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Sum Divisor Cordial Labeling of $T_p$-Tree Related Graphs

A. Lourdusamy\textsuperscript{1} S. Jenifer Wency\textsuperscript{2} and F. Patrick\textsuperscript{1,*}

\textsuperscript{1} Department of Mathematics, St. Xavier’s College (Autonomous), Palayamkottai - 627 002, Tamilnadu, India.
\textsuperscript{2} Research Scholar, Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli, Tamilnadu, India.

* Correspondence: patrickjermaiyas@gmail.com

Abstract: A sum divisor cordial labeling of a graph $G$ with vertex set $V(G)$ is a bijection $f$ from $V(G)$ to $\{1, 2, \cdots , |V(G)|\}$ such that an edge $uv$ is assigned the label 1 if 2 divides $f(u) + f(v)$ and 0 otherwise; and the number of edges labeled with 1 and the number of edges labeled with 0 differ by at most 1. A graph with a sum divisor cordial labeling is called a sum divisor cordial graph. In this paper, we discuss the sum divisor cordial labeling of transformed tree related graphs.

Keywords: Sum divisor cordial labeling, corona, transformed tree.
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1. Introduction

All graphs considered here are simple, finite, connected and undirected. The vertex set and the edge set of a graph are denoted by $V(G)$ and $E(G)$ respectively. We follow the basic notations and terminology of graph theory as in [1]. A labeling of a graph is a map that carries the graph elements to the set of numbers, usually to the set of non-negative or positive integers. If the domain is the set of vertices then the labeling is called vertex labeling. If the domain is the set of edges then the labeling is called edge labeling. If the labels are assigned to both vertices and edges then the labeling is called total labeling. A detailed survey of graph labeling is available in [2]. The concept of cordial labeling was introduced by Cahit in [3].

Lourdusamy et al. introduced the concept of sum divisor cordial labeling in [4]. They prove that paths, combs, stars, complete bipartite, $K_2 + mK_1$, bistars, jewels, crowns, flowers, gears, subdivisions of stars, the graph obtained from $K_{1,n}$ by attaching the root of $K_{1,n}$ at each pendant vertex of $K_{1,3}$, and the square $B_{n,n}$ are sum divisor cordial graphs. Also they discussed the sum divisor labeling of star related graphs, path related graphs and cycle related graphs in [5–7].

In [8–11], Sugumaran et al. investigated the behaviour of sum divisor cordial labeling of swastiks, path unions of finite number of copies of swastiks, cycles of $k$ copies of swastiks, when $k$ is odd, jelly fish, Petersen graphs, theta graphs, the fusion of any two vertices in the cycle of swastiks, duplication of any vertex in the cycle of swastiks, the switchings of a central vertex in swastiks, the path unions of two copies of a swastik, the star graph of the theta graphs, the Herschel graph, the fusion of any
two adjacent vertices of degree 3 in Herschel graphs, the duplication of any vertex of degree 3 in the Herschel graph, the switching of central vertex in Herschel graph, the path union of two copies of the Herschel graph, $H$-graph $H_n$, when $n$ is odd, $C_3 \oplus K_{1,n}$, $<F^1_n \Delta F^2_n>$ and open star of swastik graphs $S(t.Sw_n)$, when $t$ is odd.

In [12–15] Sugumaran et al. proved that the following graphs are sum divisor cordial graphs: $H$-graph $H_n$, when $n$ is even, duplication of all edges of the $H$-graph $H_n$, when $n$ is even, $H_n \oplus K_1$, $P(r.H_n)$, $C(r.H_n)$, plus graphs, umbrella graphs, path unions of odd cycles, kites, complete binary trees, drums graph, twigs, fire crackers of the form $P_n \oplus S_n$, where $n$ is even, and the double arrow graph $DA^m_n$, where $|m - n| \leq 1$ and $n$ is even. Further results on sum divisor cordial labeling are given in [16, 17].

In this paper, we discuss the sum divisor cordial labeling of transformed tree related graphs like $TbOP_n$, $TbOC_n$ ($n \equiv 1, 3, 0 \pmod{4}$), $TbOK_1$, $n$, $TbO\bar{K}_n$, $TbOQ_n$, $TbOC_n$ ($n \equiv 1, 3, 0 \pmod{4}$) and $TbOQ_n$.

We use the following definitions in the subsequent sections.

**Definition 1.** Let $G = (V(G), E(G))$ be a simple graph and $f : V(G) \to \{1, 2, \cdots, |V(G)|\}$ be a bijection. For each edge $uv$, assign the label 1 if $2|(f(u) + f(v))$ and the label 0 otherwise. The function $f$ is called a sum divisor cordial labeling if $|e_f(1) - e_f(0)| \leq 1$. A graph which admits a sum divisor cordial labeling is called a sum divisor cordial graph.

**Definition 2.** [18] Let $T$ be a tree and $u_0$ and $v_0$ be two adjacent vertices in $T$. Let there be two pendant vertices $u$ and $v$ in $T$ such that the length of $u_0 - u$ path is equal to the length of $v_0 - v$ path. If the edge $u_0v_0$ is deleted from $T$ and $u, v$ are joined by an edge $uv$, then such a transformation of $T$ is called an elementary parallel transformation (or an ept) and the edge $u_0v_0$ is called transformable edge.

If by the sequence of ept’s, $T$ can be reduced to a path, then $T$ is called a $T_p$-tree (transformed tree) and such a sequence regarded as a composition of mappings (ept’s) denoted by $P$, is called a parallel transformation of $T$. The path, the image of $T$ under $P$ is denoted as $P(T)$.

**Definition 3.** The corona $G_1 \circ G_2$ of two graphs $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ is defined as the graph obtained by taking one copy of $G_1$ and $p_1$ copies of $G_2$ and joining the $i^{th}$ vertex of $G_1$ with an edge to every vertex in the $i^{th}$ copy of $G_2$.

**Definition 4.** [19] Let $G_1$ be a graph with $p$ vertices and $G_2$ be any graph. A graph $G_1 \hat{\circ} G_2$ is obtained from $G_1$ and $p$ copies of $G_2$ by identifying one vertex of $i^{th}$ copy of $G_2$ with $i^{th}$ vertex of $G_1$. 

![Figure 1. A T_p-tree and a sequence of two ept’s reducing it to a path](image)
**Definition 5.** [19] Let $G_1$ be a graph with $p$ vertices and $G_2$ be any graph. A graph $G_1 \tilde{\circ} G_2$ is obtained from $G_1$ and $p$ copies of $G_2$ by joining one vertex of $i^{th}$ copy of $G_2$ with $i^{th}$ vertex of $G_1$ by an edge.

**Theorem 1.** [7] Every $T_p$-tree is sum divisor cordial graph.

2. $T_p$-Tree related graphs

**Theorem 2.** If $T$ be a $T_p$-tree on $m$ vertices, then the graph $\tilde{T}OP_n$ is sum divisor cordial graph.

**Proof.** Let $T$ be a $T_p$-tree with $m$ vertices. By the definition of a transformed tree there exists a parallel transformation $P$ of $T$ such that for the path $P(T)$, we have (i) $V(P(T)) = V(T)$ and (ii) $E(P(T)) = (E(T) - E_d) \cup E_p$, where $E_d$ is the set of edges deleted from $T$ and $E_p$ is the set of edges newly added through the sequence $P = (P_1, P_2, \ldots, P_k)$ of the epts $P$ used to arrive at the path $P(T)$. Clearly, $E_d$ and $E_p$ have the same number of edges. Denote the vertices of $P(T)$ successively as $v_1, v_2, \ldots, v_m$ starting from one pendant vertex of $P(T)$ right up to the other. Let $u'_1, u'_2, \ldots, u'_m (1 \leq j \leq m)$ be the vertices of $j^{th}$ copy of $P_n$ with $u'_1 = v_j$. Then $V(\tilde{T}OP_n) = \{v_j, u'_1 : 1 \leq i \leq n, 1 \leq j \leq m$ with $u'_1 = v_j\}$ and $E(\tilde{T}OP_n) = E(T) \cup \{u'_1u'_1 : 1 \leq i \leq n - 1, 1 \leq j \leq m\}$.

Define $f: V(\tilde{T}OP_n) \to \{1, 2, \ldots, mn\}$ as follows:

**Case 1.** $n$ is even.

For $1 \leq j \leq m$ and $1 \leq i \leq n$,

$$f(u'_1) = \begin{cases} 
  n(j - 1) + i + 1 & \text{if } i \equiv 1 \pmod{4} \\
  n(j - 1) + i - 1 & \text{if } i \equiv 2 \pmod{4} \\
  n(j - 1) + i & \text{if } i \equiv 3,0 \pmod{4}.
\end{cases}$$

Let $v_iv_j$ be a transformed edge in $T$, $1 \leq i < j \leq m$ and let $P_1$ be the ept obtained by deleting the edge $v_iv_j$ and adding the edge $v_iv_j$, where $t$ is the distance of $v_i$ from $v_{it}$ and the distance of $v_j$ from $v_{jt}$. Let $P$ be a parallel transformation of $T$ that contains $P_1$ as one of the constituent epts.

Since $v_iv_j$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. Therefore, $i$ and $j$ are of opposite parity.

The induced edge label of $v_iv_j$ is given by

$$f^*(v_iv_j) = f^*(v_iv_{i+2t+1})$$

$$= 2(f(v_i) + f(v_{i+2t+1}))$$

$$= 1.$$ 

The induced edge label of $v_{i+1}v_{j-1}$ is given by

$$f^*(v_{i+1}v_{j-1}) = f^*(v_{i+1}v_{i+1})$$

$$= 2(f(v_{i+1}) + f(v_{i+1}))$$

$$= 1.$$ 

Therefore, $f^*(v_iv_j) = f^*(v_{i+1}v_{j-1})$.

The induced edge labels are as follows:

$$f^*(v_jv_{j+1}) = 1, \ 1 \leq j \leq m - 1;$$

for $1 \leq i \leq n - 1$ and $1 \leq j \leq m$,

$$f^*(u'_1u'_j) = \begin{cases} 
  0 & \text{if } i \text{ is odd} \\
  1 & \text{if } i \text{ is even}.
\end{cases}$$

**Case 2.** $n$ is odd.

For $1 \leq j \leq m$, choose ‘if $j \equiv 1, 2 \pmod{4}$’,

$$f(u'_j) = \begin{cases} 
(n(j-1) + i) & \text{if } i \equiv 1, 0 \pmod{4} \text{ and } 1 \leq i \leq n \\
n(j-1) + i + 1 & \text{if } i \equiv 2 \pmod{4} \text{ and } 1 \leq i \leq n \\
n(j-1) + i - 1 & \text{if } i \equiv 3 \pmod{4} \text{ and } 1 \leq i \leq n 
\end{cases}$$

choose ‘if $j \equiv 3, 0 \pmod{4}$’ and $n \equiv 3 \pmod{4}$,

$$f(u'_j) = \begin{cases} 
(n(j-1) + i + 1) & \text{if } i \equiv 1 \pmod{4} \text{ and } 1 \leq i \leq n \\
n(j-1) + i - 1 & \text{if } i \equiv 2 \pmod{4} \text{ and } 1 \leq i \leq n \\
n(j-1) + i & \text{if } i \equiv 3 \pmod{4} \text{ and } 1 \leq i \leq n 
\end{cases}$$

choose ‘if $j \equiv 3, 0 \pmod{4}$’ and $n \equiv 1 \pmod{4}$,

$$f(u'_j) = \begin{cases} 
(n(j-1) + i + 1) & \text{if } i \equiv 1 \pmod{4} \text{ and } 1 \leq i \leq n - 2 \\
n(j-1) + i - 1 & \text{if } i \equiv 2 \pmod{4} \text{ and } 1 \leq i \leq n - 2 \\
n(j-1) + i & \text{if } i \equiv 3 \pmod{4} \text{ and } 1 \leq i \leq n - 2 \\
n(j-1) + i + 1 & \text{if } i = n - 1 \\
n(j-1) + i - 1 & \text{if } i = n 
\end{cases}$$

Let $v_iv_j$ be a transformed edge in $T$, $1 \leq i < j \leq m$ and let $P_1$ be the $ept$ obtained by deleting the edge $v_iv_j$ and adding the edge $v_{i+t}v_{j-t}$, where $t$ is the distance of $v_j$ from $v_{i+t}$ and the distance of $v_j$ from $v_{j-t}$. Let $P$ be a parallel transformation of $T$ that contains $P_1$ as one of the constituent $epts$.

Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. Therefore, $i$ and $j$ are of opposite parity.

The induced edge label of $v_iv_j$ is given by

$$f^*(v_iv_j) = f^*(v_{i+2t+1}) = \begin{cases} 
0 & \text{if } i \text{ is odd} \\
1 & \text{if } i \text{ is even} 
\end{cases}$$

The induced edge label of $v_{i+t}v_{j-t}$ is given by

$$f^*(v_{i+t}v_{j-t}) = f^*(v_{i+t+1}) = \begin{cases} 
0 & \text{if } i \text{ is odd} \\
1 & \text{if } i \text{ is even} 
\end{cases}$$

Therefore, $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.

The induced edge labels are as follows:

$$f^*(v_jv_{j+1}) = \begin{cases} 
0 & \text{if } i \text{ is odd and } 1 \leq j \leq m - 1 \\
1 & \text{if } i \text{ is even and } 1 \leq j \leq m - 1 
\end{cases}$$

for $1 \leq j \leq m$,

choose ‘if $j \equiv 1, 2 \pmod{4}$’,

$$f^*(u'_iu'_{i+1}) = \begin{cases} 
1 & \text{if } i \text{ is odd and } 1 \leq i \leq n - 1 \\
0 & \text{if } i \text{ is even and } 1 \leq i \leq n - 1 
\end{cases}$$

choose ‘if $j \equiv 3, 0 \pmod{4}$’ and $n \equiv 3 \pmod{4}$,

$$f^*(u'_iu'_{i+1}) = \begin{cases} 
0 & \text{if } i \text{ is odd and } 1 \leq i \leq n - 1 \\
1 & \text{if } i \text{ is even and } 1 \leq i \leq n - 1 
\end{cases}$$

choose ‘if $j \equiv 3, 0 \pmod{4}$’ and $n \equiv 1 \pmod{4}$,
Figure 2. Sum divisor cordial labeling of $T\bar{O}P_5$ where $T$ is a $T_p$-tree with 11 vertices

$$f^*(u^i_1u^i_{r+1}) = \begin{cases} 
0 & \text{if } i \text{ is odd and } 1 \leq i \leq n - 3 \\
1 & \text{if } i \text{ is even and } 1 \leq i \leq n - 3 \\
1 & \text{if } i = n - 2 \\
1 & \text{if } i = n - 1 \, . 
\end{cases}$$

In the above two cases, when $m$ is odd and $n$ is odd, $e_f(0) = e_f(1) = \frac{mn-1}{2}$; when $m$ is odd and $n$ is even, $e_f(0) = \left\lfloor \frac{mn-1}{2} \right\rfloor$ and $e_f(1) = \left\lfloor \frac{mn-1}{2} \right\rfloor$; when $m$ is even and $n$ is odd or even, $e_f(0) = \left\lfloor \frac{mn-1}{2} \right\rfloor$ and $e_f(1) = \left\lfloor \frac{mn-1}{2} \right\rfloor$.

Clearly $|e_f(0) - e_f(1)| \leq 1$. Hence $T\bar{O}P_n$ is sum divisor cordial graph.

**Example 1.** Sum divisor cordial labeling of $T\bar{O}P_5$ where $T$ is a $T_p$-tree with 11 vertices is shown in Figure 2.

**Theorem 3.** If $T$ is a $T_p$-tree on $m$ vertices, then the graph $T\bar{O}C_n$ is sum divisor cordial graph if $n \equiv 0, 3, 1 \pmod{4}$.

**Proof.** Let $T$ be a $T_p$-tree with $m$ vertices. By the definition of a transformed tree there exists a parallel transformation $P$ of $T$ such that for the path $P(T)$, we have (i) $V(P(T)) = V(T)$ and (ii) $E(P(T)) = (E(T) - E_d) \cup E_p$, where $E_d$ is the set of edges deleted from $T$ and $E_p$ is the set of edges newly added through the sequence $P = (P_1, P_2, \ldots, P_k)$ of the epts $P$ used to arrive at the path $P(T)$. Clearly, $E_d$ and $E_p$ have the same number of edges. Denote the vertices of $P(T)$ successively as $v_1, v_2, \ldots, v_m$ starting from one pendant vertex of $P(T)$ right up to the other. Let $u'_1, u'_2, \ldots, u'_n$ ($1 \leq j \leq m$) be the vertices of $j^{th}$ copy of $C_n$ with $u'_i = v_j$. Then $V(T\bar{O}C_n) = \{u'_i : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(T\bar{O}C_n) = E(T) \cup E(C_n)$. Define $f : V(T\bar{O}C_n) \rightarrow \{1, 2, 3, \ldots, mn\}$ as follows:

**Case 1.** $n \equiv 0 \pmod{4}$.
Choose ‘if $j \equiv 1, 2 \pmod{4}$’ and $1 \leq j \leq m$,

$$f(u'_i) = \begin{cases} 
n(j-1) + i & \text{if } i \equiv 1, 0 \pmod{4} \text{ and } 1 \leq i \leq n \\
n(j-1) + i + 1 & \text{if } i \equiv 2 \pmod{4} \text{ and } 1 \leq i \leq n \\
n(j-1) + i - 1 & \text{if } i \equiv 3 \pmod{4} \text{ and } 1 \leq i \leq n \, .
\end{cases}$$

choose ‘if $j \equiv 3, 0 \pmod{4}$’ and $1 \leq j \leq m$. 


Let $v_iv_j$ be a transformed edge in $T$, $1 \leq i < j \leq m$ and let $P_1$ be the ept obtained by deleting the edge $v_iv_j$ and adding the edge $v_{i+t}v_{j-t}$, where $t$ is the distance of $v_i$ from $v_{i+t}$ and the distance of $v_j$ from $v_{j-t}$. Let $P$ be a parallel transformation of $T$ that contains $P_1$ as one of the constituent epts.

Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. Therefore, $i$ and $j$ are of opposite parity.

The induced edge label of $v_iv_j$ is given by

$$f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$$

$$= \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even.} \end{cases}$$

The induced edge label of $v_{i+t}v_{j-t}$ is given by

$$f^*(v_{i+t}v_{j-t}) = f^*(v_{i+t}v_{i+2t+1})$$

$$= \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even.} \end{cases}$$

Therefore, $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.

The induced edge labels are as follows:

$$f^*(v_iv_{j+1}) = \begin{cases} 1 & \text{if } j \text{ is odd and } 1 \leq j \leq m - 1 \\ 0 & \text{if } j \text{ is even and } 1 \leq j \leq m - 1; \end{cases}$$

for $1 \leq j \leq m$ and $1 \leq i \leq n - 1$,

$$f^*(u^i_0u^i_0) = \begin{cases} 0 & \text{if } j \equiv 1,2 \pmod{4} \\ 1 & \text{if } j \equiv 3,0 \pmod{4}; \end{cases}$$

$$f^*(u^i_1u^i_1) = \begin{cases} 1 & \text{if } i \text{ is odd and } j \equiv 1,2 \pmod{4} \\ 0 & \text{if } i \text{ is even and } j \equiv 1,2 \pmod{4} \\ 0 & \text{if } i \text{ is odd and } j \equiv 3,0 \pmod{4} \\ 1 & \text{if } i \text{ is even and } j \equiv 3,0 \pmod{4}. \end{cases}$$

**Case 2.** $n \equiv 3 \pmod{4}$.

Choose ‘if $j \equiv 1,3 \pmod{4}$’ and $1 \leq j \leq m$,

$$f^*(u^i_1) = \begin{cases} n(j-1) + i & \text{if } i \equiv 1,0 \pmod{4} \text{ and } 1 \leq i \leq n \\ n(j-1) + i + 1 & \text{if } i \equiv 2 \pmod{4} \text{ and } 1 \leq i \leq n \\ n(j-1) + i - 1 & \text{if } i \equiv 3 \pmod{4} \text{ and } 1 \leq i \leq n; \end{cases}$$

choose ‘if $j \equiv 2,0 \pmod{4}$’ and $1 \leq j \leq m$,

$$f^*(u^i_0) = \begin{cases} n(j-1) + i + 1 & \text{if } i \equiv 1 \pmod{4} \text{ and } 1 \leq i \leq n \\ n(j-1) + i - 1 & \text{if } i \equiv 2 \pmod{4} \text{ and } 1 \leq i \leq n \\ n(j-1) + i & \text{if } i \equiv 3 \pmod{4} \text{ and } 1 \leq i \leq n. \end{cases}$$

Let $v_iv_j$ be a transformed edge in $T$, $1 \leq i < j \leq m$ and let $P_1$ be the ept obtained by deleting the edge $v_iv_j$ and adding the edge $v_{i+t}v_{j-t}$, where $t$ is the distance of $v_i$ from $v_{i+t}$ and the distance of $v_j$ from $v_{j-t}$. Let $P$ be a parallel transformation of $T$ that contains $P_1$ as one of the constituent epts.

Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. Therefore, $i$ and $j$ are of opposite parity.
The induced edge label of $v_iv_j$ is given by
\[
f^*(v_iv_j) = f^*(v_{i+2i+1}) = 2|f(v_i) + f(v_{i+2i+1})| = 1.
\]

The induced edge label of $v_{i+t}v_{j-t}$ is given by
\[
f^*(v_{i+t}v_{j-t}) = f^*(v_{i+t}) = 2|f(v_{i+t}) + f(v_{i+t+1})| = 1.
\]

Therefore, $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.

The induced edge labels are as follows:
$f^*(v_{j+1}) = 1$, $1 \leq j \leq m - 1$;
for $1 \leq j \leq m$ and $1 \leq i \leq n - 1$,
\[
f^*(u_{i}u_{i+1}) = \begin{cases} 1 & \text{if } i \text{ is odd and } j \equiv 1, 3 \pmod{4} \\ 0 & \text{if } i \text{ is even and } j \equiv 1, 3 \pmod{4} \\ 0 & \text{if } i \text{ is odd and } j \equiv 2, 0 \pmod{4} \\ 1 & \text{if } i \text{ is even and } j \equiv 2, 0 \pmod{4}. \end{cases}
\]

**Case 3.** $n \equiv 1 \pmod{4}$.

For $1 \leq j \leq m$,
\[
f(u_{i}) = \begin{cases} n(j - 1) + i & \text{if } i \equiv 1, 0 \pmod{4} \text{ and } 1 \leq i \leq n \\ n(j - 1) + i + 1 & \text{if } i \equiv 2 \pmod{4} \text{ and } 1 \leq i \leq n \\ n(j - 1) + i - 1 & \text{if } i \equiv 3 \pmod{4} \text{ and } 1 \leq i \leq n. \end{cases}
\]

Let $v_iv_j$ be a transformed edge in $T$, $1 \leq i < j \leq m$ and let $P_1$ be the ept obtained by deleting the edge $v_iv_j$ and adding the edge $v_{i+t}v_{j-t}$, where $t$ is the distance of $v_i$ from $v_{i+t}$ and the distance of $v_j$ from $v_{j-t}$. Let $P$ be a parallel transformation of $T$ that contains $P_1$ as one of the constituent epts.

Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$.

Therefore, $i$ and $j$ are of opposite parity.

The induced edge label of $v_iv_j$ is given by
\[
f^*(v_iv_j) = f^*(v_{i+2i+1}) = 0.
\]

The induced edge label of $v_{i+t}v_{j-t}$ is given by
\[
f^*(v_{i+t}v_{j-t}) = f^*(v_{i+t}) = 0.
\]

Therefore, $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.

The induced edge labels are as follows:
$f^*(v_{j+1}) = 0$, $1 \leq j \leq m - 1$;
for $1 \leq j \leq m$ and $1 \leq i \leq n - 1$,
\[
f^*(u_{i}u_{i+1}) = \begin{cases} 1 & \text{if } i \text{ is odd and } j \equiv 1, 3 \pmod{4} \\ 0 & \text{if } i \text{ is even and } j \equiv 1, 3 \pmod{4} \\ 1 & \text{if } i \text{ is odd and } j \equiv 2, 0 \pmod{4} \\ 0 & \text{if } i \text{ is even and } j \equiv 2, 0 \pmod{4}. \end{cases}
\]
The induced edge label of $e_i$ is given by $f(e_i) = f(v_i v_{i+1})$. Therefore, $i$ is an edge in the path $P(T)$ such that for the path $P(T)$, we have (i) $V(P(T)) = V(T)$ and (ii) $E(P(T)) = (E(T) - E_d) \cup E_p$, where $E_d$ is the set of edges deleted from $T$ and $E_p$ is the set of edges newly added through the sequence $P = (P_1, P_2, \ldots, P_k)$ of the epts $P$ used to arrive at the path $P(T)$. Clearly, $E_d$ and $E_p$ have the same number of edges. Denote the vertices of $P(T)$ successively as $v_1, v_2, \ldots, v_m$ starting from one pendant vertex of $P(T)$ right up to the other. Let $u_0^i, u_1^i, \ldots, u_n^i (1 \leq j \leq m)$ be the vertices of $j^{th}$ copy of $K_{1,n}$ with $u_j^i = v_j$. Then $V(T \tilde{O}K_{1,n}) = \{v_j, u_0^i, u_1^i : 1 \leq i \leq n, 1 \leq j \leq m \}$ and $E(T \tilde{O}K_{1,n}) = E(T) \cup \{u_0^i | u_j^i : 1 \leq j \leq m, 1 \leq i \leq n \}$.

Define $f : V(T \tilde{O}K_{1,n}) \rightarrow \{1, 2, \ldots, mn + m\}$ as follows:

For $1 \leq j \leq m$,

$$f(v_j) = 2j;$$

$$f(u_i^0) = 2j - 1;$$

$$f(u_i^j) = 2m + (n - 1)(j - 1) + i, \quad 1 \leq i \leq n - 1.$$

Let $v_i v_j$ be a transformed edge in $T$, $1 \leq i < j \leq m$ and let $P_1$ be the ept obtained by deleting the edge $v_i v_j$ and adding the edge $v_{i+t} v_{j-t}$ where $t$ is the distance of $v_i$ from $v_{i+t}$ and the distance of $v_j$ from $v_{j-t}$. Let $P$ be a parallel transformation of $T$ that contains $P_1$ as one of the constituent epts. Since $v_{i+t} v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. Therefore, $i$ and $j$ are of opposite parity.

The induced edge label of $v_i v_j$ is given by

$$f^*(v_i v_j) = f^*(v_i v_{i+2t+1})$$

$$= 2(f(v_i) + f(v_{i+2t+1}))$$

$$= 1.$$
The induced edge label of $v_{i+1}v_{j-t}$ is given by
\[
    f^*(v_{i+1}v_{j-t}) = f^*(v_{i+t}v_{i+t+1}) = 2(f(v_{i+t}) + f(v_{i+t+1})) = 1.
\]

Therefore, $f^*(v_iv_j) = f^*(v_{i+1}v_{j-t})$.

The induced edge labels are as follows:
\[
f^*(v_{i+1}v_{j+1}) = 1, \quad 1 \leq j \leq m - 1;
\]
for $1 \leq j \leq m$,
\[
f^*(u_i^tu_j^t) = 0;
\]
when $n$ is odd and $1 \leq i \leq n - 1$,
\[
f^*(u_i^tu_i') = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even} \end{cases};
\]
when $n$ is even and $1 \leq i \leq n - 1$,
\[
f^*(u_i^tu_i') = \begin{cases} 1 & \text{if } i \text{ is odd and } j \text{ is odd} \\ 0 & \text{if } i \text{ is even and } j \text{ is odd} \\ 0 & \text{if } i \text{ is odd and } j \text{ is even} \\ 1 & \text{if } i \text{ is even and } j \text{ is even} \end{cases}.
\]

In view of above labeling we get,
when $m$ is odd and $n$ is even,
\[
e_j(0) = e_j(1) = \frac{mn+m-1}{2};
\]
when $m$ is odd and $n$ is odd,
\[
e_j(0) = \left\lfloor \frac{mn+m-1}{2} \right\rfloor \quad \text{and} \quad e_j(1) = \left\lceil \frac{mn+m-1}{2} \right\rceil;
\]
when $m$ is even and $n$ is odd or even,
\[
e_j(0) = \left\lfloor \frac{mn+m-1}{2} \right\rfloor \quad \text{and} \quad e_j(1) = \left\lceil \frac{mn+m-1}{2} \right\rceil.
\]

Clearly $|e_j(0) - e_j(1)| \leq 1$. Hence $T\hat{O}K_{1,n}$ is sum divisor cordial graph.

**Example 3.** Sum divisor cordial labeling of $T\hat{O}K_{1,3}$ where $T$ is a $T_p$-tree with 12 vertices is shown in Figure 4.

**Theorem 5.** If $T$ be a $T_p$-tree on $m$ vertices, then the graph $T \odot \overline{K}_n$ is sum divisor cordial graph.

**Proof.** Let $T$ be a $T_p$-tree with $m$ vertices. By the definition of $T_p$-tree there exists a parallel transformation $P$ of $T$ such that for the path $P(T)$, we have (i) $V(P(T)) = V(T)$ and (ii) $E(P(T)) = (E(T) - E_d) \cup E_p$, where $E_d$ is the set of edges deleted from $T$ and $E_p$ is the set of edges newly added through the sequence $P = (P_1, P_2, \ldots, P_k)$ of the epts $P$ used to arrive at the path $P(T)$. Clearly, $E_d$ and $E_p$ have the same number of edges. Denote the vertices of $P(T)$ successively as $v_1, v_2, \ldots, v_m$ starting from one pendant vertex of $P(T)$ right up to the other. Let $u_i^t, u_i^t, \ldots, u_i^t(1 \leq j \leq m)$ be the pendant vertices joined with $v_j(1 \leq j \leq m)$ by an edge. Then $V(T \odot \overline{K}_n) = \{v_j, u_j^t : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(T \odot \overline{K}_n) = E(T) \cup \{v_ju_j^t : 1 \leq j \leq m, 1 \leq i \leq n\}$.

Define $f : V(T \odot \overline{K}_n) \to \{1, 2, \ldots, mn + m\}$ as follows:

For $1 \leq j \leq m$,
\[
    f(v_j) = 2j - 1;
\]
\[
    f(u_i^t) = 2j;
\]
\[
    f(u_i^t) = 2m + (n-1)(j-1) + i, \quad 1 \leq i \leq n - 1.
\]

Let $v_iv_j$ be a transformed edge in $T$, $1 \leq i < j \leq m$ and let $P_1$ be the ept obtained by deleting the edge $v_iv_j$ and adding the edge $v_{i+t}v_{j-t}$ where $t$ is the distance of $v_i$ from $v_{i+t}$ and the distance of $v_j$ from $v_{j-t}$. Let $P$ be a parallel transformation of $T$ that contains $P_1$ as one of the constituent epts. Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. 
Therefore, $i$ and $j$ are of opposite parity.

The induced edge label of $v_iv_j$ is given by
\[
\begin{align*}
f^*(v_iv_j) &= f^*(v_iv_{i+2r+1}) \\
&= 2(f(v_i) + f(v_{i+2r+1})) \\
&= 1.
\end{align*}
\]

The induced edge label of $v_{i+t}v_{j-t}$ is given by
\[
\begin{align*}
f^*(v_{i+t}v_{j-t}) &= f^*(v_{i+t}v_{i+t+1}) \\
&= 2(f(v_{i+t}) + f(v_{i+t+1})) \\
&= 1.
\end{align*}
\]

Therefore, $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t}).$

The induced edge labels are as follows:
\[
f^*(v_{j+1}) = 1, \ 1 \leq j \leq m - 1;
\]
\[
\begin{align*}
f^*(v_{j+1}) &= 1 \text{ if } i \text{ is odd} \\
&= 0 \text{ if } i \text{ is even};
\end{align*}
\]
\[
\begin{align*}
\text{when } n \text{ is odd and } 1 \leq i \leq n - 1, \\
\end{align*}
\[
\begin{align*}
f^*(v_{i+1}) &= 1 \text{ if } i \text{ is odd and } j \text{ is odd} \\
&= 0 \text{ if } i \text{ is even and } j \text{ is odd} \\
&= 0 \text{ if } i \text{ is odd and } j \text{ is even} \\
&= 1 \text{ if } i \text{ is even and } j \text{ is even}.
\end{align*}
\]

It can be verified that $|e_f(0) - e_f(1)| \leq 1$. Hence $T \odot K_n$ is sum divisor cordial graph. 
\qed
Example 4. Sum divisor cordial labeling of $T \odot K_4$ where $T$ is a $T_p$-tree with 10 vertices is shown in Figure 5.

Theorem 6. If $T$ be a $T_p$-tree on $m$ vertices, then the graph $T\tilde{Q}_n$ is sum divisor cordial graph.

Proof. Let $T$ be a $T_p$-tree with $m$ vertices. By the definition of a transformed tree there exists a parallel transformation $P$ of $T$ such that for the path $P(T)$, we have (i) $V(P(T)) = V(T)$ and (ii) $E(P(T)) = (E(T) - E_d) \cup E_p$, where $E_d$ is the set of edges deleted from $T$ and $E_p$ is the set of edges newly added through the sequence $P = (P_1, P_2, \ldots, P_k)$ of the epts $P$ used to arrive at the path $P(T)$. Clearly, $E_d$ and $E_p$ have the same number of edges. Denote the vertices of $P(T)$ successively as $v_1, v_2, \ldots, v_m$ starting from one pendant vertex of $P(T)$ right up to the other. Let $u'_1, u'_2, \ldots, u'_n, u'_{n+1}$ $(1 \leq j \leq m)$ be the vertices of $j^{th}$ copy of $Q_n$ with $u'_{n+1} = v_j$. Then $V(T\tilde{Q}_n) = \{u'_i : 1 \leq i \leq n + 1, 1 \leq j \leq m\} \cup \{x'_i, y'_i : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(T\tilde{Q}_n) = E(T) \cup E(Q_n)$. We note that $|V(T\tilde{Q}_n)| = 3nm + m$ and $|E(T\tilde{Q}_n)| = 4mn + m - 1$.

Define $f : V(T\tilde{Q}_n) \to \{1, 2, \ldots, 3nm + m\}$ as follows:

Case 1. $m$ is odd.

For $1 \leq j \leq m$ and $1 \leq i \leq n$,

- $f(u'_i) = m + 3n(j - 1) + 3i - 2$;
- $f(x'_i) = m + 3n(j - 1) + 3i - 1$;
- $f(y'_i) = m + 3n(j - 1) + 3i$;
- $f(v_j) = f(u'_{n+1}) = \begin{cases} j & \text{if } j \equiv 1, 0 \pmod{4} \\ j + 1 & \text{if } j \equiv 2 \pmod{4} \\ j - 1 & \text{if } j \equiv 3 \pmod{4} \end{cases}$.

Let $v_i v_j$ be a transformed edge in $T$, $1 \leq i < j \leq m$ and let $P_1$ be the ept obtained by deleting the edge $v_i v_j$ and adding the edge $v_{i+t} v_{j-t}$, where $t$ is the distance of $v_i$ from $v_{i+t}$ and the distance of $v_j$ from $v_{j-t}$. Let $P$ be a parallel transformation of $T$ that contains $P_1$ as one of the constituent epts.

Since $v_{i+t} v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. Therefore, $i$ and $j$ are of opposite parity.

Figure 5. Sum divisor cordial labeling of $T \odot K_4$ where $T$ is a $T_p$-tree with 10 vertices.
The induced edge label of $v_i v_j$ is given by

$$f^*(v_i v_j) = f^*(v_i v_{i+2r+1}) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even.} \end{cases}$$

The induced edge label of $v_{i+r} v_{j-t}$ is given by

$$f^*(v_{i+r} v_{j-t}) = f^*(v_{i+r} v_{i+t+1}) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even.} \end{cases}$$

Therefore, $f^*(v_i v_j) = f^*(v_{i+r} v_{j-t})$.

The induced edge labels are as follows:

$$f^*(v_i v_{j+1}) = \begin{cases} 1 & \text{if } i \text{ is odd and } 1 \leq j \leq m - 1 \\ 0 & \text{if } i \text{ is even and } 1 \leq j \leq m - 1; \end{cases}$$

for $1 \leq j \leq m$,

$$f^*(u_i^j v_j) = 0, \quad 1 \leq i \leq n;$$
$$f^*(u_i^j v_{j+1}) = 1, \quad 1 \leq i \leq n;$$
$$f^*(x_i^j u_{i+1}^j) = 1, \quad 1 \leq i \leq n - 1;$$
$$f^*(y_i^j u_{i+1}^j) = 0, \quad 1 \leq i \leq n - 1;$$

$$f^*(x_i^j v_j) = \begin{cases} 1 & \text{if } n \text{ is odd and } j \equiv 1, 0 \text{ (mod 4)} \\ 0 & \text{if } n \text{ is odd and } j \equiv 2, 3 \text{ (mod 4)} \\ 0 & \text{if } n \text{ is even and } j \equiv 1, 2 \text{ (mod 4)} \\ 1 & \text{if } n \text{ is even and } j \equiv 3, 0 \text{ (mod 4);} \end{cases}$$

$$f^*(y_i^j v_j) = \begin{cases} 1 & \text{if } n \text{ is odd and } j \equiv 2, 3 \text{ (mod 4)} \\ 1 & \text{if } n \text{ is even and } j \equiv 1, 2 \text{ (mod 4)} \\ 0 & \text{if } n \text{ is even and } j \equiv 3, 0 \text{ (mod 4).} \end{cases}$$

**Case 2.** $m$ is even.

For $1 \leq j \leq m$ and $1 \leq i \leq n$,

$$f(u_i^j) = m + 3n(j - 1) + 3i - 2;$$
$$f(x_i^j) = m + 3n(j - 1) + 3i - 1;$$
$$f(y_i^j) = m + 3n(j - 1) + 3i;$$

$$f(v_j) = f(u_{i+1}^j) = \begin{cases} j & \text{if } j \equiv 1, 2 \text{ (mod 4)} \\ j + 1 & \text{if } j \equiv 3 \text{ (mod 4)} \\ j - 1 & \text{if } j \equiv 0 \text{ (mod 4).} \end{cases}$$

Let $v_i v_j$ be a transformed edge in $T$, $1 \leq i < j \leq m$ and let $P_1$ be the ept obtained by deleting the edge $v_i v_j$ and adding the edge $v_{i+r} v_{j-t}$, where $t$ is the distance of $v_i$ from $v_{i+r}$ and the distance of $v_j$ from $v_{j-t}$. Let $P$ be a parallel transformation of $T$ that contains $P_1$ as one of the constituent epts.

Since $v_{i+r} v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2r + 1$. Therefore, $i$ and $j$ are of opposite parity.

The induced edge label of $v_i v_j$ is given by

$$f^*(v_i v_j) = f^*(v_i v_{i+2r+1}) = \begin{cases} 0 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \text{ is even.} \end{cases}$$
The induced edge label of $v_{i+1}v_{j-1}$ is given by

$$f^*(v_{i+1}v_{j-1}) = f^*(v_{i+2}v_{i+1})$$

$$= \begin{cases} 
0 & \text{if } i \text{ is odd} \\
1 & \text{if } i \text{ is even.}
\end{cases}$$

Therefore, $f^*(v_iv_j) = f^*(v_{i+1}v_{j-1})$.

The induced edge labels are as follows:

$$f^*(v_iv_j) = \begin{cases} 
0 & \text{if } i \text{ is odd and } 1 \leq j \leq m - 1 \\
1 & \text{if } i \text{ is even and } 1 \leq j \leq m - 1;
\end{cases}$$

for $1 \leq j \leq m$,

$$f^*(u_i'x_j') = 0, \quad 1 \leq i \leq n;$$

$$f^*(u_i'y_j') = 1, \quad 1 \leq i \leq n;$$

$$f^*(x_i'u_{i+1}') = 1, \quad 1 \leq i \leq n - 1;$$

$$f^*(y_i'u_{i+1}') = 0, \quad 1 \leq i \leq n - 1;$$

$$f^*(x_i'v_j) = \begin{cases} 
0 & \text{if } n \text{ is odd and } j \equiv 1, 2 \pmod{4} \\
1 & \text{if } n \text{ is odd and } j \equiv 3, 0 \pmod{4} \\
1 & \text{if } n \text{ is even and } j \equiv 1, 0 \pmod{4} \\
0 & \text{if } n \text{ is even and } j \equiv 2, 3 \pmod{4};
\end{cases}$$

$$f^*(y_i'v_j) = \begin{cases} 
1 & \text{if } n \text{ is odd and } j \equiv 1, 2 \pmod{4} \\
0 & \text{if } n \text{ is odd and } j \equiv 3, 0 \pmod{4} \\
1 & \text{if } n \text{ is even and } j \equiv 2, 3 \pmod{4} \\
0 & \text{if } n \text{ is even and } j \equiv 1, 0 \pmod{4}.
\end{cases}$$

In the above two cases, when $m$ is odd,

$$e_f(1) = e_f(0) = \frac{4mn^2 + m - 1}{2};$$

when $m$ is even,

$$e_f(1) = \left\lfloor \frac{4mn^2 + m - 1}{2} \right\rfloor$$

and

$$e_f(0) = \left\lceil \frac{4mn^2 + m - 1}{2} \right\rceil.$$ 

It can be verified that $|e_f(0) - e_f(1)| \leq 1$. Hence $T\overline{O}Q_n$ is sum divisor cordial graph.

**Example 5.** Sum divisor cordial labeling of $T\overline{O}Q_2$ where $T$ is a $T_p$-tree with 8 vertices is shown in Figure 6.

**Theorem 7.** If $T$ be a $T_p$-tree on $m$ vertices, then the graph $T\overline{O}C_n$ is sum divisor cordial graph if $n \equiv 0, 3, 1 \pmod{4}$.

**Proof.** Let $T$ be a $T_p$-tree with $m$ vertices. By the definition of a transformed tree there exists a parallel transformation $P$ of $T$ such that for the path $P(T)$, we have (i) $V(P(T)) = V(T)$ and (ii) $E(P(T)) = (E(T) - E_d) \cup E_p$, where $E_d$ is the set of edges deleted from $T$ and $E_p$ is the set of edges newly added through the sequence $P = (P_1, P_2, \ldots, P_k)$ of the epts $P$ used to arrive at the path $P(T)$. Clearly, $E_d$ and $E_p$ have the same number of edges. Denote the vertices of $P(T)$ successively as $v_1, v_2, \ldots, v_m$ starting from one pendant vertex of $P(T)$ right up to the other. Let $u_1', u_2', \ldots, u_m'(1 \leq j \leq m)$ be the vertices of $j^{th}$ copy of $C_n$. Then

$$V(T\overline{O}C_n) = \{v_j, u_j' : 1 \leq i \leq n, 1 \leq j \leq m\} \text{ and } E(T\overline{O}C_n) = E(T) \cup E(C_n) \cup \{v_ju_i' : 1 \leq j \leq m\}.$$ 

Define $f : V(T\overline{O}C_n) \rightarrow \{1, 2, \ldots, mn + m\}$ as follows:

**Case 1.** $n \equiv 0 \pmod{4}$.

$$f(v_j) = (n + 1)j, \quad 1 \leq j \leq m;$$

for $1 \leq j \leq m$ and $1 \leq i \leq n,$
Let $v_i v_j$ be a transformed edge in $T$, $1 \leq i < j \leq m$ and let $P_1$ be the ept obtained by deleting the edge $v_i v_j$ and adding the edge $v_{i+t} v_{j-t}$ where $t$ is the distance of $v_i$ from $v_{i+t}$ and also the distance of $v_j$ from $v_{j-t}$. Let $P$ be a parallel transformation of $T$ that contains $P_1$ as one of the constituent epts. Since $v_{i+t} v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. Therefore, $i$ and $j$ are of opposite parity.

The induced edge label of $v_i v_j$ is given by

$$f^*(v_i v_j) = f^*(v_{i+t} v_{j-t+1})$$

$$= 0.$$ 

The induced edge label of $v_{i+t} v_{j-t}$ is given by

$$f^*(v_{i+t} v_{j-t}) = f^*(v_{i+t} v_{i+t+1})$$

$$= 0.$$ 

Therefore, $f^*(v_i v_j) = f^*(v_{i+t} v_{j-t})$.

The induced edge labels are as follows:

$$f^*(v_j v_{j+1}) = 0, \ 1 \leq j \leq m - 1;$$
$$f^*(u_i^j v_j) = 1, \ 1 \leq j \leq m;$$
$$f^*(u_i^j u_{i+1}^j) = 0, \ 1 \leq j \leq m;$$

for $1 \leq j \leq m$,

$$f^*(u_i^j u_{i+1}^j) = \begin{cases} 1 & \text{if } i \text{ is odd and } 1 \leq i \leq n - 1 \\ 0 & \text{if } j \text{ is even and } 1 \leq i \leq n - 1. \end{cases}$$

**Case 2.** $n \equiv 3 \pmod{4}.$
Let $v_iv_j$ be a transformed edge in $T$, $1 \leq i < j \leq m$ and let $P_1$ be the ept obtained by deleting the edge $v_iv_j$ and adding the edge $v_{i+t}v_{j-t}$ where $t$ is the distance of $v_i$ from $v_{i+t}$ and the distance of $v_j$ from $v_{j-t}$. Let $P$ be a parallel transformation of $T$ that contains $P_1$ as one of the constituent epts. Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. Therefore, $i$ and $j$ are of opposite parity.

The induced edge label of $v_iv_j$ is given by

$$f^*(v_iv_j) = f^*(v_{i+2t+1})$$

$$= \begin{cases} 
1 & \text{if } i \text{ is odd} \\
0 & \text{if } i \text{ is even.} 
\end{cases}$$

The induced edge label of $v_{i+t}v_{j-t}$ is given by

$$f^*(v_{i+t}v_{j-t}) = f^*(v_{i+2t+1})$$

$$= \begin{cases} 
1 & \text{if } i \text{ is odd} \\
0 & \text{if } i \text{ is even.} 
\end{cases}$$

Therefore, $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.

The induced edge labels are as follows:

$$f^*(v_iv_j) = \begin{cases} 
1 & \text{if } i \text{ is odd and } 1 \leq j \leq m - 1 \\
0 & \text{if } i \text{ is even and } 1 \leq j \leq m - 1 \end{cases}$$

for $1 \leq j \leq m$, $f^*(u^i_iv^j_i) = 1$; $f^*(u^i_jv^j_i) = 0$; $f^*(u^i_iu^j_i) = \begin{cases} 
0 & \text{if } i \text{ is odd and } 1 \leq i \leq n - 1 \\
1 & \text{if } i \text{ is even and } 1 \leq i \leq n - 1 \end{cases}$.

**Case 3, $n \equiv 1 \pmod{4}$**.

$$f(v_i) = \begin{cases} 
(n + 1)j & \text{if } j \equiv 1, 2 \pmod{4} \text{ and } 1 \leq j \leq m \\
(n + 1)(j - 1) + 1 & \text{if } j \equiv 3, 0 \pmod{4} \text{ and } 1 \leq j \leq m 
\end{cases}$$

choose ‘if $j \equiv 1, 2 \pmod{4}$’ and $1 \leq j \leq m$,

$$f(u^i_i) = \begin{cases} 
(n + 1)(j - 1) + i & \text{if } i \equiv 1, 0 \pmod{4} \text{ and } 1 \leq i \leq n \\
(n + 1)(j - 1) + i + 1 & \text{if } i \equiv 2 \pmod{4} \text{ and } 1 \leq i \leq n \\
(n + 1)(j - 1) + i - 1 & \text{if } i \equiv 3 \pmod{4} \text{ and } 1 \leq i \leq n 
\end{cases}$$

choose ‘if $j \equiv 3, 0 \pmod{4}$’ and $1 \leq j \leq m$,

$$f(u^j_i) = \begin{cases} 
(n + 1)(j - 1) + i + 1 & \text{if } i \equiv 1, 0 \pmod{4} \text{ and } 1 \leq i \leq n \\
(n + 1)(j - 1) + i + 2 & \text{if } i \equiv 2 \pmod{4} \text{ and } 1 \leq i \leq n \\
(n + 1)(j - 1) + i & \text{if } i \equiv 3 \pmod{4} \text{ and } 1 \leq i \leq n 
\end{cases}$$
Let \( v_i v_j \) be a transformed edge in \( T \), \( 1 \leq i < j \leq m \) and let \( P_1 \) be the ept obtained by deleting the edge \( v_i v_j \) and adding the edge \( v_{i+t} v_{j-t} \), where \( t \) is the distance of \( v_i \) from \( v_{i+t} \) and the distance of \( v_j \) from \( v_{j-t} \). Let \( P \) be a parallel transformation of \( T \) that contains \( P_1 \) as one of the constituent epts. Since \( v_{i+t} v_{j-t} \) is an edge in the path \( P(T) \), it follows that \( i + t + 1 = j - t \) which implies \( j = i + 2t + 1 \). Therefore, \( i \) and \( j \) are of opposite parity.

The induced edge label of \( v_i v_j \) is given by

\[
f'(v_i v_j) = f'(v_{i+2t+1})
\]

\[
= \begin{cases} 
1 & \text{if } i \text{ is odd} \\
0 & \text{if } i \text{ is even.}
\end{cases}
\]

The induced edge label of \( v_{i+t} v_{j-t} \) is given by

\[
f'(v_{i+t} v_{j-t}) = f'(v_{i+t+1})
\]

\[
= \begin{cases} 
1 & \text{if } i \text{ is odd} \\
0 & \text{if } i \text{ is even.}
\end{cases}
\]

Therefore, \( f'(v_i v_j) = f'(v_{i+t} v_{j-t}) \).

The induced edge labels are as follows:

\[
f'(v_{i+j+1}) = \begin{cases} 
1 & \text{if } i \text{ is odd and } 1 \leq j \leq m - 1 \\
0 & \text{if } i \text{ is even and } 1 \leq j \leq m - 1.
\end{cases}
\]

for \( 1 \leq j \leq m \),

\[
f'(u_i^l v_j) = 0; \\
f'(u_i^r u_{i+1}^l) = 1.
\]

In the above three cases, when \( m \) is odd,

\[
e_f(1) = \begin{cases} 
\frac{mn+2m-1}{2} & \text{if } n \equiv 1, 3 \pmod{4} \\
\frac{mn+2m-1}{2} & \text{if } n \equiv 0 \pmod{4},
\end{cases}
\]

\[
e_f(0) = \begin{cases} 
\frac{mn+2m-1}{2} & \text{if } n \equiv 1, 3 \pmod{4} \\
\frac{mn+2m-1}{2} & \text{if } n \equiv 0 \pmod{4};
\end{cases}
\]

when \( m \) is even and \( n \equiv 1, 3, 0 \pmod{4} \),

\[
e_f(1) = \left\lfloor \frac{mn+2m-1}{2} \right\rfloor \text{ and } e_f(0) = \left\lfloor \frac{mn+2m-1}{2} \right\rfloor.
\]

It can be verified that \( |e_f(0) - e_f(1)| \leq 1 \). Hence \( T\tilde{O}C_n \) is sum divisor cordial graph. \( \square \)

**Example 6.** Sum divisor cordial labeling of \( T\tilde{O}C_5 \) where \( T \) is a \( T_p \)-tree with 8 vertices is shown in Figure 7.

**Theorem 8.** If \( T \) be a \( T_p \)-tree on \( m \) vertices, then the graph \( T\tilde{O}Q_n \) is sum divisor cordial graph.

**Proof.** Let \( T \) be a \( T_p \)-tree with \( m \) vertices. By the definition of a transformed tree there exists a parallel transformation \( P \) of \( T \) such that for the path \( P(T) \), we have (i) \( V(P(T)) = V(T) \) and (ii) \( E(P(T)) = (E(T) - E_d) \cup E_p \), where \( E_d \) is the set of edges deleted from \( T \) and \( E_p \) is the set of edges newly added through the sequence \( P = (P_1, P_2, \cdots, P_k) \) of the epts \( P \) used to arrive at the path \( P(T) \). Clearly, \( E_d \) and \( E_p \) have the same number of edges. Denote the vertices of \( P(T) \) successively as \( v_1, v_2, \cdots, v_m \) starting from one pendant vertex of \( P(T) \) right up to the other. Let \( u_i^l, u_i^r, \cdots, u_{n+1}^l, u_{n+1}^r (1 \leq j \leq m) \) be the vertices of \( j^{th} \) copy of \( Q_n \). Then \( V(T\tilde{O}Q_n) = \{v_j, u_i^l : 1 \leq i \leq n + 1, 1 \leq j \leq m\} \cup \{x_i^l, y_i^l : 1 \leq i \leq n, 1 \leq j \leq m\} \) and
**Figure 7.** Sum divisor cordial labeling of $T\tilde{O}C_5$ where $T$ is a $T_p$-tree with 8 vertices

$$E(T\tilde{O}Q_n) = E(T) \cup E(Q_n) \cup \{v_j u_{j+1} : 1 \leq j \leq m\}.$$ We note that $|V(T\tilde{O}Q_n)| = m(3n + 2)$ and $|E(T\tilde{O}Q_n)| = 4mn + 2m - 1$. Define $f : V(T\tilde{O}Q_n) \rightarrow \{1, 2, \ldots, m(3n + 2)\}$ as follows:

**Case 1.** $n$ is odd.

- $f(v_j) = (3n + 2)j$, $1 \leq j \leq m$;
- for $1 \leq j \leq m$ and $1 \leq i \leq n + 1$,
  
  $$f(u_{ij}) = \begin{cases} 
  (3n + 2)(j - 1) + 3i - 1 & \text{if } i \text{ is odd} \\
  (3n + 2)(j - 1) + 3i - 3 & \text{if } i \text{ is even} 
  \end{cases};$$
- for $1 \leq j \leq m$ and $1 \leq i \leq n$,
  
  $$f(x_{ij}) = \begin{cases} 
  (3n + 2)(j - 1) + 3i - 2 & \text{if } i \text{ is odd} \\
  (3n + 2)(j - 1) + 3i - 1 & \text{if } i \text{ is even} 
  \end{cases};$$
- $$f(y_{ij}) = \begin{cases} 
  (3n + 2)(j - 1) + 3i + 1 & \text{if } i \text{ is odd} \\
  (3n + 2)(j - 1) + 3i & \text{if } i \text{ is even}. 
  \end{cases}$$

Let $v_i v_j$ be a transformed edge in $T$, $1 \leq i < j \leq m$ and let $P_1$ be the ept obtained by deleting the edge $v_i v_j$ and adding the edge $v_{i+t} v_{j-t}$ where $t$ is the distance of $v_i$ from $v_{i+t}$ and also the distance of $v_j$ from $v_{j-t}$. Let $P$ be a parallel transformation of $T$ that contains $P_1$ as one of the constituent epts. Since $v_{i+t} v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. Therefore, $i$ and $j$ are of opposite parity.

The induced edge label of $v_i v_j$ is given by

$$f^*(v_i v_j) = f^*(v_i v_{i+2t+1}) = 0.$$

The induced edge label of $v_{i+t} v_{j-t}$ is given by

$$f^*(v_{i+t} v_{j-t}) = f^*(v_{i+t} v_{i+t+1}).$$
Therefore, $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.

The induced edge labels are as follows:

- $f^*(v_iv_{j+1}) = 0$, $1 \leq j \leq m - 1$;
- for $1 \leq j \leq m$,
  - $f^*(u_{n+1}^jv_j) = 1$;
  - $f^*(u_i^jx_i^j) = 0$, $1 \leq i \leq n$;
  - $f^*(u_i^jy_i^j) = 1$, $1 \leq i \leq n$;
  - $f^*(x_i^ju_{i+1}^j) = 1$, $1 \leq i \leq n$;
  - $f^*(y_i^ju_{i+1}^j) = 0$, $1 \leq i \leq n$.

Case 2. $n$ is even.

- $f(v_j) = (3n + 2)j$, $1 \leq j \leq m$;
- for $1 \leq j \leq m$,
  - $f(u_i^j) = (3n + 2)(j - 1) + 3i - 2$, $1 \leq i \leq n + 1$;
  - $f(x_i^j) = (3n + 2)(j - 1) + 3i - 1$, $1 \leq i \leq n$;
  - $f(y_i^j) = (3n + 2)(j - 1) + 3i$, $1 \leq i \leq n$.

Let $v_iv_j$ be a transformed edge in $T$, $1 \leq i < j \leq m$ and let $P_1$ be the $ept$ obtained by deleting the edge $v_iv_j$ and adding the edge $v_{i+t}v_{j-t}$ where $i$ is the distance of $v_i$ from $v_{i+t}$ and also the distance of $v_j$ from $v_{j-t}$. Let $P$ be a parallel transformation of $T$ that contains $P_1$ as one of the constituent $epts$. Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. Therefore, $i$ and $j$ are of opposite parity.

The induced edge label of $v_iv_j$ is given by

$$f^*(v_iv_j) = f^*(v_{iV_{i+2t+1}}) = 2(f(v_i) + f(v_{i+2t+1})) = 1.$$ 

The induced edge label of $v_{i+t}v_{j-t}$ is given by

$$f^*(v_{i+t}v_{j-t}) = f^*(v_{i+t}v_{i+t+1}) = 2(f(v_{i+t}) + f(v_{i+t+1})) = 1.$$ 

Therefore, $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.

The induced edge labels are as follows:

- $f^*(v_iv_{j+1}) = 1$, $1 \leq j \leq m - 1$;
- for $1 \leq j \leq m$,
  - $f^*(u_{n+1}^jv_j) = 0$;
  - $f^*(u_i^jx_i^j) = 0$, $1 \leq i \leq n$;
  - $f^*(u_i^jy_i^j) = 1$, $1 \leq i \leq n$;
  - $f^*(x_i^ju_{i+1}^j) = 1$, $1 \leq i \leq n$;
  - $f^*(y_i^ju_{i+1}^j) = 0$, $1 \leq i \leq n$.

In above two cases, it can be verified that $|e_f(1) - e_f(0)| \leq 1$. Hence $\tilde{TQ}_n$ is sum divisor cordial graph.

\[ \square \]

**Example 7.** Sum divisor cordial labeling of $\tilde{TQ}_2$ where $T$ is a $T_p$-tree with 8 vertices is shown in Figure 8.
Figure 8. Sum divisor cordial labeling of $T\bar{Q}_2$ where $T$ is a $T_p$-tree with 8 vertices

Conflict of Interest

The authors declare no conflict of interests.

References


