M(C_s)$ be the middle graph of C_s. Then*

$$rvc(M(C_s)) = \begin{cases} \frac{s}{2} & \text{if } s \text{ is even;} \\ \frac{s+1}{2} & \text{if } s \text{ is odd.} \end{cases}$$

and

$$trc(M(C_s)) = \begin{cases} s+1 & \text{if } s \text{ is even;} \\ s \text{ or } s+1 & \text{if } s \text{ is odd.} \end{cases}$$

Proof. The graph $M(C_s)$ is depicted in Figure 2. First we prove that

$$\mathrm{rvc}(M(C_s)) = \begin{cases} \frac{s}{2} & \text{if } s \text{ is even;} \\ \frac{s+1}{2} & \text{if } s \text{ is odd.} \end{cases}$$

Suppose s is even with $s = 2t$. Since $\mathrm{diam}(M(C_s)) = t+1$, we have $\mathrm{rvc}(M(C_s)) \geq t$. Let c be a vertex-colouring of $M(C_s)$ defined as follows: $c(u_i) = c(v_i) = i$ for $1 \leq i \leq t$, $c(u_i) = c(v_i) = i-t$

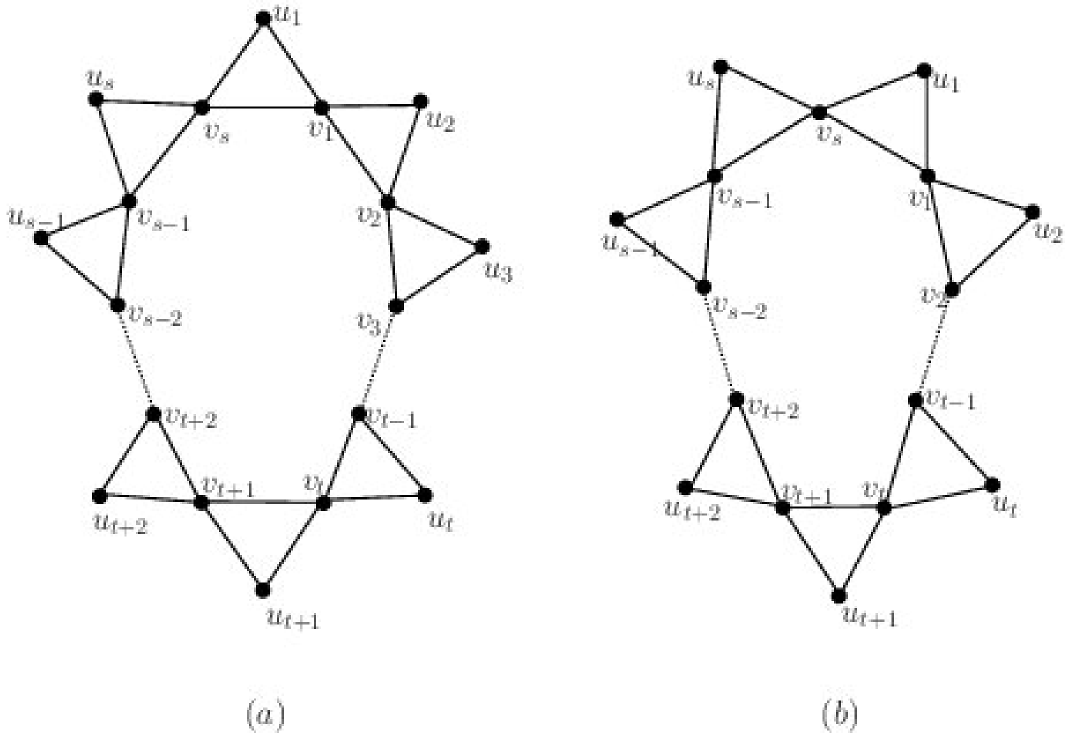

Figure 2. The middle graph of C_s.

for $t+1 \leq i \leq s$. We know that $M(C_s)$ is rainbow vertex connected, and so $\text{rvc}(M(C_s)) \leq t$. Thus $\text{rvc}(M(C_s)) = t$.

Suppose s is odd with $s = 2t+1$. Since $\text{diam}(M(C_s)) = t+1$, it follows that $\text{rvc}(M(C_s)) \geq t$. Now we show that $\text{rvc}(M(C_s)) \neq t$. To the contrary, assume that $M(C_s)$ exists a rainbow vertex-colouring c with t colours. Considering u_1 and u_{t+1}, $u_1v_1v_2\cdots v_{t-1}v_tu_{t+1}$ must be a vertex rainbow $u_1 - u_{t+1}$ path. Without loss of generality, let $c(v_i) = i$ for $1 \leq i \leq t$. Considering u_2 and u_{t+2}, $u_2v_2v_3\cdots v_tv_{t+1}u_{t+2}$ must be a vertex rainbow $u_2 - u_{t+2}$ path. Thus $c(v_{t+1}) = 1$. By the same steps, we know that $c(v_{t+i}) = i$ for $2 \leq i \leq t$. Considering u_{t+2} and u_1, $u_{t+2}v_{t+2}v_{t+3}\cdots v_{s-1}v_su_1$ must be a vertex rainbow $u_{t+2} - u_1$ path, and so $c(v_s) = 1$. But then, there does not exist a vertex rainbow path connecting u_{t+3} and u_2, a contradiction. Thus $\text{rvc}(M(C_s)) \neq t$. Let c be a vertex-colouring of $M(C_n)$ defined as follows: $c(v_i) = c(u_i) = i$ for $1 \leq i \leq t$, $c(v_i) = c(u_i) = i-t$ for $t+1 \leq i \leq 2t, c(v_s) = c(u_s) = t+1$. We will see that $M(C_s)$ is rainbow vertex connected, and so $\text{rvc}(M(C_s)) = t+1$.

Now we prove that

$$\text{trc}(M(C_s)) = \begin{cases} s+1 & \text{if } s \text{ is even;} \\ s \text{ or } s+1 & \text{if } s \text{ is odd.} \end{cases}$$

Suppose s is even with $s = 2t$. Since $\text{diam}(M(C_s)) = t+1$, we have $\text{trc}(M(C_s)) \geq 2t+1$. Let c be a total-colouring of $M(C_s)$ defined as follows: $c(u_1v_1) = c(v_sv_1) = 1, c(u_iv_i) = c(v_{i-1}v_i) = i$ for $2 \leq i \leq t$, $c(u_iv_i) = c(v_{i-1}v_i) = i-t$ for $t+1 \leq i \leq s$, assign all other edges with $t+1$, $c(v_i) = c(u_i) = t+i+1$ for $1 \leq i \leq t$, $c(v_i) = c(u_i) = i+1$ for $t+1 \leq i \leq s$. We will see that $M(C_s)$ is total rainbow connected, and so $\text{trc}(M(C_s)) \leq s+1$. Thus $\text{trc}(M(C_s)) = s+1$.

Suppose s is odd with $s = 2t+1$. Let c be a total-colouring of $M(C_s)$ defined as follows: $c(u_1v_1) = c(v_sv_1) = 1, c(u_1v_s) = t+1, c(v_iu_i) = c(v_{i-1}v_i) = i$ and $c(v_{i-1}u_i) = t+1$ for $2 \leq i \leq t$, $c(v_tv_{t+1}) = c(v_tu_{t+1}) = c(u_{t+1}v_{t+1}) = t+1$, $c(v_{t+i}u_{t+i+1}) = c(v_{t+i}v_{t+i+1}) = c(v_{t+i+1}u_{t+i+1}) = i$ for $1 \leq i \leq t$, $c(v_i) = c(u_i) = t+i+1$ for $1 \leq i \leq t$, $c(v_i) = c(u_i) = i+1$ for $t+1 \leq i \leq s$. We obtained that $M(C_s)$ is total rainbow connected. Since $\text{diam}(M(C_s)) = t+1$, we have $\text{trc}(M(C_s)) \geq 2t+1 = s$, which follows that $s \leq \text{trc}(M(C_s)) \leq s+1$. □

Proposition 3. *Let $M(K_{1,s})$ be the middle graph of $K_{1,s}$. Then $rvc(M(K_{1,s}))$ $= s$ and $trc(M(K_{1,s})) = 2s$.*

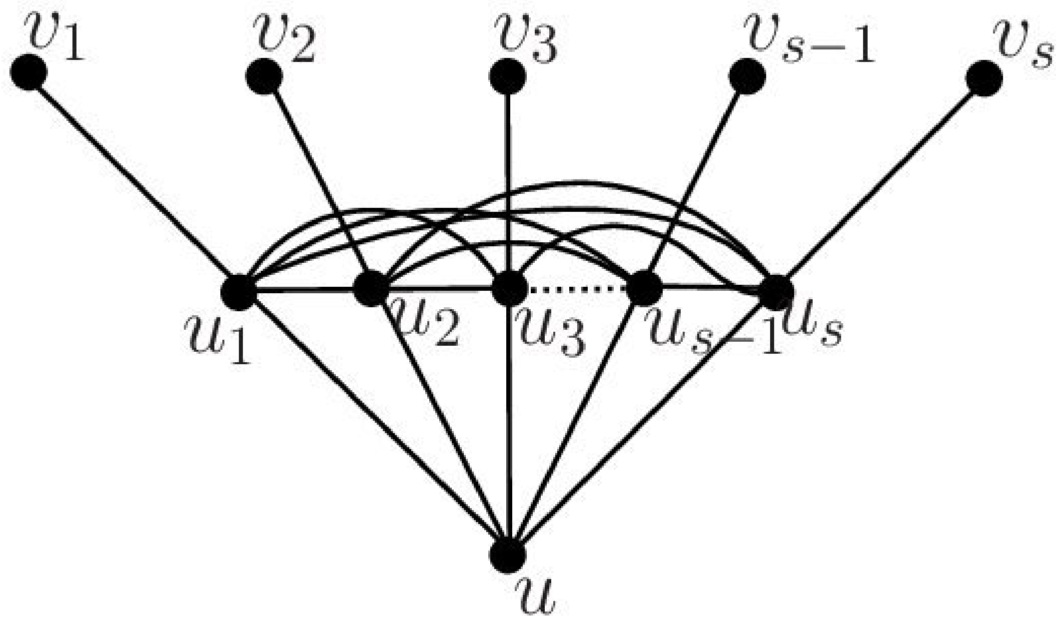

Figure 3. The middle graph of $K_{1,s}$.

Proof. The graph $M(K_{1,s})$ is depicted in Figure 3. First we prove that $\text{rvc}(M(K_{1,s})) = s$. Since the cut vertices must be coloured by different colours, we have $\text{rvc}(M(K_{1,s})) \geq s$. Assign u_i with i for $1 \leq i \leq s$, and assign all other vertices with 1. We will see that $M(K_{1,s})$ is rainbow vertex connected, and so $\text{rvc}(M(P_s)) \leq s$. This implies $\text{rvc}(M(K_{1,s})) = s$. Now we prove that $\text{trc}(M(K_{1,s})) = 2s$. Since the cut edges and cut vertices must be coloured by different colours, we obtain $\text{trc}(M(K_{1,s})) \geq 2s$. Let c be a total-colouring of $M(K_{1,s})$ defined as follows: $c(u_iv_i) = i$ for $1 \leq i \leq s$, $c(uu_1) = s, c(uu_i) = i-1$ for $2 \leq i \leq s$, $c(u_iu_{i+1}) = i+2$ for $1 \leq i \leq s-2$, $c(u_{s-1}u_s) = 1, c(u_iu_j) = j-1$ for $1 \leq i, j \leq s$ and $j-i \geq 2$, $c(u_i) = s+i$ for $1 \leq i \leq s$, assign all other vertices with $s+1$. Note that for any two different vertices v_i and v_j for $1 \leq i, j \leq s$, we know that $v_iu_iu_jv_j$ is a total rainbow $v_i - v_j$ path. Thus $M(K_{1,s})$ is total rainbow connected, and hence $\text{trc}(M(K_{1,s})) \leq 2s$. Therefore, $\text{trc}(M(K_{1,s})) = 2s$. □

Proposition 4. *Let $M(K_s)$ be the middle graph of K_s. Then $rvc(M(K_s))$ $= 1$ and $3 \leq trc(M(K_s)) \leq s+1$.*

Proof. Note that $\text{diam}(M(K_s)) = 2$, we have $\text{rvc}(M(K_s)) = 1$. Now we prove that $3 \leq \text{trc}(M(K_s)) \leq s+1$. Obviously, $\text{trc}(M(K_s)) \geq 3$ since $\text{diam}(M(K_s)) = 2$. The structure of $M(K_s)$ is depicted as follows: $M(K_s) = H_1 \cup H_2 \cup \cdots \cup H_s$, where $H_i \cong K_s$ for $1 \leq i \leq s$, and for any $i, j \in \{1, 2, \cdots, s\}$, H_i and H_j only intersect a different vertex. Let c be a total-colouring of $M(K_s)$ defined as follows: For any $e \in H_i, c(e) = i$, assign $s+1$ to all vertices. We will see that $M(K_s)$ is total rainbow connected, and so $\text{trc}(M(K_s)) \leq s+1$. Hence $3 \leq \text{trc}(M(K_s)) \leq s+1$. □

Combining Propositions 1, 2, 3, 4, Theorem 1 is immediate.

3. Proof of Theorem 2

Proposition 5. *Let $T(P_s)$ be the total graph of P_s. Then $rvc(T(P_s)) = s-2$ and $trc(T(P_s)) = 2s-3$.*

Proof. The graph $T(P_s)$ is depicted in Figure 4. First we prove that $\text{rvc}(T(P_s)) = s-2$. Since $\text{diam}(T(P_s)) = s-1$, we have $\text{rvc}(T(P_s)) \geq s-2$. Let c be a vertex-colouring of $T(P_s)$ defined as follows: $c(v_i) = c(u_{i+1}) = i$ for $1 \leq i \leq s-2$, assign all other vertices with 1. We will see that $T(P_s)$ is rainbow vertex connected, and so $\text{rvc}(T(P_s)) \leq s-2$. Thus $\text{rvc}(T(P_s)) = s-1$.

Now we prove that $\text{trc}(T(P_s)) = 2s-3$. Let c be a total-colouring of $T(P_s)$ defined as follows: $c(u_iv_i) = c(u_iu_{i+1}) = c(v_iu_{i+1}) = i$ for $1 \leq i \leq s-1$, assign all other edges with 1, $c(v_i) = c(u_{i+1}) =$

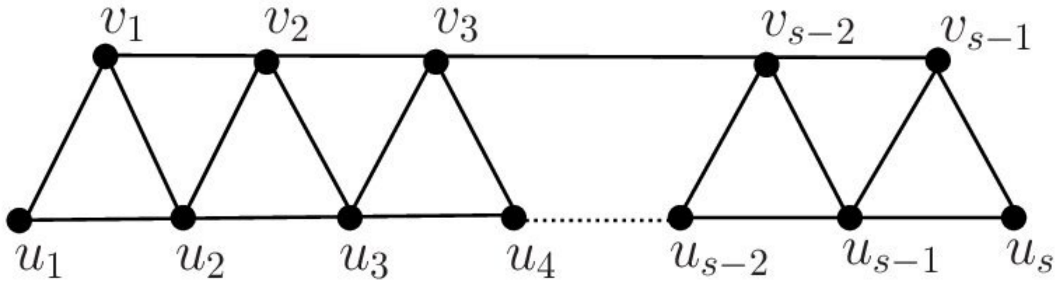

Figure 4. The total graph of P_s.

$s + i - 1$ for $1 \leq i \leq s - 2$, all other vertices coloured with s. We will see that $T(P_s)$ is total rainbow connected with the above total-colouring. Thus $\mathrm{trc}(T(P_s)) \leq 2s-3$. On the other hand, since $\mathrm{diam}(T(P_s)) = s - 1$, this implies $\mathrm{trc}(T(P_s)) \geq 2s - 3$. This follows that $\mathrm{trc}(T(P_s)) = 2s - 3$. □

Proposition 6. *Let $T(C_s)$ be the total graph of C_s. Then*

$$rvc(T(C_s)) = \begin{cases} \frac{s}{2} - 1 \text{ or } \frac{s}{2} & \text{if } s \text{ is even;} \\ \frac{s-1}{2} \text{ or } \frac{s+1}{2} & \text{if } s \text{ is odd.} \end{cases}$$

and

$$trc(T(C_s)) = \begin{cases} s - 1, s \text{ or } s + 1 & \text{if } s \text{ is even;} \\ s \text{ or } s + 1 & \text{if } s \text{ is odd.} \end{cases}$$

Proof. By [24], we know that

$$\mathrm{diam}(T(C_s)) = \begin{cases} \frac{s}{2} & \text{if } s \text{ is even;} \\ \frac{s+1}{2} & \text{if } s \text{ is odd.} \end{cases}$$

On the other hand, note that $M(C_s)$ is a connected spanning subgraph of $T(C_s)$, this proposition follows from Proposition 2. □

Proposition 7. *Let $T(K_{1,s})$ be the total graph of $K_{1,s}$. Then $rvc(T(K_{1,s}))$ $= 1, trc(T(K_{1,2})) = 3, trc(T(K_{1,3})) = 4$ and $trc(T(K_{1,s})) = 5$ with $s \geq 4$.*

Proof. Since $\mathrm{diam}(T(K_{1,s})) = 2$, we have $\mathrm{rvc}(T(K_{1,s})) = 1$ and $\mathrm{trc}(T(K_{1,s}))$ ≥ 3 for $s \geq 2$.

Suppose $s = 2$. We can easily verify that the total-colouring shown in Figure 5(a) is total rainbow. Thus $\mathrm{trc}(T(K_{1,2})) = 3$.

Suppose $s = 3$. Let c be a total-colouring of $T(K_{1,3})$ defined as follows: $c(uv_1) = c(uu_1) = 1, c(uv_2) = c(uu_2) = 2, c(uv_3) = c(uu_3) = 3$, assign all other edges with 1, and assign all vertices with 4. We can easily verify that $T(K_{1,3})$ is total rainbow connected with the above total-colouring, and so $\mathrm{trc}(T(K_{1,3})) \leq 4$. Now we only need to prove that $\mathrm{trc}(T(K_{1,3})) \neq 3$. To the contrary, assume that $T(K_{1,3})$ exists a total rainbow colouring with 3 colours. Considering v_1 and v_2, the total rainbow $v_1 - v_2$ path must be v_1uv_2. Without loss of generality, assume that $c(v_1u) = 1, c(u) = 2, c(uv_2) = 3$. Considering v_1 and v_3, the total rainbow $v_1 - v_3$ path must be v_1uv_3. Thus $c(uv_3) = 3$. But then, there is no total rainbow $v_2 - v_3$ path, a contradiction. Hence $\mathrm{trc}(T(K_{1,3})) \neq 3$, and so $\mathrm{trc}(T(K_{1,3})) = 4$.

Suppose $s \geq 4$. Let c be a total-colouring of $T(K_{1,s})$ defined as follows: assign 1 to the edges uv_i for $1 \leq i \leq s$, assign 2 to the edges u_iv_i for $1 \leq i \leq s$, assign 3 to all other edges, assign 4 to u, and assign 5 to all other vertices. We will see that $T(K_{1,s})$ is total rainbow connected with the above total-colouring. Thus $\mathrm{trc}(T(K_{1,s})) \leq 5$. Now we only need to prove that $\mathrm{trc}(T(K_{1,s})) \neq 4$. To the contrary, assume that $T(K_{1,s})$ exists a total rainbow colouring with 4 colours. Considering v_1 and v_2, v_1uv_2 must be a total rainbow v_1-v_2 path. Without loss of generality, assume that $c(u) = 1, c(v_1u) = 2, c(uv_2) = 3$.

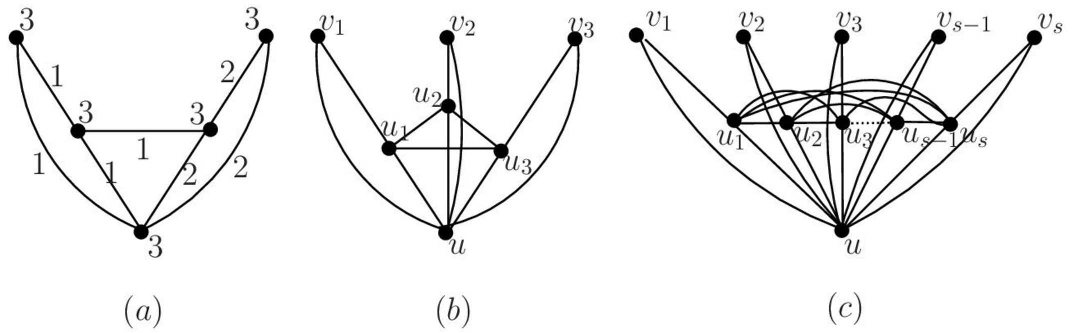

Figure 5. The total graph of $K_{1,s}$.

Considering v_1 and v_3, v_1uv_3 must be a total rainbow $v_1 - v_3$ path. Thus $c(uv_3) = 4$, otherwise there is no total rainbow $v_2 - v_3$ path. Considering v_1 and v_4, v_1uv_4 must be a total rainbow $v_1 - v_4$ path. Hence $c(uv_4) = 3$ or 4. But then there is no total rainbow $v_2 - v_4$ path or $v_3 - v_4$ path. Hence $\text{trc}(T(K_{1,s})) \neq 4$, and so $\text{trc}(T(K_{1,s})) = 5$. □

Proposition 8. *Let $T(K_s)$ be the total graph of K_s. Then $rvc(T(K_s)) = 1$ and $3 \leq trc(T(K_s)) \leq s + 1$.*

Proof. Note that $\text{diam}(T(K_s)) = 2$. Then $\text{rvc}(T(K_s)) = 1$. Obviously, $M(K_s)$ is a connected spanning subgraph of $T(K_s)$. By Proposition 4, we have $3 \leq \text{trc}(T(K_s)) \leq s + 1$. □

Combining Propositions 5, 6, 7, 8, Theorem 2 is immediate.

4. Conclusion

The concept of total rainbow connection number was proposed in recent years. Moreover, Chen et al. [16] proved that the calculating of $\text{trc}(\Gamma)$ is NP-hard. Subsequently, there is a great interest towards determining the total rainbow connection numbers of some graph classes. In this paper, we mainly consider the total rainbow connection numbers of middle and total graphs.

Conflict of Interest

The authors declare no conflict of interests.

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References

1. Bondy, J.A. and Murty, U.S.R., 2008. *Graph Theory*. Springer, Berlin.
2. Chartrand, G., Johns, G.L., McKeon, K.A. and Zhang, P., 2008. Rainbow connection in graphs. Mathematica bohemica, 133(1), pp.85-98.
3. Krivelevich, M. and Yuster, R., 2010. The rainbow connection of a graph is (at most) reciprocal to its minimum degree. Journal of Graph Theory, 63(3), pp.185-191.
4. Sunil Chandran, L., Das, A., Rajendraprasad, D. and Varma, N.M., 2012. Rainbow connection number and connected dominating sets. *Journal of Graph Theory, 71*(2), pp.206-218.
5. Huang, X., Li, X. and Shi, Y., 2015. Note on the hardness of rainbow connections for planar and line graphs. *Bulletin of the Malaysian Mathematical Sciences Society, 38*(3), pp.1235-1241.

6. Huang, X., Li, X., Shi, Y., Yue, J. and Zhao, Y., 2014. Rainbow connections for outerplanar graphs with diameter 2 and 3. *Applied Mathematics and Computation, 242*, pp.277-280.
7. Lei, H., Li, S., Liu, H. and Shi, Y., 2018. Rainbow vertex connection of digraphs. *Journal of Combinatorial Optimization, 35*, pp.86-107.
8. Li, X., Liu, S., Chandran, L.S., Mathew, R. and Rajendraprasad, D., 2012. Rainbow connection number and connectivity. *The Electronic Journal Of Combinatorics,* pp.P20.
9. Li, X. and Shi, Y., 2013. Rainbow connection in 3-connected graphs. *Graphs and Combinatorics, 29*(5), pp.1471-1475.
10. Li, X. and Shi, Y., 2013. On the rainbow vertex-connection. *Discussiones Mathematicae Graph Theory, 33*(2), pp.307-313.
11. Schiermeyer, I., 2013. On minimally rainbow k-connected graphs. *Discrete Applied Mathematics, 161*(4-5), pp.702-705.
12. Li, X., Shi, Y. and Sun, Y., 2013. Rainbow connections of graphs: A survey. *Graphs and combinatorics, 29*, pp.1-38.
13. Li, X. and Sun, Y., 2012. *Rainbow connections of graphs*. Springer Science & Business Media.
14. Uchizawa, K., Aoki, T., Ito, T., Suzuki, A. and Zhou, X., 2013. On the rainbow connectivity of graphs: complexity and FPT algorithms. *Algorithmica, 67*, pp.161-179.
15. Liu, H., Mestre, Â. and Sousa, T., 2014. Total rainbow k-connection in graphs. *Discrete Applied Mathematics, 174*, pp.92-101.
16. Chen, L., Huo, B. and Ma, Y., 2016. Hardness results for total rainbow connection of graphs. *Discussiones Mathematicae Graph Theory, 36*(2), pp.355-362.
17. Jiang, H., Li, X. and Zhang, Y., 2016. Upper bounds for the total rainbow connection of graphs. *Journal of Combinatorial Optimization, 32*, pp.260-266.
18. Lei, H., Liu, H., Magnant, C. and Shi, Y., 2018. Total rainbow connection of digraphs. *Discrete Applied Mathematics, 236*, pp.288-305.
19. Ma, Y., Chen, L. and Li, H., 2017. Graphs with small total rainbow connection number. *Frontiers of Mathematics in China, 12*, pp.921-936.
20. Ma, Y., Chen, L. and Li, H., 2017. Graphs with small total rainbow connection number. *Frontiers of Mathematics in China, 12*, pp.921-936.
21. Ma, Y.B., Chen, L., Li, H.Z. and Li, H.F., 2016. The total rainbow connection numbers of cubic graphs. utilitas mathematica, 99, pp.143-150.
22. Sun, Y., 2015. On rainbow total-coloring of a graph. *Discrete Applied Mathematics, 194,* pp.171-177.
23. Hamada, T. and Yoshimura, I., 1976. Traversability and connectivity of the middle graph of a graph. Discrete Mathematics, 14(3), pp.247-255.
24. Li, F., 2013. Rainbow connection numbers of middle and total graphs. Utilitas Mathematica, 91, 243-260.
25. Behzad, M., 1967, July. A criterion for the planarity of the total graph of a graph. In *Mathematical Proceedings of the Cambridge Philosophical Society* (Vol. 63, No. 3, pp. 679-681). Cambridge University Press.