Ars Combinatoria, 157: 3–22
DOI:10.61091/ars157-01
http://www.combinatorialpress.com/ars
Received 22 April 2020, Accepted 20 January 2021, Published 22 November 2023

Combinatorial Press

Article

Sum Divisor Cordial Labeling of T_p-Tree Related Graphs

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Abstract: A sum divisor cordial labeling of a graph G with vertex set $V(G)$ is a bijection f from $V(G)$ to $\{1, 2, \cdots, |V(G)|\}$ such that an edge uv is assigned the label 1 if 2 divides $f(u) + f(v)$ and 0 otherwise; and the number of edges labeled with 1 and the number of edges labeled with 0 differ by at most 1. A graph with a sum divisor cordial labeling is called a sum divisor cordial graph. In this paper, we discuss the sum divisor cordial labeling of transformed tree related graphs.

Keywords: Sum divisor cordial labeling, corona, transformed tree.
Mathematics Subject Classification: 05C78.

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1. Introduction

All graphs considered here are simple, finite, connected and undirected. The vertex set and the edge set of a graph are denoted by $V(G)$ and $E(G)$ respectively. We follow the basic notations and terminology of graph theory as in [1]. A labeling of a graph is a map that carries the graph elements to the set of numbers, usually to the set of non-negative or positive integers. If the domain is the set of vertices then the labeling is called vertex labeling. If the domain is the set of edges then the labeling is called edge labeling. If the labels are assigned to both vertices and edges then the labeling is called total labeling. A detailed survey of graph labeling is available in [2]. The concept of cordial labeling was introduced by Cahit in [3].

Lourdusamy et al. introduced the concept of sum divisor cordial labeling in [4]. They prove that paths, combs, stars, complete bipartite, $K_2 + mK_1$, bistars, jewels, crowns, flowers, gears, subdivisions of stars, the graph obtained from $K_{1,3}$ by attaching the root of $K_{1,n}$ at each pendent vertex of $K_{1,3}$, and the square $B_{n,n}$ are sum divisor cordial graphs. Also they discussed the sum divisor labeling of star related graphs, path related graphs and cycle related graphs in [5–7].

In [8–11], Sugumaran et al. investigated the behaviour of sum divisor cordial labeling of swastiks, path unions of finite number of copies of swastiks, cycles of k copies of swastiks, when k is odd, jelly fish, Petersen graphs, theta graphs, the fusion of any two vertices in the cycle of swastiks, duplication of any vertex in the cycle of swastiks, the switchings of a central vertex in swastiks, the path unions of two copies of a swastik, the star graph of the theta graphs, the Herschel graph, the fusion of any

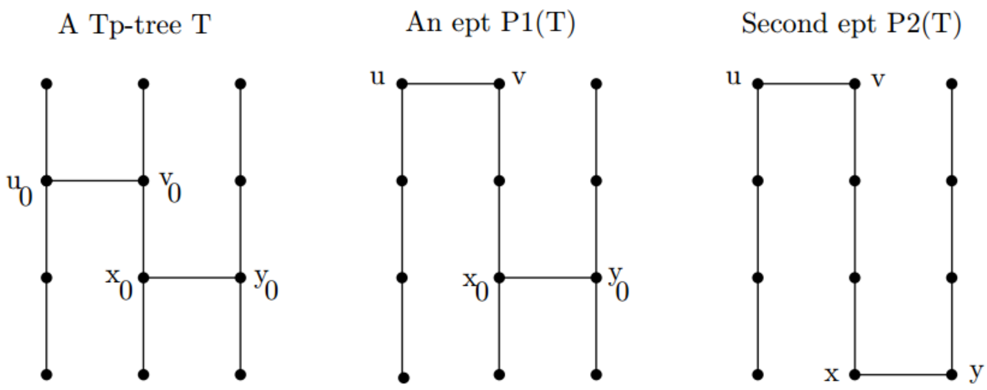

Figure 1. A T_P-tree and a sequence of two ept's reducing it to a path

two adjacent vertices of degree 3 in Herschel graphs, the duplication of any vertex of degree 3 in the Herschel graph, the switching of central vertex in Herschel graph, the path union of two copies of the Herschel graph, H-graph H_n, when n is odd, $C_3@K_{1,n}$, $< F_n^1 \Delta F_n^2 >$ and open star of swastik graphs $S(t.Sw_n)$, when t is odd.

In [12–15] Sugumaran et al. proved that the following graphs are sum divisor cordial graphs: H-graph H_n, when n is even, duplication of all edges of the H-graph H_n, when n is even, $H_n \odot K_1$, $P(r.H_n)$, $C(r.H_n)$, plus graphs, umbrella graphs, path unions of odd cycles, kites, complete binary trees, drums graph, twigs, fire crackers of the form $P_n \odot S_n$, where n is even, and the double arrow graph DA_m^n, where $|m - n| \leq 1$ and n is even. Further results on sum divisor cordial labeling are given in [16, 17].

In this paper, we discuss the sum divisor cordial labeling of transformed tree related graphs like $T\widehat{OP}_n$, $T\widehat{OC}_n$ ($n \equiv 1, 3, 0 \ (mod\ 4)$), $T\widehat{OK}_{1,n}$, $T\widehat{O\bar{K}}_n$, $T\widehat{OQ}_n$, $T\widetilde{OC}_n$ ($n \equiv 1, 3, 0 \ (mod\ 4)$) and $T\widetilde{OQ}_n$. We use the following definitions in the subsequent sections.

Definition 1. *Let $G = (V(G), E(G))$ be a simple graph and $f : V(G) \to \{1, 2, \cdots, |V(G)|\}$ be a bijection. For each edge uv, assign the label 1 if $2|(f(u) + f(v))$ and the label 0 otherwise. The function f is called a sum divisor cordial labeling if $|e_f(1) - e_f(0)| \leq 1$. A graph which admits a sum divisor cordial labeling is called a sum divisor cordial graph.*

Definition 2. *[18] Let T be a tree and u_0 and v_0 be two adjacent vertices in T. Let there be two pendant vertices u and v in T such that the length of $u_0 - u$ path is equal to the length of $v_0 - v$ path. If the edge u_0v_0 is deleted from T and u, v are joined by an edge uv, then such a transformation of T is called an elementary parallel transformation (or an ept) and the edge u_0v_0 is called transformable edge.*

If by the sequence of ept's, T can be reduced to a path, then T is called a T_p-tree (transformed tree) and such a sequence regarded as a composition of mappings (ept's) denoted by P, is called a parallel transformation of T. The path, the image of T under P is denoted as $P(T)$.

Definition 3. *The corona $G_1 \odot G_2$ of two graphs $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ is defined as the graph obtained by taking one copy of G_1 and p_1 copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2.*

Definition 4. *[19] Let G_1 be a graph with p vertices and G_2 be any graph. A graph $G_1 \hat{O} G_2$ is obtained from G_1 and p copies of G_2 by identifying one vertex of i^{th} copy of G_2 with i^{th} vertex of G_1.*

Definition 5. *[19] Let G_1 be a graph with p vertices and G_2 be any graph. A graph $G_1\tilde{O}G_2$ is obtained from G_1 and p copies of G_2 by joining one vertex of i^{th} copy of G_2 with i^{th} vertex of G_1 by an edge.*

Theorem 1. *[7] Every T_p-tree is sum divisor cordial graph.*

2. T_p-Tree related graphs

Theorem 2. *If T be a T_p-tree on m vertices, then the graph $T\widehat{O}P_n$ is sum divisor cordial graph.*

Proof. Let T be a T_p-tree with m vertices. By the definition of a transformed tree there exists a parallel transformation P of T such that for the path $P(T)$, we have (i) $V(P(T)) = V(T)$ and (ii) $E(P(T)) = (E(T) - E_d) \bigcup E_p$, where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequence $P = (P_1, P_2, \cdots, P_k)$ of the *epts* P used to arrive at the path $P(T)$. Clearly, E_d and E_p have the same number of edges. Denote the vertices of $P(T)$ successively as $v_1, v_2, \cdots, v_m$ starting from one pendant vertex of $P(T)$ right up to the other. Let $u_1^j, u_2^j, \cdots, u_n^j (1 \le j \le m)$ be the vertices of j^{th} copy of P_n with $u_1^j = v_j$. Then $V(T\widehat{O}P_n) = \{v_j, u_i^j : 1 \le i \le n, 1 \le j \le m$ with $u_1^j = v_j\}$ and $E(T\widehat{O}P_n) = E(T) \bigcup \{u_i^j u_{i+1}^j : 1 \le i \le n-1, 1 \le j \le m\}$.
Define $f : V(T\widehat{O}P_n) \to \{1, 2, \ldots, mn\}$ as follows:
Case 1. n is even.
For $1 \le j \le m$ and $1 \le i \le n$,

$$f(u_i^j) = \begin{cases} n(j-1)+i+1 & \text{if } i \equiv 1\ (mod\ 4) \\ n(j-1)+i-1 & \text{if } i \equiv 2\ (mod\ 4) \\ n(j-1)+i & \text{if } i \equiv 3, 0\ (mod\ 4)\,. \end{cases}$$

Let $v_i v_j$ be a transformed edge in T, $1 \le i < j \le m$ and let P_1 be the *ept* obtained by deleting the edge $v_i v_j$ and adding the edge $v_{i+t} v_{j-t}$ where t is the distance of v_i from v_{i+t} and the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent *epts*.

Since $v_{i+t} v_{j-t}$ is an edge in the path $P(T)$, it follows that $i+t+1 = j-t$ which implies $j = i+2t+1$. Therefore, i and j are of opposite parity.
The induced edge label of $v_i v_j$ is given by

$$\begin{aligned} f^*(v_i v_j) &= f^*(v_i v_{i+2t+1}) \\ &= 2|(f(v_i) + f(v_{i+2t+1})) \\ &= 1. \end{aligned}$$

The induced edge label of $v_{i+t} v_{j-t}$ is given by

$$\begin{aligned} f^*(v_{i+t} v_{j-t}) &= f^*(v_{i+t} v_{i+t+1}) \\ &= 2|(f(v_{i+t}) + f(v_{i+t+1})) \\ &= 1. \end{aligned}$$

Therefore, $f^*(v_i v_j) = f^*(v_{i+t} v_{j-t})$.
The induced edge labels are as follows:

$$f^*(v_j v_{j+1}) = 1,\ 1 \le j \le m-1;$$

for $1 \le i \le n-1$ and $1 \le j \le m$,

$$f^*(u_i^j u_{i+1}^j) = \begin{cases} 0 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \text{ is even}\,. \end{cases}$$

Case 2. n is odd.

For $1 \leq j \leq m$,
choose 'if $j \equiv 1, 2\ (mod\ 4)$',

$$f(u_i^j) = \begin{cases} n(j-1)+i & \text{if } i \equiv 1, 0\ (mod\ 4) \text{ and } 1 \leq i \leq n \\ n(j-1)+i+1 & \text{if } i \equiv 2\ (mod\ 4) \text{ and } 1 \leq i \leq n \\ n(j-1)+i-1 & \text{if } i \equiv 3\ (mod\ 4) \text{ and } 1 \leq i \leq n\ ; \end{cases}$$

choose 'if $j \equiv 3, 0\ (mod\ 4)$' and $n \equiv 3\ (mod\ 4)$,

$$f(u_i^j) = \begin{cases} n(j-1)+i+1 & \text{if } i \equiv 1\ (mod\ 4) \text{ and } 1 \leq i \leq n \\ n(j-1)+i-1 & \text{if } i \equiv 2\ (mod\ 4) \text{ and } 1 \leq i \leq n \\ n(j-1)+i & \text{if } i \equiv 3, 0\ (mod\ 4) \text{ and } 1 \leq i \leq n\ ; \end{cases}$$

choose 'if $j \equiv 3, 0\ (mod\ 4)$' and $n \equiv 1\ (mod\ 4)$,

$$f(u_i^j) = \begin{cases} n(j-1)+i+1 & \text{if } i \equiv 1\ (mod\ 4) \text{ and } 1 \leq i \leq n-2 \\ n(j-1)+i-1 & \text{if } i \equiv 2\ (mod\ 4) \text{ and } 1 \leq i \leq n-2 \\ n(j-1)+i & \text{if } i \equiv 3, 0\ (mod\ 4) \text{ and } 1 \leq i \leq n-2 \\ n(j-1)+i+1 & \text{if } i = n-1 \\ n(j-1)+i-1 & \text{if } i = n\ . \end{cases}$$

Let $v_i v_j$ be a transformed edge in T, $1 \leq i < j \leq m$ and let P_1 be the *ept* obtained by deleting the edge $v_i v_j$ and adding the edge $v_{i+t} v_{j-t}$ where t is the distance of v_i from v_{i+t} and the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent *epts*.

Since $v_{i+t} v_{j-t}$ is an edge in the path $P(T)$, it follows that $i+t+1 = j-t$ which implies $j = i+2t+1$. Therefore, i and j are of opposite parity.
The induced edge label of $v_i v_j$ is given by

$$\begin{aligned} f^*(v_i v_j) &= f^*(v_i v_{i+2t+1}) \\ &= \begin{cases} 0 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \text{ is even.} \end{cases} \end{aligned}$$

The induced edge label of $v_{i+t} v_{j-t}$ is given by

$$\begin{aligned} f^*(v_{i+t} v_{j-t}) &= f^*(v_{i+t} v_{i+t+1}) \\ &= \begin{cases} 0 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \text{ is even.} \end{cases} \end{aligned}$$

Therefore, $f^*(v_i v_j) = f^*(v_{i+t} v_{j-t})$.
The induced edge labels are as follows:

$$f^*(v_j v_{j+1}) = \begin{cases} 0 & \text{if } i \text{ is odd and } 1 \leq j \leq m-1 \\ 1 & \text{if } i \text{ is even and } 1 \leq j \leq m-1\ ; \end{cases}$$

for $1 \leq j \leq m$,
choose 'if $j \equiv 1, 2\ (mod\ 4)$',

$$f^*(u_i^j u_{i+1}^j) = \begin{cases} 1 & \text{if } i \text{ is odd and } 1 \leq i \leq n-1 \\ 0 & \text{if } i \text{ is even and } 1 \leq i \leq n-1\ ; \end{cases}$$

choose 'if $j \equiv 3, 0\ (mod\ 4)$' and $n \equiv 3\ (mod\ 4)$,

$$f^*(u_i^j u_{i+1}^j) = \begin{cases} 0 & \text{if } i \text{ is odd and } 1 \leq i \leq n-1 \\ 1 & \text{if } i \text{ is even and } 1 \leq i \leq n-1\ ; \end{cases}$$

choose 'if $j \equiv 3, 0\ (mod\ 4)$' and $n \equiv 1\ (mod\ 4)$,

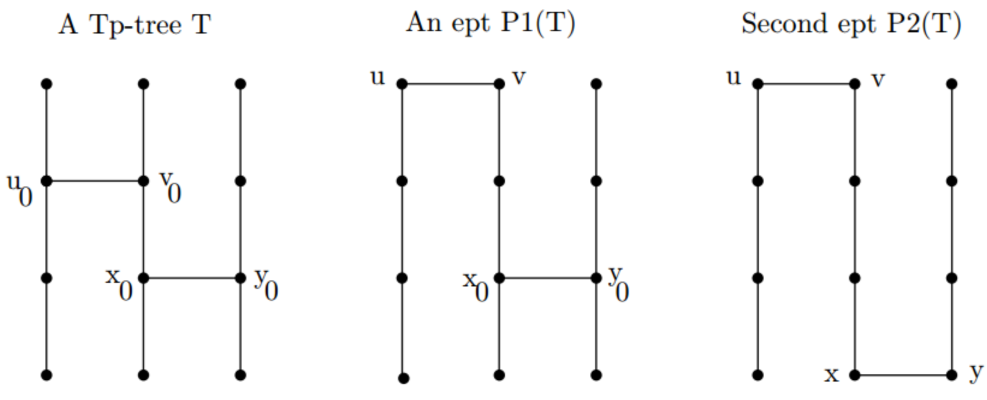

Figure 2. Sum divisor cordial labeling of $T\widehat{O}P_5$ where T is a T_p-tree with 11 vertices

$$f^*(u_i^j u_{i+1}^j) = \begin{cases} 0 & \text{if } i \text{ is odd and } 1 \leq i \leq n-3 \\ 1 & \text{if } i \text{ is even and } 1 \leq i \leq n-3 \\ 1 & \text{if } i = n-2 \\ 1 & \text{if } i = n-1\,. \end{cases}$$

In the above two cases,
when m is odd and n is odd,

$$e_f(0) = e_f(1) = \tfrac{mn-1}{2};$$

when m is odd and n is even,

$$e_f(0) = \left\lceil \tfrac{mn-1}{2} \right\rceil \text{ and } e_f(1) = \left\lfloor \tfrac{mn-1}{2} \right\rfloor;$$

when m is even and n is odd or even,

$$e_f(0) = \left\lceil \tfrac{mn-1}{2} \right\rceil \text{ and } e_f(1) = \left\lfloor \tfrac{mn-1}{2} \right\rfloor.$$

Clearly $|e_f(0) - e_f(1)| \leq 1$. Hence $T\widehat{O}P_n$ is sum divisor cordial graph. □

Example 1. *Sum divisor cordial labeling of $T\widehat{O}P_5$ where T is a T_p-tree with 11 vertices is shown in Figure 2.*

Theorem 3. *If T be a T_p-tree on m vertices, then the graph $T\widehat{O}C_n$ is sum divisor cordial graph if $n \equiv 0, 3, 1\ (mod\ 4)$.*

Proof. Let T be a T_p-tree with m vertices. By the definition of a transformed tree there exists a parallel transformation P of T such that for the path $P(T)$, we have (i) $V(P(T)) = V(T)$ and (ii) $E(P(T)) = (E(T) - E_d) \bigcup E_p$, where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequence $P = (P_1, P_2, \cdots, P_k)$ of the *epts* P used to arrive at the path $P(T)$. Clearly, E_d and E_p have the same number of edges. Denote the vertices of $P(T)$ successively as $v_1, v_2, \cdots, v_m$ starting from one pendant vertex of $P(T)$ right up to the other. Let $u_1^j, u_2^j, \cdots, u_n^j$ $(1 \leq j \leq m)$ be the vertices of j^{th} copy of C_n with $u_1^j = v_j$. Then $V(T\widehat{O}C_n) = \{u_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(T\widehat{O}C_n) = E(T) \bigcup E(C_n)$. Define
$f : V(T\widehat{O}C_n) \to \{1, 2, 3, \ldots, mn\}$ as follows:

Case 1. $n \equiv 0\ (mod\ 4)$.

Choose 'if $j \equiv 1, 2\ (mod\ 4)$' and $1 \leq j \leq m$,

$$f(u_i^j) = \begin{cases} n(j-1) + i & \text{if } i \equiv 1, 0\ (mod\ 4) \text{ and } 1 \leq i \leq n \\ n(j-1) + i + 1 & \text{if } i \equiv 2\ (mod\ 4) \text{ and } 1 \leq i \leq n \\ n(j-1) + i - 1 & \text{if } i \equiv 3\ (mod\ 4) \text{ and } 1 \leq i \leq n; \end{cases}$$

choose 'if $j \equiv 3, 0\ (mod\ 4)$' and $1 \leq j \leq m$,

$$f(u_i^j) = \begin{cases} n(j-1)+i+1 & \text{if } i \equiv 1\ (mod\ 4) \text{ and } 1 \le i \le n \\ n(j-1)+i-1 & \text{if } i \equiv 2\ (mod\ 4) \text{ and } 1 \le i \le n \\ n(j-1)+i & \text{if } i \equiv 3,0\ (mod\ 4) \text{ and } 1 \le i \le n. \end{cases}$$

Let v_iv_j be a transformed edge in T, $1 \le i < j \le m$ and let P_1 be the *ept* obtained by deleting the edge v_iv_j and adding the edge $v_{i+t}v_{j-t}$ where t is the distance of v_i from v_{i+t} and the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent *epts*.

Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i+t+1 = j-t$ which implies $j = i+2t+1$. Therefore, i and j are of opposite parity.
The induced edge label of v_iv_j is given by

$$\begin{aligned} f^*(v_iv_j) &= f^*(v_iv_{i+2t+1}) \\ &= \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even.} \end{cases} \end{aligned}$$

The induced edge label of $v_{i+t}v_{j-t}$ is given by

$$\begin{aligned} f^*(v_{i+t}v_{j-t}) &= f^*(v_{i+t}v_{i+t+1}) \\ &= \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even.} \end{cases} \end{aligned}$$

Therefore, $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.
The induced edge labels are as follows:

$$f^*(v_jv_{j+1}) = \begin{cases} 1 & \text{if } j \text{ is odd and } 1 \le j \le m-1 \\ 0 & \text{if } j \text{ is even and } 1 \le j \le m-1; \end{cases}$$

for $1 \le j \le m$ and $1 \le i \le n-1$,

$$f^*(u_n^ju_1^j) = \begin{cases} 0 & \text{if } j \equiv 1,2\ (mod\ 4) \\ 1 & \text{if } j \equiv 3,0\ (mod\ 4); \end{cases}$$

$$f^*(u_i^ju_{i+1}^j) = \begin{cases} 1 & \text{if } i \text{ is odd and } j \equiv 1,2\ (mod\ 4) \\ 0 & \text{if } i \text{ is even and } j \equiv 1,2\ (mod\ 4) \\ 0 & \text{if } i \text{ is odd and } j \equiv 3,0\ (mod\ 4) \\ 1 & \text{if } i \text{ is even and } j \equiv 3,0\ (mod\ 4). \end{cases}$$

Case 2. $n \equiv 3\ (mod\ 4)$.
Choose 'if $j \equiv 1,3\ (mod\ 4)$' and $1 \le j \le m$,

$$f(u_i^j) = \begin{cases} n(j-1)+i & \text{if } i \equiv 1,0\ (mod\ 4) \text{ and } 1 \le i \le n \\ n(j-1)+i+1 & \text{if } i \equiv 2\ (mod\ 4) \text{ and } 1 \le i \le n \\ n(j-1)+i-1 & \text{if } i \equiv 3\ (mod\ 4) \text{ and } 1 \le i \le n; \end{cases}$$

choose 'if $j \equiv 2,0\ (mod\ 4)$' and $1 \le j \le m$,

$$f(u_i^j) = \begin{cases} n(j-1)+i+1 & \text{if } i \equiv 1\ (mod\ 4) \text{ and } 1 \le i \le n \\ n(j-1)+i-1 & \text{if } i \equiv 2\ (mod\ 4) \text{ and } 1 \le i \le n \\ n(j-1)+i & \text{if } i \equiv 3,0\ (mod\ 4) \text{ and } 1 \le i \le n. \end{cases}$$

Let v_iv_j be a transformed edge in T, $1 \le i < j \le m$ and let P_1 be the *ept* obtained by deleting the edge v_iv_j and adding the edge $v_{i+t}v_{j-t}$ where t is the distance of v_i from v_{i+t} and the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent *epts*.

Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i+t+1 = j-t$ which implies $j = i+2t+1$. Therefore, i and j are of opposite parity.

The induced edge label of v_iv_j is given by

$$\begin{aligned} f^*(v_iv_j) &= f^*(v_iv_{i+2t+1}) \\ &= 2|(f(v_i) + f(v_{i+2t+1})) \\ &= 1. \end{aligned}$$

The induced edge label of $v_{i+t}v_{j-t}$ is given by

$$\begin{aligned} f^*(v_{i+t}v_{j-t}) &= f^*(v_{i+t}v_{i+t+1}) \\ &= 2|(f(v_{i+t}) + f(v_{i+t+1})) \\ &= 1. \end{aligned}$$

Therefore, $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.
The induced edge labels are as follows:

$f^*(v_jv_{j+1}) = 1,\ 1 \le j \le m-1;$

for $1 \le j \le m$ and $1 \le i \le n-1$,

$f^*(u_n^ju_1^j) = 0;$

$$f^*(u_i^ju_{i+1}^j) = \begin{cases} 1 & \text{if } i \text{ is odd and } j \equiv 1,3\ (mod\ 4) \\ 0 & \text{if } i \text{ is even and } j \equiv 1,3\ (mod\ 4) \\ 0 & \text{if } i \text{ is odd and } j \equiv 2,0\ (mod\ 4) \\ 1 & \text{if } i \text{ is even and } j \equiv 2,0\ (mod\ 4). \end{cases}$$

Case 3. $n \equiv 1\ (mod\ 4)$.
For $1 \le j \le m$,

$$f(u_i^j) = \begin{cases} n(j-1)+i & \text{if } i \equiv 1,0\ (mod\ 4) \text{ and } 1 \le i \le n \\ n(j-1)+i+1 & \text{if } i \equiv 2\ (mod\ 4) \text{ and } 1 \le i \le n \\ n(j-1)+i-1 & \text{if } i \equiv 3\ (mod\ 4) \text{ and } 1 \le i \le n. \end{cases}$$

Let v_iv_j be a transformed edge in T, $1 \le i < j \le m$ and let P_1 be the *ept* obtained by deleting the edge v_iv_j and adding the edge $v_{i+t}v_{j-t}$ where t is the distance of v_i from v_{i+t} and the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent *epts*.

Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i+t+1 = j-t$ which implies $j = i+2t+1$. Therefore, i and j are of opposite parity.
The induced edge label of v_iv_j is given by

$$\begin{aligned} f^*(v_iv_j) &= f^*(v_iv_{i+2t+1}) \\ &= 0. \end{aligned}$$

The induced edge label of $v_{i+t}v_{j-t}$ is given by

$$\begin{aligned} f^*(v_{i+t}v_{j-t}) &= f^*(v_{i+t}v_{i+t+1}) \\ &= 0. \end{aligned}$$

Therefore, $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.
The induced edge labels are as follows:

$f^*(v_jv_{j+1}) = 0,\ 1 \le j \le m-1;$

for $1 \le j \le m$ and $1 \le i \le n-1$,

$f^*(u_n^ju_1^j) = 1;$

$$f^*(u_i^ju_{i+1}^j) = \begin{cases} 1 & \text{if } i \text{ is odd and } j \equiv 1,3\ (mod\ 4) \\ 0 & \text{if } i \text{ is even and } j \equiv 1,3\ (mod\ 4) \\ 1 & \text{if } i \text{ is odd and } j \equiv 2,0\ (mod\ 4) \\ 0 & \text{if } i \text{ is even and } j \equiv 2,0\ (mod\ 4). \end{cases}$$

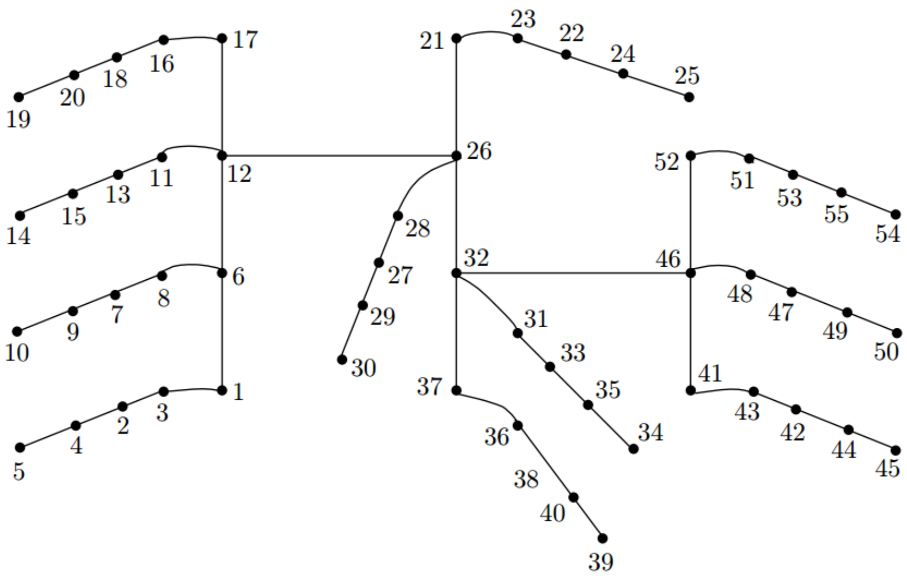

Figure 3. Sum divisor cordial labeling of $T\widehat{O}C_7$ where T is a T_p-tree with 8 vertices

In the above three cases, it can be verified that $|e_f(0) - e_f(1)| \leq 1$. Hence $T\widehat{O}C_n$ is sum divisor cordial graph. □

Example 2. *Sum divisor cordial labeling of $T\widehat{O}C_7$ where T is a T_p-tree with 8 vertices is shown in Figure 3.*

Theorem 4. *If T be a T_p-tree on m vertices, then the graph $T\widehat{O}K_{1,n}$ is sum divisor cordial graph.*

Proof. Let T be a T_p-tree with m vertices. By the definition of a transformed tree there exists a parallel transformation P of T such that for the path $P(T)$, we have (i) $V(P(T)) = V(T)$ and (ii) $E(P(T)) = (E(T) - E_d) \bigcup E_p$, where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequence $P = (P_1, P_2, \cdots, P_k)$ of the *epts* P used to arrive at the path $P(T)$. Clearly, E_d and E_p have the same number of edges. Denote the vertices of $P(T)$ successively as $v_1, v_2, \cdots, v_m$ starting from one pendant vertex of $P(T)$ right up to the other. Let $u_0^j, u_1^j, \cdots, u_n^j (1 \leq j \leq m)$ be the vertices of j^{th} copy of $K_{1,n}$ with $u_n^j = v_j$. Then $V(T\widehat{O}K_{1,n}) = \{v_j, u_0^j, u_i^j : 1 \leq i \leq n, 1 \leq j \leq m \text{ with } v_j = u_n^j\}$ and $E(T\widehat{O}K_{1,n}) = E(T) \bigcup \{u_0^j u_i^j : \ 1 \leq j \leq m, \ 1 \leq i \leq n\}$.
Define $f : V(T\widehat{O}K_{1,n}) \to \{1, 2, \cdots, mn + m\}$ as follows:
For $1 \leq j \leq m$,

$$f(v_j) = 2j;$$
$$f(u_0^j) = 2j - 1;$$
$$f(u_i^j) = 2m + (n - 1)(j - 1) + i, \ 1 \leq i \leq n - 1.$$

Let $v_i v_j$ be a transformed edge in T, $1 \leq i < j \leq m$ and let P_1 be the *ept* obtained by deleting the edge $v_i v_j$ and adding the edge $v_{i+t} v_{j-t}$ where t is the distance of v_i from v_{i+t} and the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent *epts*. Since $v_{i+t} v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. Therefore, i and j are of opposite parity.
The induced edge label of $v_i v_j$ is given by

$$\begin{aligned} f^*(v_i v_j) &= f^*(v_i v_{i+2t+1}) \\ &= 2|(f(v_i) + f(v_{i+2t+1})) \\ &= 1. \end{aligned}$$

The induced edge label of $v_{i+t}v_{j-t}$ is given by

$$\begin{aligned} f^*(v_{i+t}v_{j-t}) &= f^*(v_{i+t}v_{i+t+1}) \\ &= 2|(f(v_{i+t}) + f(v_{i+t+1})) \\ &= 1. \end{aligned}$$

Therefore, $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.
The induced edge labels are as follows:
$f^*(v_jv_{j+1}) = 1,\ 1 \le j \le m-1;$
for $1 \le j \le m$,
$f^*(u_0^ju_n^j) = 0;$
when n is odd and $1 \le i \le n-1$,
$$f^*(u_0^ju_i^j) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even;} \end{cases}$$
when n is even and $1 \le i \le n-1$,
$$f^*(u_0^ju_i^j) = \begin{cases} 1 & \text{if } i \text{ is odd and } j \text{ is odd} \\ 0 & \text{if } i \text{ is even and } j \text{ is odd} \\ 0 & \text{if } i \text{ is odd and } j \text{ is even} \\ 1 & \text{if } i \text{ is even and } j \text{ is even.} \end{cases}$$
In view of above labeling we get,
when m is odd and n is even,
$e_f(0) = e_f(1) = \frac{mn+m-1}{2};$
when m is odd and n is odd,
$e_f(0) = \left\lceil \frac{mn+m-1}{2} \right\rceil$ and $e_f(1) = \left\lfloor \frac{mn+m-1}{2} \right\rfloor;$
when m is even and n is odd or even,
$e_f(0) = \left\lceil \frac{mn+m-1}{2} \right\rceil$ and $e_f(1) = \left\lfloor \frac{mn+m-1}{2} \right\rfloor.$
Clearly $|e_f(0) - e_f(1)| \le 1$. Hence $T\widehat{O}K_{1,n}$ is sum divisor cordial graph. □

Example 3. *Sum divisor cordial labeling of $T\widehat{O}K_{1,3}$ where T is a T_p-tree with 12 vertices is shown in Figure 4.*

Theorem 5. *If T be a T_p-tree on m vertices, then the graph $T \odot \overline{K_n}$ is sum divisor cordial graph.*

Proof. Let T be a T_p-tree with m vertices. By the definition of T_p-tree there exists a parallel transformation P of T such that for the path $P(T)$, we have (i) $V(P(T)) = V(T)$ and (ii) $E(P(T)) = (E(T) - E_d) \bigcup E_p$, where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequence $P = (P_1, P_2, \cdots, P_k)$ of the *epts* P used to arrive at the path $P(T)$. Clearly, E_d and E_p have the same number of edges. Denote the vertices of $P(T)$ successively as $v_1, v_2, \cdots, v_m$ starting from one pendant vertex of $P(T)$ right up to the other. Let $u_1^j, u_2^j, \cdots, u_n^j (1 \le j \le m)$ be the pendant vertices joined with $v_j (1 \le j \le m)$ by an edge. Then $V(T \odot \overline{K_n}) = \{v_j, u_i^j : 1 \le i \le n, 1 \le j \le m\}$ and $E(T \odot \overline{K_n}) = E(T) \bigcup \{v_ju_i^j :\ 1 \le j \le m,\ 1 \le i \le n\}$.
Define $f : V(T \odot \overline{K_n}) \to \{1, 2, \cdots, mn+m\}$ as follows:
For $1 \le j \le m$,
$f(v_j) = 2j - 1;$
$f(u_n^j) = 2j;$
$f(u_i^j) = 2m + (n-1)(j-1) + i,\ 1 \le i \le n-1.$

Let v_iv_j be a transformed edge in T, $1 \le i < j \le m$ and let P_1 be the *ept* obtained by deleting the edge v_iv_j and adding the edge $v_{i+t}v_{j-t}$ where t is the distance of v_i from v_{i+t} and the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent *epts*. Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$.

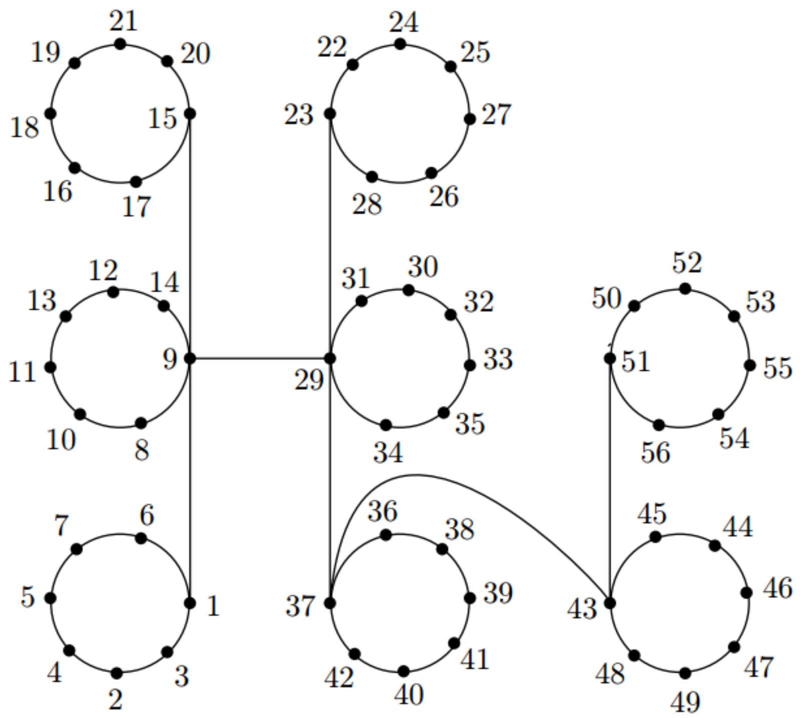

Figure 4. Sum divisor cordial labeling of $T\widehat{O}K_{1,3}$ where T is a T_p-tree with 12 vertices

Therefore, i and j are of opposite parity.
The induced edge label of v_iv_j is given by

$$\begin{aligned} f^*(v_iv_j) &= f^*(v_iv_{i+2t+1}) \\ &= 2|(f(v_i) + f(v_{i+2t+1})) \\ &= 1. \end{aligned}$$

The induced edge label of $v_{i+t}v_{j-t}$ is given by

$$\begin{aligned} f^*(v_{i+t}v_{j-t}) &= f^*(v_{i+t}v_{i+t+1}) \\ &= 2|(f(v_{i+t}) + f(v_{i+t+1})) \\ &= 1. \end{aligned}$$

Therefore, $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.
The induced edge labels are as follows:

$$f^*(v_jv_{j+1}) = 1, \ 1 \le j \le m-1;$$

for $1 \le j \le m$,

$$f^*(v_ju_n^j) = 0;$$

when n is odd and $1 \le i \le n-1$,

$$f^*(v_ju_i^j) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even;} \end{cases}$$

when n is even and $1 \le i \le n-1$,

$$f^*(v_ju_i^j) = \begin{cases} 1 & \text{if } i \text{ is odd and } j \text{ is odd} \\ 0 & \text{if } i \text{ is even and } j \text{ is odd} \\ 0 & \text{if } i \text{ is odd and } j \text{ is even} \\ 1 & \text{if } i \text{ is even and } j \text{ is even.} \end{cases}$$

It can be verified that $|e_f(0) - e_f(1)| \le 1$. Hence $T \odot \overline{K_n}$ is sum divisor cordial graph. □

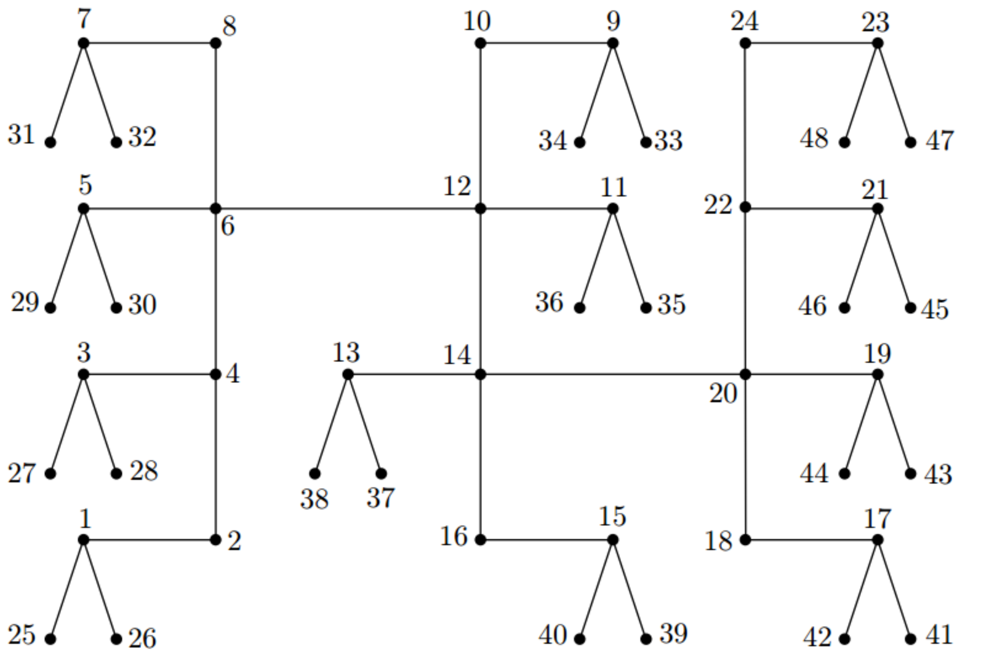

Figure 5. Sum divisor cordial labeling of $T \odot \overline{K_4}$ where T is a T_p-tree with 10 vertices

Example 4. *Sum divisor cordial labeling of $T \odot \overline{K_4}$ where T is a T_p-tree with 10 vertices is shown in Figure 5.*

Theorem 6. *If T be a T_p-tree on m vertices, then the graph $T\widehat{O}Q_n$ is sum divisor cordial graph.*

Proof. Let T be a T_p-tree with m vertices. By the definition of a transformed tree there exists a parallel transformation P of T such that for the path $P(T)$, we have (i) $V(P(T)) = V(T)$ and (ii) $E(P(T)) = (E(T) - E_d) \bigcup E_p$, where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequence $P = (P_1, P_2, \cdots, P_k)$ of the *epts* P used to arrive at the path $P(T)$. Clearly, E_d and E_p have the same number of edges. Denote the vertices of $P(T)$ successively as $v_1, v_2, \cdots, v_m$ starting from one pendant vertex of $P(T)$ right up to the other. Let $u_1^j, u_2^j, \cdots, u_n^j, u_{n+1}^j (1 \le j \le m)$ be the vertices of j^{th} copy of Q_n with $u_{n+1}^j = v_j$. Then $V(T\widehat{O}Q_n) = \{u_i^j : 1 \le i \le n+1, 1 \le j \le m\} \bigcup \{x_i^j, y_i^j : 1 \le i \le n, 1 \le j \le m\}$ and $E(T\widehat{O}Q_n) = E(T) \bigcup E(Q_n)$. We note that $\left|V(T\widehat{O}Q_n)\right| = 3nm + m$ and $\left|E(T\widehat{O}Q_n)\right| = 4mn + m - 1$. Define $f : V(T\widehat{O}Q_n) \to \{1, 2, \cdots, 3mn + m\}$ as follows:

Case 1. m is odd.

For $1 \le j \le m$ and $1 \le i \le n$,

$$f(u_i^j) = m + 3n(j-1) + 3i - 2;$$
$$f(x_i^j) = m + 3n(j-1) + 3i - 1;$$
$$f(y_i^j) = m + 3n(j-1) + 3i;$$
$$f(v_j) = f(u_{n+1}^j) = \begin{cases} j & \text{if } j \equiv 1, 0\ (mod\ 4) \\ j+1 & \text{if } j \equiv 2\ (mod\ 4) \\ j-1 & \text{if } j \equiv 3\ (mod\ 4)\ . \end{cases}$$

Let $v_i v_j$ be a transformed edge in T, $1 \le i < j \le m$ and let P_1 be the *ept* obtained by deleting the edge $v_i v_j$ and adding the edge $v_{i+t} v_{j-t}$ where t is the distance of v_i from v_{i+t} and the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent *epts*.

Since $v_{i+t} v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. Therefore, i and j are of opposite parity.

The induced edge label of v_iv_j is given by

$$\begin{aligned} f^*(v_iv_j) &= f^*(v_iv_{i+2t+1}) \\ &= \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even.} \end{cases} \end{aligned}$$

The induced edge label of $v_{i+t}v_{j-t}$ is given by

$$\begin{aligned} f^*(v_{i+t}v_{j-t}) &= f^*(v_{i+t}v_{i+t+1}) \\ &= \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even.} \end{cases} \end{aligned}$$

Therefore, $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.
The induced edge labels are as follows:

$$f^*(v_jv_{j+1}) = \begin{cases} 1 & \text{if } i \text{ is odd and } 1 \le j \le m-1 \\ 0 & \text{if } i \text{ is even and } 1 \le j \le m-1; \end{cases}$$

for $1 \le j \le m$,

$f^*(u_i^jx_i^j) = 0,\ 1 \le i \le n$;
$f^*(u_i^jy_i^j) = 1,\ 1 \le i \le n$;
$f^*(x_i^ju_{i+1}^j) = 1,\ 1 \le i \le n-1$;
$f^*(y_i^ju_{i+1}^j) = 0,\ 1 \le i \le n-1$;

$$f^*(x_n^jv_j) = \begin{cases} 1 & \text{if } n \text{ is odd and } j \equiv 1, 0\ (mod\ 4) \\ 0 & \text{if } n \text{ is odd and } j \equiv 2, 3\ (mod\ 4) \\ 0 & \text{if } n \text{ is even and } j \equiv 1, 2\ (mod\ 4) \\ 1 & \text{if } n \text{ is even and } j \equiv 3, 0\ (mod\ 4); \end{cases}$$

$$f^*(y_n^jv_j) = \begin{cases} 0 & \text{if } n \text{ is odd and } j \equiv 1, 0\ (mod\ 4) \\ 1 & \text{if } n \text{ is odd and } j \equiv 2, 3\ (mod\ 4) \\ 1 & \text{if } n \text{ is even and } j \equiv 1, 2\ (mod\ 4) \\ 0 & \text{if } n \text{ is even and } j \equiv 3, 0\ (mod\ 4). \end{cases}$$

Case 2. m is even.
For $1 \le j \le m$ and $1 \le i \le n$,

$f(u_i^j) = m + 3n(j-1) + 3i - 2$;
$f(x_i^j) = m + 3n(j-1) + 3i - 1$;
$f(y_i^j) = m + 3n(j-1) + 3i$;

$$f(v_j) = f(u_{n+1}^j) = \begin{cases} j & \text{if } j \equiv 1, 2\ (mod\ 4) \\ j+1 & \text{if } j \equiv 3\ (mod\ 4) \\ j-1 & \text{if } j \equiv 0\ (mod\ 4)\ . \end{cases}$$

Let v_iv_j be a transformed edge in T, $1 \le i < j \le m$ and let P_1 be the *ept* obtained by deleting the edge v_iv_j and adding the edge $v_{i+t}v_{j-t}$ where t is the distance of v_i from v_{i+t} and the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent *epts*.

Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i+t+1 = j-t$ which implies $j = i+2t+1$. Therefore, i and j are of opposite parity.
The induced edge label of v_iv_j is given by

$$\begin{aligned} f^*(v_iv_j) &= f^*(v_iv_{i+2t+1}) \\ &= \begin{cases} 0 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \text{ is even.} \end{cases} \end{aligned}$$

The induced edge label of $v_{i+t}v_{j-t}$ is given by

$$\begin{aligned} f^*(v_{i+t}v_{j-t}) &= f^*(v_{i+t}v_{i+t+1}) \\ &= \begin{cases} 0 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \text{ is even.} \end{cases} \end{aligned}$$

Therefore, $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.
The induced edge labels are as follows:

$$f^*(v_jv_{j+1}) = \begin{cases} 0 & \text{if } i \text{ is odd and } 1 \le j \le m-1 \\ 1 & \text{if } i \text{ is even and } 1 \le j \le m-1; \end{cases}$$

for $1 \le j \le m$,

$f^*(u_i^jx_i^j) = 0,\ 1 \le i \le n$;
$f^*(u_i^jy_i^j) = 1,\ 1 \le i \le n$;
$f^*(x_i^ju_{i+1}^j) = 1,\ 1 \le i \le n-1$;
$f^*(y_i^ju_{i+1}^j) = 0,\ 1 \le i \le n-1$;

$$f^*(x_n^jv_j) = \begin{cases} 0 & \text{if } n \text{ is odd and } j \equiv 1,2\ (mod\ 4) \\ 1 & \text{if } n \text{ is odd and } j \equiv 3,0\ (mod\ 4) \\ 1 & \text{if } n \text{ is even and } j \equiv 1,0\ (mod\ 4) \\ 0 & \text{if } n \text{ is even and } j \equiv 2,3\ (mod\ 4); \end{cases}$$

$$f^*(y_n^jv_j) = \begin{cases} 1 & \text{if } n \text{ is odd and } j \equiv 1,2\ (mod\ 4) \\ 0 & \text{if } n \text{ is odd and } j \equiv 3,0\ (mod\ 4) \\ 1 & \text{if } n \text{ is even and } j \equiv 2,3\ (mod\ 4) \\ 0 & \text{if } n \text{ is even and } j \equiv 1,0\ (mod\ 4). \end{cases}$$

In the above two cases,
when m is odd,

$e_f(1) = e_f(0) = \frac{4mn+m-1}{2}$;

when m is even,

$e_f(1) = \left\lfloor \frac{4mn+m-1}{2} \right\rfloor$ and $e_f(0) = \left\lceil \frac{4mn+m-1}{2} \right\rceil$.

It can be verified that $|e_f(0) - e_f(1)| \le 1$. Hence $T\widehat{O}Q_n$ is sum divisor cordial graph. □

Example 5. *Sum divisor cordial labeling of $T\widehat{O}Q_2$ where T is a T_p-tree with 8 vertices is shown in Figure* 6.

Theorem 7. *If T be a T_p-tree on m vertices, then the graph $T\widetilde{O}C_n$ is sum divisor cordial graph if $n \equiv 0,3,1\ (mod\ 4)$.*

Proof. Let T be a T_p-tree with m vertices. By the definition of a transformed tree there exists a parallel transformation P of T such that for the path $P(T)$, we have (i) $V(P(T)) = V(T)$ and (ii) $E(P(T)) = (E(T) - E_d) \bigcup E_p$, where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequence $P = (P_1, P_2, \cdots, P_k)$ of the *epts* P used to arrive at the path $P(T)$. Clearly, E_d and E_p have the same number of edges. Denote the vertices of $P(T)$ successively as $v_1, v_2, \cdots, v_m$ starting from one pendant vertex of $P(T)$ right up to the other. Let $u_1^j, u_2^j, \cdots, u_n^j (1 \le j \le m)$ be the vertices of j^{th} copy of C_n. Then $V(T\widetilde{O}C_n) = \{v_j, u_i^j : 1 \le i \le n,\ 1 \le j \le m\}$ and $E(T\widetilde{O}C_n) = E(T) \bigcup E(C_n) \bigcup \{v_ju_1^j : 1 \le j \le m\}$. Define $f : V(T\widetilde{O}C_n) \to \{1, 2, \ldots, mn+m\}$ as follows:
Case 1. $n \equiv 0\ (mod\ 4)$.

$f(v_j) = (n+1)j, 1 \le j \le m$;

for $1 \le j \le m$ and $1 \le i \le n$,

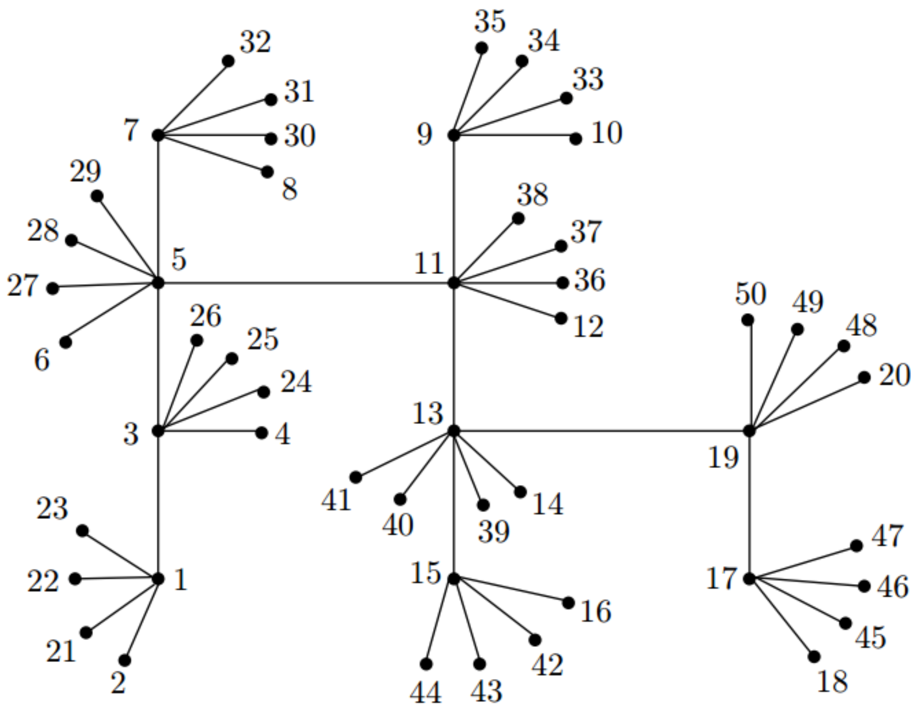

Figure 6. Sum divisor cordial labeling of $T\widehat{O}Q_2$ where T is a T_p-tree with 8 vertices

$$f(u_i^j) = \begin{cases} (n+1)(j-1)+i & \text{if } \ i \equiv 1, 0\ (mod\ 4) \\ (n+1)(j-1)+i+1 & \text{if } \ i \equiv 2\ (mod\ 4) \\ (n+1)(j-1)+i-1 & \text{if } \ i \equiv 3\ (mod\ 4)\ . \end{cases}$$

Let v_iv_j be a transformed edge in T, $1 \le i < j \le m$ and let P_1 be the *ept* obtained by deleting the edge v_iv_j and adding the edge $v_{i+t}v_{j-t}$ where t is the distance of v_i from v_{i+t} and also the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent *epts*. Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i+t+1 = j-t$ which implies $j = i+2t+1$. Therefore, i and j are of opposite parity.
The induced edge label of v_iv_j is given by

$$\begin{aligned} f^*(v_iv_j) &= f^*(v_iv_{i+2t+1}) \\ &= 0. \end{aligned}$$

The induced edge label of $v_{i+t}v_{j-t}$ is given by

$$\begin{aligned} f^*(v_{i+t}v_{j-t}) &= f^*(v_{i+t}v_{i+t+1}) \\ &= 0. \end{aligned}$$

Therefore, $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.
The induced edge labels are as follows:

$f^*(v_jv_{j+1}) = 0,\ 1 \le j \le m-1;$
$f^*(u_1^jv_j) = 1,\ 1 \le j \le m;$
$f^*(u_n^ju_1^j) = 0,\ 1 \le j \le m;$

for $1 \le j \le m$,

$$f^*(u_i^ju_{i+1}^j) = \begin{cases} 1 & \text{if } \ i \text{ is odd and } 1 \le i \le n-1 \\ 0 & \text{if } \ j \text{ is even and } 1 \le i \le n-1. \end{cases}$$

Case 2. $n \equiv 3\ (mod\ 4)$.

$$f(v_j) = \begin{cases} (n+1)j & \text{if } j \equiv 1,2 \ (mod\ 4) \text{ and } 1 \le j \le m \\ (n+1)(j-1)+1 & \text{if } j \equiv 3,0 \ (mod\ 4) \text{ and } 1 \le j \le m \ ; \end{cases}$$

choose 'if $j \equiv 1,2\ (mod\ 4)$' and $1 \le j \le m$,

$$f(u_i^j) = \begin{cases} (n+1)(j-1)+i+1 & \text{if } i \equiv 1 \ (mod\ 4) \text{ and } 1 \le i \le n \\ (n+1)(j-1)+i-1 & \text{if } i \equiv 2 \ (mod\ 4) \text{ and } 1 \le i \le n \\ (n+1)(j-1)+i & \text{if } i \equiv 3,0 \ (mod\ 4) \text{ and } 1 \le i \le n \ ; \end{cases}$$

choose 'if $j \equiv 3,0\ (mod\ 4)$' and $1 \le j \le m$,

$$f(u_i^j) = \begin{cases} (n+1)(j-1)+i+2 & \text{if } i \equiv 1 \ (mod\ 4) \text{ and } 1 \le i \le n \\ (n+1)(j-1)+i & \text{if } i \equiv 2 \ (mod\ 4) \text{ and } 1 \le i \le n \\ (n+1)(j-1)+i+1 & \text{if } i \equiv 3,0 \ (mod\ 4) \text{ and } 1 \le i \le n \ . \end{cases}$$

Let v_iv_j be a transformed edge in T, $1 \le i < j \le m$ and let P_1 be the *ept* obtained by deleting the edge v_iv_j and adding the edge $v_{i+t}v_{j-t}$ where t is the distance of v_i from v_{i+t} and the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent *epts*. Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i+t+1 = j-t$ which implies $j = i+2t+1$. Therefore, i and j are of opposite parity.
The induced edge label of v_iv_j is given by

$$\begin{aligned} f^*(v_iv_j) &= f^*(v_iv_{i+2t+1}) \\ &= \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even.} \end{cases} \end{aligned}$$

The induced edge label of $v_{i+t}v_{j-t}$ is given by

$$\begin{aligned} f^*(v_{i+t}v_{j-t}) &= f^*(v_{i+t}v_{i+t+1}) \\ &= \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even.} \end{cases} \end{aligned}$$

Therefore, $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.
The induced edge labels are as follows:

$$f^*(v_jv_{j+1}) = \begin{cases} 1 & \text{if } i \text{ is odd and } 1 \le j \le m-1 \\ 0 & \text{if } i \text{ is even and } 1 \le j \le m-1 \ ; \end{cases}$$

for $1 \le j \le m$,

$f^*(u_1^jv_j) = 1;$

$f^*(u_n^ju_1^j) = 0;$

$$f^*(u_i^ju_{i+1}^j) = \begin{cases} 0 & \text{if } i \text{ is odd and } 1 \le i \le n-1 \\ 1 & \text{if } i \text{ is even and } 1 \le i \le n-1 \ . \end{cases}$$

Case 3. $n \equiv 1\ (mod\ 4)$.

$$f(v_j) = \begin{cases} (n+1)j & \text{if } j \equiv 1,2 \ (mod\ 4) \text{ and } 1 \le j \le m \\ (n+1)(j-1)+1 & \text{if } j \equiv 3,0 \ (mod\ 4) \text{ and } 1 \le j \le m \ ; \end{cases}$$

choose 'if $j \equiv 1,2\ (mod\ 4)$' and $1 \le j \le m$,

$$f(u_i^j) = \begin{cases} (n+1)(j-1)+i & \text{if } i \equiv 1,0 \ (mod\ 4) \text{ and } 1 \le i \le n \\ (n+1)(j-1)+i+1 & \text{if } i \equiv 2 \ (mod\ 4) \text{ and } 1 \le i \le n \\ (n+1)(j-1)+i-1 & \text{if } i \equiv 3 \ (mod\ 4) \text{ and } 1 \le i \le n \ ; \end{cases}$$

choose 'if $j \equiv 3,0\ (mod\ 4)$' and $1 \le j \le m$,

$$f(u_i^j) = \begin{cases} (n+1)(j-1)+i+1 & \text{if } i \equiv 1,0 \ (mod\ 4) \text{ and } 1 \le i \le n \\ (n+1)(j-1)+i+2 & \text{if } i \equiv 2 \ (mod\ 4) \text{ and } 1 \le i \le n \\ (n+1)(j-1)+i & \text{if } i \equiv 3 \ (mod\ 4) \text{ and } 1 \le i \le n \ ; \end{cases}$$

Let v_iv_j be a transformed edge in T, $1 \leq i < j \leq m$ and let P_1 be the *ept* obtained by deleting the edge v_iv_j and adding the edge $v_{i+t}v_{j-t}$ where t is the distance of v_i from v_{i+t} and the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent *epts*. Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i+t+1 = j-t$ which implies $j = i+2t+1$. Therefore, i and j are of opposite parity.
The induced edge label of v_iv_j is given by

$$\begin{aligned} f^*(v_iv_j) &= f^*(v_iv_{i+2t+1}) \\ &= \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even.} \end{cases} \end{aligned}$$

The induced edge label of $v_{i+t}v_{j-t}$ is given by

$$\begin{aligned} f^*(v_{i+t}v_{j-t}) &= f^*(v_{i+t}v_{i+t+1}) \\ &= \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even.} \end{cases} \end{aligned}$$

Therefore, $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.
The induced edge labels are as follows:

$$f^*(v_jv_{j+1}) = \begin{cases} 1 & \text{if } i \text{ is odd and } 1 \leq j \leq m-1 \\ 0 & \text{if } i \text{ is even and } 1 \leq j \leq m-1 \end{cases};$$

for $1 \leq j \leq m$,

$f^*(u_1^jv_j) = 0;$
$f^*(u_n^ju_1^j) = 1;$

$$f^*(u_i^ju_{i+1}^j) = \begin{cases} 1 & \text{if } i \text{ is odd and } 1 \leq i \leq n-1 \\ 0 & \text{if } i \text{ is even and } 1 \leq i \leq n-1 \end{cases}.$$

In the above three cases,
when m is odd,

$$e_f(1) = \begin{cases} \frac{mn+2m-1}{2} & \text{if } n \equiv 1,3 \ (mod\ 4) \\ \left\lceil \frac{mn+2m-1}{2} \right\rceil & \text{if } n \equiv 0 \ (mod\ 4), \end{cases}$$

$$e_f(0) = \begin{cases} \frac{mn+2m-1}{2} & \text{if } n \equiv 1,3 \ (mod\ 4) \\ \left\lfloor \frac{mn+2m-1}{2} \right\rfloor & \text{if } n \equiv 0 \ (mod\ 4); \end{cases}$$

when m is even and $n \equiv 1,3,0 \ (mod\ 4)$,

$e_f(1) = \left\lceil \frac{mn+2m-1}{2} \right\rceil$ and $e_f(0) = \left\lfloor \frac{mn+2m-1}{2} \right\rfloor$.

It can be verified that $|e_f(0) - e_f(1)| \leq 1$. Hence $T\widetilde{O}C_n$ is sum divisor cordial graph. □

Example 6. *Sum divisor cordial labeling of $T\widetilde{O}C_5$ where T is a T_p-tree with 8 vertices is shown in Figure* 7.

Theorem 8. *If T be a T_p-tree on m vertices, then the graph $T\widetilde{O}Q_n$ is sum divisor cordial graph.*

Proof. Let T be a T_p-tree with m vertices. By the definition of a transformed tree there exists a parallel transformation P of T such that for the path $P(T)$, we have (i) $V(P(T)) = V(T)$ and (ii) $E(P(T)) = (E(T) - E_d) \bigcup E_p$, where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequence $P = (P_1, P_2, \cdots, P_k)$ of the *epts* P used to arrive at the path $P(T)$. Clearly, E_d and E_p have the same number of edges. Denote the vertices of $P(T)$ successively as $v_1, v_2, \cdots, v_m$ starting from one pendant vertex of $P(T)$ right up to the other. Let $u_1^j, u_2^j, \cdots, u_n^j, u_{n+1}^j (1 \leq j \leq m)$ be the vertices of j^{th} copy of Q_n. Then $V(T\widetilde{O}Q_n) = \{v_j, u_i^j : 1 \leq i \leq n+1, 1 \leq j \leq m\} \bigcup \{x_i^j, y_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$ and

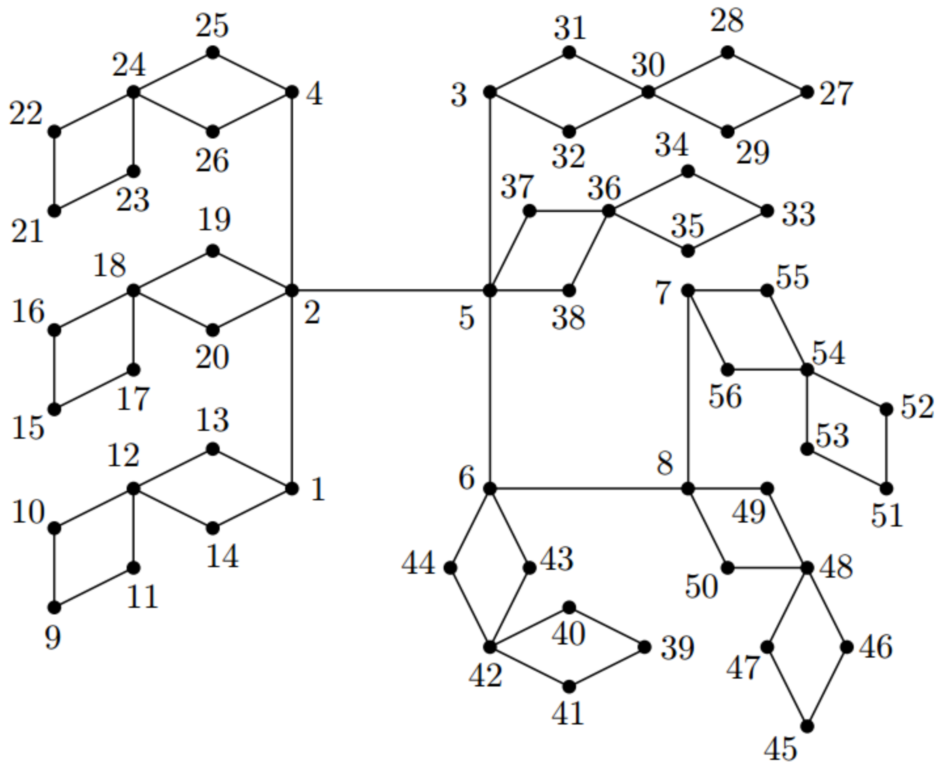

Figure 7. Sum divisor cordial labeling of $T\widetilde{O}C_5$ where T is a T_p-tree with 8 vertices

$E(T\widetilde{O}Q_n) = E(T)\bigcup E(Q_n)\bigcup\{v_ju^j_{n+1} : 1 \le j \le m\}$. We note that $\left|V(T\widetilde{O}Q_n)\right| = m(3n+2)$ and $\left|E(T\widetilde{O}Q_n)\right| = 4mn+2m-1$. Define $f : V(T\widetilde{O}Q_n) \to \{1,2,\cdots,m(3n+2)\}$ as follows:

Case 1. n is odd.

$f(v_j) = (3n+2)j,\ 1 \le j \le m;$

for $1 \le j \le m$ and $1 \le i \le n+1$,

$$f(u^j_i) = \begin{cases} (3n+2)(j-1)+3i-1 & \text{if } i \text{ is odd} \\ (3n+2)(j-1)+3i-3 & \text{if } i \text{ is even ;} \end{cases}$$

for $1 \le j \le m$ and $1 \le i \le n$,

$$f(x^j_i) = \begin{cases} (3n+2)(j-1)+3i-2 & \text{if } i \text{ is odd} \\ (3n+2)(j-1)+3i-1 & \text{if } i \text{ is even ;} \end{cases}$$

$$f(y^j_i) = \begin{cases} (3n+2)(j-1)+3i+1 & \text{if } i \text{ is odd} \\ (3n+2)(j-1)+3i & \text{if } i \text{ is even .} \end{cases}$$

Let v_iv_j be a transformed edge in T, $1 \le i < j \le m$ and let P_1 be the *ept* obtained by deleting the edge v_iv_j and adding the edge $v_{i+t}v_{j-t}$ where t is the distance of v_i from v_{i+t} and also the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent *epts*. Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i+t+1 = j-t$ which implies $j = i+2t+1$. Therefore, i and j are of opposite parity.

The induced edge label of v_iv_j is given by

$$\begin{aligned} f^*(v_iv_j) &= f^*(v_iv_{i+2t+1}) \\ &= 0. \end{aligned}$$

The induced edge label of $v_{i+t}v_{j-t}$ is given by

$$f^*(v_{i+t}v_{j-t}) = f^*(v_{i+t}v_{i+t+1})$$

$$= 0.$$

Therefore, $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.
The induced edge labels are as follows:

$f^*(v_jv_{j+1}) = 0,\ 1 \le j \le m-1;$

for $1 \le j \le m$,

$f^*(u^j_{n+1}v_j) = 1;$
$f^*(u^j_ix^j_i) = 0,\ 1 \le i \le n;$
$f^*(u^j_iy^j_i) = 1,\ 1 \le i \le n;$
$f^*(x^j_iu^j_{i+1}) = 1,\ 1 \le i \le n;$
$f^*(y^j_iu^j_{i+1}) = 0,\ 1 \le i \le n.$

Case 2. n is even.

$f(v_j) = (3n+2)j,\ 1 \le j \le m;$

for $1 \le j \le m$,

$f(u^j_i) = (3n+2)(j-1) + 3i - 2,\ 1 \le i \le n+1;$
$f(x^j_i) = (3n+2)(j-1) + 3i - 1,\ 1 \le i \le n;$
$f(y^j_i) = (3n+2)(j-1) + 3i,\ 1 \le i \le n.$

Let v_iv_j be a transformed edge in T, $1 \le i < j \le m$ and let P_1 be the *ept* obtained by deleting the edge v_iv_j and adding the edge $v_{i+t}v_{j-t}$ where t is the distance of v_i from v_{i+t} and also the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent *epts*. Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i+t+1 = j-t$ which implies $j = i+2t+1$. Therefore, i and j are of opposite parity.
The induced edge label of v_iv_j is given by

$$\begin{aligned} f^*(v_iv_j) &= f^*(v_iv_{i+2t+1}) \\ &= 2|(f(v_i) + f(v_{i+2t+1})) \\ &= 1. \end{aligned}$$

The induced edge label of $v_{i+t}v_{j-t}$ is given by

$$\begin{aligned} f^*(v_{i+t}v_{j-t}) &= f^*(v_{i+t}v_{i+t+1}) \\ &= 2|(f(v_{i+t}) + f(v_{i+t+1})) \\ &= 1. \end{aligned}$$

Therefore, $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.
The induced edge labels are as follows:

$f^*(v_jv_{j+1}) = 1,\ 1 \le j \le m-1;$

for $1 \le j \le m$,

$f^*(u^j_{n+1}v_j) = 0;$
$f^*(u^j_ix^j_i) = 0,\ 1 \le i \le n;$
$f^*(u^j_iy^j_i) = 1,\ 1 \le i \le n;$
$f^*(x^j_iu^j_{i+1}) = 1,\ 1 \le i \le n;$
$f^*(y^j_iu^j_{i+1}) = 0,\ 1 \le i \le n.$

In above two cases, it can be verified that $|e_f(1) - e_f(0)| \le 1$. Hence $T\widetilde{O}Q_n$ is sum divisor cordial graph. □

Example 7. *Sum divisor cordial labeling of $T\widetilde{O}Q_2$ where T is a T_p-tree with 8 vertices is shown in Figure 8.*

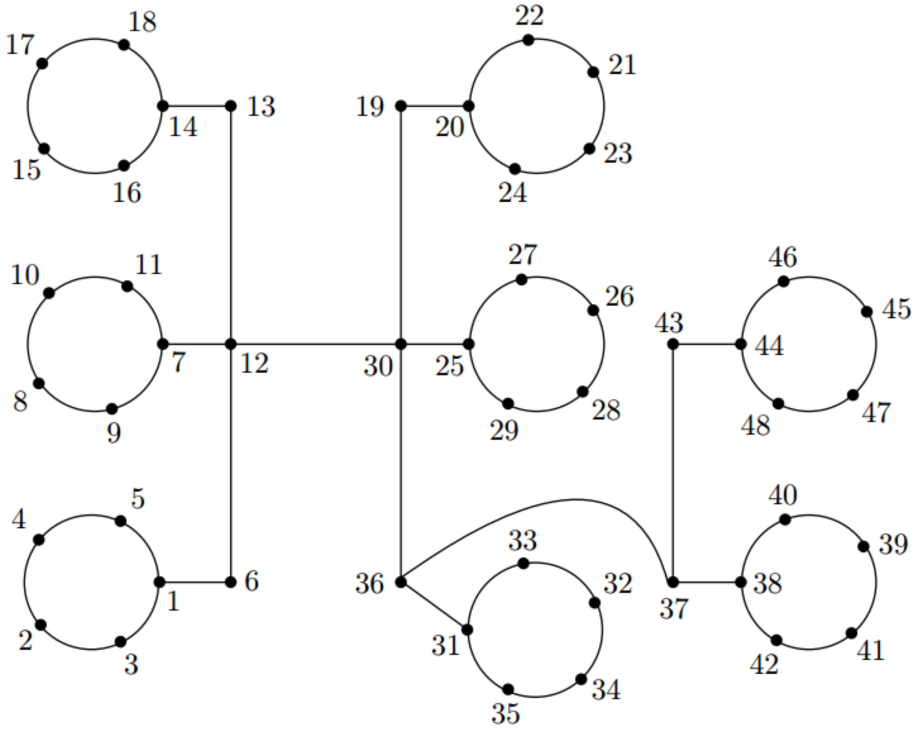

Figure 8. Sum divisor cordial labeling of $T\widetilde{O}Q_2$ where T is a T_p-tree with 8 vertices

Conflict of Interest

The authors declare no conflict of interests.

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