On the Existence of $(v, 4, 1)$-PMD

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ABSTRACT. F.E. Bennett has proved that a $(v,4,1)$-RPMD exists for every positive integer $v \equiv 1 \pmod 4$ with the possible exception of $v = 33, 57, 93$ and 133. In this paper, we shall first introduce the concept of an incomplete PMD and use it to establish some construction methods for Mendelsohn designs; then we shall give the following results: (1) a $(v,4,1)$-PMD exists for every positive integer $V \equiv 0 \pmod 4$ with the exception of $v = 4$ and the possible exception of $v = 8, 12$; (2) a $(v,4,1)$-PMD exists if $v = 57, 93$ or 133.

1. Introduction

The concept of a perfect cyclic design was introduced by N.S. Mendelsohn [9] and further studied in a subsequent paper [3]. These designs were also called Mendelsohn designs by Hsu and Keedwell in [8]. The following are some definitions on Mendelsohn designs.

Definition 1.1: A set of k distinct elements $\{a_1, a_2, \ldots, a_k\}$ is said to be cyclically ordered by $a_1 < a_2 < \cdots < a_k < a_1$ and the pair a_i, a_{i+t} are said to be t-apart in a cyclic k-tuple $(a_1, a_2, \ldots, a_k)$ where $i + t$ is taken modulo k.

Definition 1.2: Let v, k and λ be positive integers. A (v,k,λ)-Mendelsohn design (briefly (v,k,λ)-MD) is a pair $(X, \mathbf{B})$ where X is a v-set (of points) and $\mathbf{B}$ is a collection of cyclically ordered subsets of X (called blocks) with size k such that every ordered pair of points of X are consecutive in exactly λ blocks of $\mathbf{B}$.

Definition 1.3: Let $(X, \mathbf{B})$ be a (v,k,λ)-MD. The design is called perfect and denoted by (v,k,λ)-PMD if each ordered pair (x, y) of points of X appears t-apart in exactly λ of the blocks of $\mathbf{B}$ for all $t = 1, 2, \ldots, k-1$.

It is known [2] that a necessary condition for the existence of a (v,k,λ)-MD is $\lambda v(v-1) \equiv 0 \pmod k$. We next define the notion of resolvability of a $(v,k,1)$-PMD where $v(v-1) \equiv 0 \pmod k$.

Definition 1.4: If the blocks of a $(v,k,1)$-PMD for which $v \equiv 1 \pmod k$ can be partitioned into v sets each containing $(v-1)/k$ blocks which are pairwise disjoint (as sets), we say that the $(v,k,1)$-PMD is resolvable (briefly $(v,k,1)$-RPMD).

Definition 1.5: If the blocks of a $(v,k,1)$-PMD for which $v \equiv 0 \pmod k$ can be partitioned into $v-1$ sets each containing v/k blocks which are pairwise disjoint (as sets), we shall also say that the $(v,k,1)$-PMD is resolvable (briefly $(v,k,1)$-RPMD).

The following are the known results on $(v,k,1)$-PMDs, of which a survey can be found in [2].

Theorem 1.1. *A* $(v,3,1)$ *-RPMD exists if and only if* $v \equiv 0$ *or* $1 \pmod 3$, $v \neq 6$.

Theorem 1.2. *A* $(v,4,1)$ *-RPMD exists for every positive integer* $v \equiv 1 \pmod 4$ *with the possible exception of* $v = 33, 57, 93$ *and 133 [1, Theorem 4.2].*

Theorem 1.3. *Let* p *be an odd prime and* $r \geq 1$, *then there exists a* $(p^r, p, 1)$*-PMD.*

Theorem 1.4. *Let* $v = p^r$ *be any prime power and* $k > 2$ *be such that* $k|(v-1)$, *then there exists a* $(v,k,1)$ *-RPMD.*

Theorem 1.5. *A* $(v,k,1)$ *-RPMD exists for all sufficiently large* v *with* $k \geq 3$ *and* $v \equiv 1 \pmod k$.

Theorem 1.6. *A* $(v,k,1)$ *-PMD exists with* $v(v-1) \equiv 0 \pmod k$ *for the case when* k *is an odd prime and* v *is sufficiently large.*

In this paper, we shall introduce in section 2 the concept of an incomplete PMD and use it to establish some construction methods for PMDs, and further obtain in section 3 the following result: A $(v,4,1)$-PMD exists for every positive integer $v \equiv 0$ or $1 \pmod 4$ with the exception of $v = 4$ and the possible exception of $v = 8, 12, 33$.

We mention some definitions and known facts on PBDs and related designs for later use.

Definition 1.6: Let X be a set of v points. Let $\mathbf{A}$ be a collection of some subsets (called blocks) of X. A pair $(X, \mathbf{A})$ is called a pairwise balanced design (briefly PBD) of index 1 if any two distinct points of X are contained in exactly one block of $\mathbf{A}$, and denoted by $(v, K, 1)$-PMD where K is a set of some integers containing all the block sizes of $\mathbf{A}$. Let $\mathbf{D} \subset \mathbf{A}$. $\mathbf{D}$ is called a parallel class of a PBD $(X, \mathbf{A})$ if $\mathbf{D}$ forms a partition of X. The PBD is called resolvable if $\mathbf{A}$ can be partitioned into some disjoint parallel classes. If a $(v, K, 1)$-PBD is resolvable, we denote it by $(v, K, 1)$-RPBD.

A transversal design TD$[k, 1; n]$ of $(X, \mathbf{G}, \mathbf{B})$ can be viewed as a $(kn, \{k, n\}, 1)$-PBD of $(X, \mathbf{G} \cup \mathbf{B})$ where $\mathbf{G}$ forms a parallel class of k blocks (called groups) of size n and $\mathbf{B}$ consists of blocks of size k. If $\mathbf{B}$ can be partitioned into some disjoint parallel classes, the transversal design TD$[k, 1; n]$ is called resolvable.

Let $N(n)$ denote the maximum number of mutually orthogonal Latin squares of order v. The following results are well known (see [6,7]).

Lemma 1.1. *The existence of a TD*$[k, 1; n]$ *is equivalent to* $N(n) \geq k-2$.

Lemma 1.2. *The existence of a TD*$[k+1, 1; n]$ *implies the existence of a resolvable TD*$[k, 1; n]$.

Lemma 1.3. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ be the factorization of n into powers of distinct primes p_i, then $N(n) \geq \min_{1 \leq i \leq k} \{p_i^{\alpha_i}\} - 1$.*

Lemma 1.4. *If $n > 10632$, then $N(n) \geq 14$ (see [5]).*

Lemma 1.5. *If $n \notin \{2, 6\}$, then $N(n) \geq 2$ (see [4]).*

2. IPMD and construction methods

We first introduce the concept of an incomplete PMD.

Definition 2.1: Let v, k and n be positive integers. A $(v, k, 1)$-incomplete PMD with emptiness n (briefly IPMD$[v, k, n]$) is a triple $(X, Y, \mathbf{B})$ where X is a v-set (of points), Y $(\subset X)$ is a n-set (of points) and $\mathbf{B}$ is a collection of cyclically ordered subsets of X (called blocks) with block size k such that (1) every ordered pair (x, y) of points of X with $\{x, y\} \not\subset Y$ appears t-apart in a unique block of $\mathbf{B}$ for $t = 1, 2, \dots, k-1$; (2) every ordered pair (x, y) of points of X with $\{x, y\} \subset Y$ appears in no block of $\mathbf{B}$.

We now establish several constructions for Mendelsohn design.

Theorem 2.1. *Suppose there exists an IPMD$[v, k, n]$ and a $(n, k, 1)$-PMD. Then there exists a $(v, k, 1)$-PMD.*

Proof: Let $(X, Y, \mathbf{B}_1)$ be an IPMD$[v, k, n]$ and $(Y, \mathbf{B}_2)$ be a $(n, k, 1)$-PMD. It is easy to see that $(X, \mathbf{B}_1 \cup \mathbf{B}_2)$ is a $(v, k, 1)$-PMD.

Theorem 2.2. *Suppose there exists: (1) a $(v, k, 1)$-PMD, (2) a $(u+l, k, 1)$-PMD, (3) a TD$[k, 1; u]$, (4) an IPMD$[u+l, k, l]$, where v, k and u are positive integers and l is a nonnegative integer. Then there exists a $(vu+l, k, 1)$-PMD.*

Proof: Let X, Y and Z be three disjoint sets of points where $X = \{x_1, x_2, \dots, x_v\}$, $Y = \{y_1, y_2, \dots, y_u\}$ and $|Z| = l$. From condition (1), we can let $(X, \mathbf{A})$ be a $(v, k, 1)$-PMD. Let $M_A = \{x_{i_1}, x_{i_2}, \dots, x_{i_k}\}$ for any $A = (x_{i_1}, x_{i_2}, \dots, x_{i_k}) \in \mathbf{A}$. From condition (3), we can let $(M_A \times Y, \mathbf{G}, \mathbf{B}_A)$ be a TD$[k, 1; u]$. Let $\widetilde{B} = ((x_{i_1}, y_{j_1}), (x_{i_2}, y_{j_2}) \dots, (x_{i_k}, y_{j_k}))$ for every $B = \{(x_{i_1}, y_{j_1}), (x_{i_2}, y_{j_2}) \dots, (x_{i_k}, y_{j_k})\} \in \mathbf{B}_A$ and $\widetilde{\mathbf{B}}_A = \cup_{B \in \mathbf{B}_A} \widetilde{B}$. From condition (2) we can let $((\{x_1\} \times Y) \cup Z, \mathbf{C}_1)$ be a $(u+l, k, 1)$-PMD. From condition (4) we can let $((\{x_i\} \times Y) \cup Z, Z, \mathbf{C}_i)$ be an IPMD$[u+l, k, l]$ where $i = 2, 3, \dots, v$. We are to prove that $((X \times Y) \cup Z, (\cup_{A \in \mathbf{A}} \widetilde{\mathbf{B}}_A) \cup (\cup_{i=1}^{v} \mathbf{C}_i))$ is a $(vu+l, k, 1)$-PMD. Let (w_1, w_2) be an ordered pair of points of $(X \times Y) \cup Z$. We consider the following cases.

(a) If $w_1, w_2 \in (\{x_i\} \times Y) \cup Z$ and $\{w_1, w_2\} \not\subset Z$, then (w_1, w_2) appears t-apart in a unique block of $\mathbf{C}_i$ for any $t = 1, 2, \dots, k-1$.

(b) If $\{w_1, w_2\} \subset Z$, then (w_1, w_2) appears t-apart in a unique block of $\mathbf{C}_1$ for any $t = 1, 2, \dots, k-1$.

(c) If $w_1 \in \{x_i\} \times Y$ and $w_2 \in \{x_j\} \times Y$ with $i \neq j$, then the ordered pair (x_i, x_j) appears t-apart in a unique block A of $\mathbf{A}$ for any $t = 1, 2, \ldots, k-1$. Therefore the ordered pair (w_1, w_2) appears t-apart in a unique block of $\widetilde{\mathbf{B}}_A$ for any $t = 1, 2, \ldots, k-1$.

The three cases described above are mutually exclusive and cover all possibilities.

Theorem 2.3. *Suppose there exist: (1) a $(v, k, 1)$-PMD, (2) a resolvable TD$[k, 1; u]$, (3) an IPMD$[u+l, k, l]$, (4) an IPMD$[v+m, k, m]$, (5) a $(l+m, k, 1)$-PMD, where v, k and u are positive integers, l and m are nonnegative integers. Then there exists a $(vu+l+m, k, 1)$-PMD.*

Proof: We adapt the notations in the proof of Theorem 2.2. Let S be a m-set with $S \cap X = S \cap Y = S \cap (X \times Y) = \emptyset$. For every $A = (x_{i_1}, x_{i_2}, \ldots x_{i_k}) \in \mathbf{A}$, since there exists a resolvable TD$[k, 1; u]$, we can let $\mathbf{D}_A = \{((x_{i_1}, y_j), (x_{i_2}, y_j), \ldots, (x_{i_k}, y_j)) \mid j = 1, 2, \ldots, u\} \subset \widetilde{\mathbf{B}}_A$. From condition (4) we can let $((X \times \{y_i\}) \cup S, S, \mathbf{E}_i)$ be an IPMD$[v+m, k, m]$ where $i = 1, 2, \ldots, u$. From condition (3) we can let $((\{x_1\} \times Y) \cup Z, Z, \mathbf{D}_1)$ be an IPMD$[u+l, k, l]$. From condition (5) we can let $(S \cup Z, \mathbf{F})$ be a $(m+l, k, 1)$-PMD. It is easy to see that

$$\left((X \times Y) \cup Z \cup S,\ \left(\bigcup_{A \in \mathbf{A}} (\widetilde{\mathbf{B}}_A \backslash \mathbf{D}_A)\right) \cup \left(\bigcup_{i=2}^{v} \mathbf{C}_i\right) \cup \mathbf{D}_1 \cup \mathbf{F} \cup \left(\bigcup_{i=1}^{u} \mathbf{E}_i\right)\right)$$

is a $(vu+l+m, k, 1)$-PMD.

Theorem 2.4. *Suppose there exist:*

(1) *a $(v, k, 1)$-RPBD of $(X, \mathbf{A})$ where $\mathbf{A}$ can be partitioned into s parallel classes $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_s$,*
(2) *an IPMD $[|A|+l_i, k, l_i]$ for every $A \in \mathbf{A}_i$ where $l_i \geq 0, 1 \leq i \leq s$,*
(3) *a $(\sum_{i=1}^{s} l_i, k, 1)$-PMD.*

Then there exists a $(v + \sum_{i=1}^{s} l_i, k, 1)$-PMD.

Proof: Let $Y_1, Y_2, \ldots, Y_s$ be s sets of points where $|Y_i| = l_i$ and $Y_i \cap X = \emptyset$ for $1 \leq i \leq s$ and $Y_i \cap Y_j = \emptyset$ if $i \neq j$. Let $Y = \cup_{i=1}^{s} Y_i$. From condition (2) we can let $(A \cup Y_i, Y_i, \mathbf{C}_A^i)$ be an IPMD$[|A|+l_i, k, l_i]$ where $A \in \mathbf{A}_i$ for $i = 1, 2, \ldots, s$. Let $\mathbf{C}^i = \cup_{A \in \mathbf{A}_i} \mathbf{C}_A^i$. From condition (3) we can let $(Y, \mathbf{B})$ be a $(\sum_{i=1}^{s} l_i, k, 1)$-PMD. We now prove that $(X \cup Y,\ (\cup_{i=1}^{s} \mathbf{C}^i) \cup \mathbf{B})$ is a $(v + \sum_{i=1}^{s} l_i, k, 1)$-PMD. Let (w_1, w_2) ba an ordered pair of points of $X \cup Y$.

(a) If $w_1 \in X$, $w_2 \in Y_i$ where $1 \leq i \leq s$, then there exists a unique block A of $\mathbf{A}_i$ such that $w_1 \in A$, therefore (w_1, w_2) appears t-apart in a unique block of $\mathbf{C}_A^i$ for $t = 1, 2, \ldots, k-1$. If $w_1 \in Y_i$, $w_2 \in X$, the proof is similar.

(b) If $\{w_1, w_2\} \subset Y$, then (w_1, w_2) appears t-apart in a unique block of $\mathbf{B}$ for $t = 1, 2, \ldots, k-1$.

(c) If $\{w_1, w_2\} \subset X$, then the pair $\{w_1, w_2\}$ appears in a unique block A of $\mathbf{A}_i$ where $1 \leq i \leq s$ from condition (1). Therefore, the ordered pair (w_1, w_2) appears t-apart in a unique block of $\mathbf{C}_A^i$ for $t = 1, 2, \ldots, k-1$.

The three cases described above are mutually exclusive and cover all possibilities.

3. New results for $(v, 4, 1)$-PMD

In this section, we need the following notations:

PMD = $\{v \mid$ there exists a $(v, 4, 1)$-PMD$\}$

IPMD[n] = $\{v \mid$ there exists an IPMD[$v, 4, n$]$\}$

The following facts are obvious.

(1) The existence of a $(v, 4, 1)$-RPMD implies the existence of a $(v, 4, 1)$-PMD.

(2) There does not exist any $(4, 4, 1)$-PMD.

(3) A $(v, 4, 1)$-MD is perfect if any ordered pair of points appears t-apart in a unique block for $t = 1, 2$.

Lemma 3,1. *Let $2s+1$ be a prime power where s is odd and $s > 1$, then $3s+1 \in$ IPMD[s].*

Proof: Let w be a primitive root of $GF(2s+1)$. Let $A_i = (\infty_i, 0, w^{2i}, w^{2i}(1+w))$ be base blocks where $i = 0, 1, 2, \ldots, s-1$. It is readily checked that 1-apart difference $D_1 = \cup_{i=0}^{s-1}\{w^{2i}, w^{2i+1}\} = GF(2s+1)\backslash\{0\}$. Since s is odd and $s > 1$, we have 2-apart difference $D_2 = \cup_{i=0}^{s-1}\{w^{2i}(1+w), -w^{2i}(1+w)\} = GF(2s+1\backslash\{0\}$. Therefore, $3s+1 \in$ IPMD[s].

Lemma 3.2.

$$\{13, 17\} \subset IPMD[4], \quad \{20, 24, 36\} \subset IPMD[5], \quad 25 \in IPMD[8],$$
$$32 \in IPMD[9], \quad \{44, 48\} \subset IPMD[13], \{52, 56\} \subset IPMD[17],$$
$$\{68, 72\} \subset IPMD[21], \quad 84 \in IPMD[25], \quad 92 \in IPMD[29],$$
$$132 \in IPMD[41].$$

Proof: Here, we always take the additive group of integers mod $(v-n)$. Let

$A_i = (\infty_i, 0, a_i, b_i)$ where $i = 1, 2, \ldots, n$ and

$B_j = (0, c_j, d_j, e_j)$ where $j = 1, 2, \ldots, s$ (s is a nonegative integer)

be base blocks. For brevity we denote these blocks by

$$\begin{bmatrix} & & & & c_1 & c_2 & \ldots & c_s \\ a_1 & a_2 & \ldots & a_n & d_1 & d_2 & \ldots & d_s \\ b_1 & b_2 & \ldots & b_n & e_1 & e_2 & \ldots & e_s \end{bmatrix}.$$

It is easy to see that $v \in \mathrm{IPMD}[n]$ whenever

$$D_1 = \left(\bigcup_{i=1}^{n}\{b_i - a_i, a_i\}\right) \cup \left(\bigcup_{i=1}^{s}\{c_i, d_i - c_i, e_i - d_i, -e_i\}\right) = Z_{v-n}\backslash\{0\} \text{ and}$$

$$D_2 = \left(\bigcup_{i=1}^{n}\{\pm b_i\}\right) \cup \left(\bigcup_{i=1}^{s}\{\pm d_i, \pm(e_i - c_i)\}\right) = Z_{v-n}\backslash\{0\}.$$

It is readily checked that the following parameters all satisfy the condition $D_1 = D_2 = Z_{v-n}\backslash\{0\}$.

$$Z_{13-4} \begin{bmatrix} -2 & 2 & -3 & 4 \\ -1 & -4 & 2 & 3 \end{bmatrix}, \quad Z_{17-4} \begin{bmatrix} & & & & -3 \\ -5 & 4 & -2 & 3 & 6 \\ 1 & 3 & 5 & 4 & -5 \end{bmatrix},$$

$$Z_{20-5} \begin{bmatrix} & & & & & 1 \\ 4 & 6 & 5 & -7 & -4 & 3 \\ -4 & 1 & 2 & 7 & -6 & 6 \end{bmatrix}, \quad Z_{24-5} \begin{bmatrix} & & & & & 1 & 5 \\ 6 & 4 & 8 & 9 & -5 & 3 & -7 \\ -1 & -4 & 6 & 8 & -9 & 6 & 3 \end{bmatrix},$$

$$Z_{36-5} \begin{bmatrix} & & & & & 1 & 3 & 7 & 15 & 6 \\ 10 & -13 & -10 & 9 & 5 & 15 & 7 & 3 & 1 & 14 \\ 5 & 9 & 10 & -11 & -13 & 7 & 1 & 15 & 3 & 2 \end{bmatrix},$$

$$Z_{25-8} \begin{bmatrix} 2 & 8 & -2 & 4 & 6 & -7 & 3 & -4 \\ -1 & 2 & 3 & -4 & 5 & -6 & -7 & 8 \end{bmatrix},$$

$$Z_{32-9} \begin{bmatrix} & & & & & & & & & 1 \\ 8 & 9 & -2 & -10 & -1 & 11 & -4 & 10 & 7 & 3 \\ -1 & -2 & 4 & 5 & -7 & 8 & -9 & -10 & -11 & 7 \end{bmatrix},$$

$$Z_{44-13} \begin{bmatrix} & & & & & & & & & & & & & 2 \\ 12 & 8 & 13 & 1 & -2 & -8 & -10 & 3 & 4 & 9 & -3 & -5 & 5 & 8 \\ -2 & -3 & -4 & -5 & -6 & 7 & 9 & -10 & 11 & -12 & 13 & -14 & -15 & 1 \end{bmatrix},$$

$$Z_{48-13} \begin{bmatrix} & & & & & & & & & & & & & 1 & 3 \\ -15 & -16 & 17 & 11 & -12 & -13 & 12 & 9 & -10 & -3 & -5 & -6 & 14 & 3 & 8 \\ 1 & 2 & -5 & -9 & 12 & 13 & -15 & -16 & 17 & -4 & -7 & -10 & -14 & 7 & 14 \end{bmatrix},$$

$$Z_{52-17} \begin{bmatrix} 14 & 13 & 11 & 3 & -1 & -6 & 6 & -9 & -10 & 2 & 7 & -3 & -5 & -2 & 5 & 1 & 8 \\ -1 & 2 & -3 & -4 & -5 & 6 & -7 & 8 & 9 & -10 & 11 & 12 & -13 & 14 & 15 & -16 & 17 \end{bmatrix},$$

$$Z_{68-21} \begin{bmatrix} & & & & & & & & & & & & & & & & & & & & & 3 \\ 17 & 18 & -2 & -8 & -15 & -7 & -1 & -13 & -5 & 6 & 16 & -10 & -3 & 10 & -4 & 2 & 4 & -6 & 1 & -9 & 9 & 10 \\ -1 & -2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & -11 & -12 & 13 & -14 & -15 & -16 & -17 & -19 & -20 & -21 & 22 & -23 & 21 \end{bmatrix},$$

$$Z_{56-17}\left[\begin{array}{cccccccccccccccccc}
 & & & & & & & & & & & & & & & & & 3\\
6 & -13 & -1 & -17 & -15 & -19 & -12 & 12 & 1 & -2 & -3 & -6 & -4 & -9 & -7 & 7 & 2 & 11\\
1 & 2 & 3 & 4 & -5 & -6 & 7 & -9 & 10 & 12 & 13 & -14 & -15 & 16 & -17 & 18 & 19 & -5
\end{array}\right],$$

$$Z_{72-21}\left[\begin{array}{ccccccccccccccccccccccc}
 & & & & & & & & & & & & & & & & & & & & & 1 & 3\\
-17 & -24 & 22 & 11 & 9 & 15 & -15 & -13 & 10 & 13 & 17 & 20 & -12 & -18 & -19 & 19 & -16 & -3 & -5 & -6 & 14 & 3 & 8\\
1 & 2 & -5 & -9 & -12 & -13 & 14 & 15 & -16 & -17 & -18 & -19 & -20 & 22 & 23 & -24 & 25 & -4 & -7 & -10 & 21 & 7 & 14
\end{array}\right],$$

$$Z_{84-25}\left[\begin{array}{ccccccccccccccccccccccccccc}
 & & & & & & & & & & & & & & & & & & & & & & & & & 1 & 3\\
9 & -25 & 26 & -13 & -8 & -16 & -23 & 23 & -15 & -24 & -21 & -11 & -12 & 19 & -17 & -26 & 24 & 25 & 12 & 11 & 16 & -3 & -5 & -6 & 14 & 3 & 8\\
-1 & 2 & -5 & 9 & 12 & 13 & 14 & -15 & 16 & 17 & 18 & 19 & 20 & -22 & 23 & 24 & -25 & -26 & 27 & 28 & 29 & -4 & -7 & -10 & 21 & 7 & 14
\end{array}\right],$$

$$Z_{92-29}\left[\begin{array}{ccccccccccccccccccccccccccccc}
 & & & & & & & & & & & & & & & & & & & & & & & & & & & & 1\\
10 & 26 & -5 & 27 & -8 & -3 & 12 & -20 & -25 & 18 & -21 & -16 & 14 & -15 & 19 & 22 & -4 & -6 & 6 & -27 & -22 & 23 & -13 & 15 & -10 & -19 & 21 & 25 & 28 & 3\\
-1 & -2 & 4 & -7 & 9 & 10 & -11 & 12 & 14 & -15 & 16 & 17 & -18 & 19 & -20 & -21 & -5 & -8 & 13 & 22 & 23 & -24 & -25 & 26 & -27 & -28 & 29 & 30 & 31 & 7
\end{array}\right],$$

$$Z_{132-41}\left[\begin{array}{ccccccccccccccccccccccccccccccccccccccccccc}
 & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & 1 & 3\\
-37 & -41 & -27 & 12 & 35 & 11 & -17 & -38 & 38 & 33 & 10 & 26 & 15 & -40 & -19 & 9 & -9 & -26 & -34 & -32 & 22 & -25 & -18 & 20 & -15 & -23 & -44 & 34 & -8 & 8 & -42 & 27 & 25 & -22 & 24 & 29 & 28 & -3 & -5 & -6 & 14 & 3 & 8\\
-1 & 2 & 5 & -9 & -12 & -13 & 14 & -15 & -16 & -17 & -18 & -19 & -20 & 22 & 23 & -24 & -25 & 26 & 27 & 28 & -29 & 30 & -31 & -32 & 33 & -34 & 35 & -36 & 37 & 38 & 39 & 40 & 41 & -42 & 43 & -44 & 45 & -4 & -7 & -10 & 21 & 7 & 14
\end{array}\right].$$

Lemma 3.3. *Suppose*

(1) $N(u) \geq v - 1$,

(2) $u + m \in IPMD[m]$ *and* $v + l_i \in IPMD[l_i]$ *where* $i = 1, 2, \ldots, u$,

(3) $m + \sum_{i=1}^{u} l_i \in PMD$.

Then $uv + m + \sum_{i=1}^{u} l_i \in PMD$.

Proof: Since $N(u) \geq v-1$, we have a resolvable TD$[v,1;u]$ of $(X,\mathbf{G},\mathbf{B})$ and then a $(uv,\{u,v\},1)$-RPMD of $(X,\mathbf{G}\cup\mathbf{B})$. Therefore, we have $uv+m+\sum_{i=1}^{u} l_i \in$ PMD by Theorem 2.4.

Since $v \in$ PMD, $v \in$ IPMD[0] and $v \in$ IPMD[1] are pairwise equivalent, we have from Lemma 3.3 the following Corollaries.

Corollary 3.4. *Suppose*

(1) $N(u) \geq v-1$,

(2) $v \in PMD, v+l \in IPMD[l]$ *and* $u+m \in IPMD[m]$,

(3) $m+sl \in PMD$ *where* $0 \leq s \leq u$.

Then $uv+m+sl \in PMD$.

Corollary 3.5. *Suppose*

(1) $N(u) \geq v-1$,

(2) $v+1 \in PMD, v+5 \in IPMD[5]$ *and* $u \in PMD$,

(3) $u+4s \in PMD$ *where* $0 \leq s \leq u$.

Then $uv+u+4s \in PMD$.

Theorem 3.6. *(1) If* $16 \leq v \leq 272$ *and* $v \equiv 0 \pmod 4$, *then* $v \in PMD$. *(2)* $\{57,93,133\} \subset PMD$.

Proof: (a) Taking $s = 5,9,13,21,29$ in Lemma 3.1 and using Theorem 1.2 and Theorem 2.1, we have $\{16,28,40,64,88\} \subset$ PMD. By using Lemma 3.2, Theorem 1.2 and Theorem 2.1, we obtain $\{20,24,32,36,44,48,52,56,68,72,84,92,132\} \subset$ PMD.

(b) We have $\{5,9,13,17,25,16,20,24,32\} \subset$ PMD from (a) and Theorem 1.2. We also have $\{17,13\} \subset$ IPMD[4] and $25 \in$ IPMD[8] from Lemma 3.2 and $16 \in$ IPMD[5] from Lemma 3.1. Since

$$60 = 5\times 11+5,\quad 76 = 5\times 15+1,\quad 80 = 5\times 16,\quad 96 = 5\times 19+1,$$
$$100 = 20\times 5,\quad 156 = 5\times 31+1,\ 212 = 16\times 13+4,\ 216 = 24\times 9,$$
$$220 = 24\times 9+4,\quad 93 = 5\times 17+8,$$

it is easy to see by using Theorem 2.2 that $\{60,76,80,96,100,156,212,216,220,93\} \subset$ PMD.

(c) Since $N(11) \geq 3$, $16 \in$ IPMD[5], $17 \in$ IPMD[4] and $13,9 \in$ PMD, we have $152 = 13\times 11+5+4 \in$ PMD by using Theorem 2.3. Since $N(4) \geq 3$, $13,5 \in$ PMD and $17 \in$ IPMD[4], we have $57 = 13\times 4+1+4 \in$ PMD by using Theorem 2.3. Take $u=13$, $v=9$, $l=4$, $m=0$, and $s=4$ in Corollary 3.4, since $16 \in$ PMD from (a), we have $133 = 13\times 9+16 \in$ PMD.

(d) Take $0 \leq s \leq 11$ and $s \neq 7$. We have $5+4s \in$ PMD from Theorem 1.2, and then $11\times 9+5+4s \in$ PMD by Corollary 3.4. We also have $132 \in$

PMD from (a), therefore, $\{v|104 \leq v \leq 148 \text{ and } v \equiv 0 \pmod 4\} \subset$ PMD. For $4 \leq s \leq 16$ we have $4s \in$ PMD from (a) and (b). Using Corollary 3.4 we obtain $16 \times 9 + 4s \in$ PMD and $16 \times 13 + 4s \in$ PMD. That is,

$$\{v|160 \leq v \leq 208 \text{ and } v \equiv 0 \pmod 4\} \subset \text{ PMD and}$$
$$\{v|224 \leq v \leq 272 \text{ and } v \equiv 0 \pmod 4\} \subset \text{ PMD.}$$

The proof is now complete.

The following corollary is straightforward by combining Theorem 1.2 and Theorem 3.6.

Corollary 3.7. *If $v \equiv 1 \pmod 4$ and $v \neq 33$, then $v \in$ PMD.*

Corollary 3.8. *If (1) $N(u) \geq 14$, and (2) $u \equiv 1 \pmod 4$ and $u + 4s \neq 33$ where $0 \leq s \leq u$. Then $16u + 4s \in$ PMD.*

Proof: Take $v = 15$ in Corollary 3.5. Since $16 \in$ PMD from Theorem 3.6, $20 \in$ IPMD[5] from Lemma 3.2 and $\{u, u + 4s\} \subset$ PMD from Corollary 3.7, we then have $16u + 4s \in$ PMD.

Theorem 3.9. *If $v \geq 276$ and $v \equiv 0 \pmod 4$, then $v \in$ PMD.*

Proof: (a) Taking $5 \leq s \leq 8$, $u = 16$ and $v = 15$ in Corollary 3.5, we have $16 \times 15 + 16 + 4s \in$ PMD, that is, $\{v|276 \leq v \leq 288 \text{ and } v \equiv 0 \pmod 4\} \subset$ PMD. By using Corollary 3.4 and Corollary 3.7, we have $23 \times 13 + 1 + 4s \in$ PMD for $10 \leq s \leq 23$, i.e., $\{v|340 \leq v \leq 392 \text{ and } v \equiv 0 \pmod 4\} \subset$ PMD, $27 \times 13 + 1 + 4s \in$ PMD for $10 \leq s \leq 27$, i.e., $\{v|392 \leq v \leq 460 \text{ and } v \equiv 0 \pmod 4\} \subset$ PMD, $43 \times 13 + 21 + 4s \in$ PMD for $0 \leq s \leq 2$, i.e., $\{v|580 \leq v \leq 588 \text{ and } v \equiv 0 \pmod 4\} \subset$ PMD. By using Corollary 3.8, we have $17 \times 15 + 17 + 4s \in$ PMD for $5 \leq s \leq 17$, i.e., $\{v|292 \leq v \leq 340 \text{ and } v \equiv 0 \pmod 4\} \subset$ PMD, $25 \times 15 + 25 + 4s \in$ PMD for $15 \leq s \leq 25$, i.e., $\{v|460 \leq v \leq 500 \text{ and } v \equiv 0 \pmod 4\} \subset$ PMD, $29 \times 15 + 29 + 4s \in$ PMD for $9 \leq s \leq 29$, i.e., $\{v|500 \leq v \leq 580 \text{ and } v \equiv 0 \pmod 4\} \subset$ PMD. Now, we have proved that $\{v|276 \leq v \leq 588 \text{ and } v \equiv 0 \pmod 4\} \subset$ PMD.

(b) Let $t_1 = 37$, $t_2 = 41$, $t_3 = 49$, $t_4 = 61$, $t_5 = 73$, $t_6 = 81$, $t_7 = 97$, $t_8 = 101$, $t_9 = 113$, $t_{10} = 137$, $t_{11} = 149$, $t_{12} = 181$, $t_{13} = 197$, $t_{14} = 229$, $t_{15} = 277$, $t_{16} = 337$, $t_{17} = 409$, $t_{18} = 509$, $t_{19} = 617$, $t_{20} = 761$. It is clear that t_i is prime power and $t_i = 1 \pmod 4$ for $1 \leq i \leq 20$. Then $N(t_i) \geq 14$ and $N(25t_i) \geq 14$ for $1 \leq i \leq 20$ from Lemma 1.3. By using Corollary 3.8 we have

$$\{v|16t_i \leq v \leq 16t_i + 4t_i \text{ and } v \equiv 0 \pmod 4\} \subset \text{ PMD and}$$
$$\{v|16 \cdot 25t_i \leq v \leq 16 \cdot 25t_i + 4 + 25t_i \text{ and } v \equiv 0 \pmod 4\} \subset \text{ PMD}$$

for $1 \leq i \leq 20$. It is readily checked that $4(t_{i+1} - t_i) \leq t_i$, that is, $16t_i + 4t_i \geq 16t_{i+1}$ and $16 \cdot 25t_i + 4 \cdot 25t_i \geq 16 \cdot 25t_{i+1}$ for $1 \leq i \leq 19$. Since $16t_1 =$

592, $16t_{20}+4t_{20} = 15220$, $16 \cdot 25t_1 = 14800$, $16 \cdot 25t_{20}+4 \cdot 25t_{20} = 380500$, we have obtained that $\{v|592 \le v \le 380500 \text{ and } v \equiv 0 \pmod 4\} \subset$ PMD.

(c) Let $t_0 = 2700$, $t_i = t_0 + i$ for $i \ge 1$.

It is easy to see that $4t_i + 1 > 10632$ for $i \ge 0$. Therefore, we have $N(4t_i + 1) \ge 14$ from Lemma 1.4. Taking $0 \le s \le 16$ in Corollary 3.8, we have $\{v|16(4t_i + 1) \le v \le 16(4t_i + 1) + 64 \text{ and } v \equiv 0 \pmod 4\} \subset$ PMD. Since $16(4t_i + 1) + 64 = 16(4t_{i+1} + 1)$, we have that $v \in$ PMD if $v \equiv 0 \pmod 4$ and $v \ge 16(4t_0 + 1) = 172816$.

Combining (a), (b) and (c) completes the proof.

From Theorem 1.2, Theorem 3.6 and Theorem 3.9 we conclude with the following theorem.

Theorem 3.10. *A* $(v, 4, 1)$*-PMD exists for every positive integer* $v \equiv 0$ *or* $1 \pmod 4$ *with the exception of* $v = 4$ *and the possible exception of* $v = 8, 12, 33$.

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