An Application of Partitioned Balanced Tournament Designs to the Construction of Semiframes with Block Size Two

Rolf Rees

Dept. Math. and Computer Science
Mount Allison University

Abstract. We employ a well-known class of designs to give a complete solution to the problem of determining the spectrum of uniform semiframes with block size two. As a corollary we prove that the complete graph K_{gu} admits a one-factorization with an orthogonal set of u disjoint sub-one-factorizations of K_g if and only if g is even and $u \geq 3$.

1. Introduction

In a recent paper [7] the author introduced a particular generalization of a frame, which we call a semiframe. We begin with a brief review of the definitions and constructions contained therein.

A *group-divible design* (GDD) is a triple (X, G, B) where X is a set of *points*, G is a partition of X into *groups* and B is a collection of subsets of X (called *blocks*) such that any pair of distinct points from X occurs in one group *or* one block, but not in both. A *parallel class* of blocks in a GDD is a subset $B' \subseteq B$ which partitions X; a *holey parallel class* of blocks is a subset $B'' \subseteq B$ which partitions $X - G_j$ for some group $G_j \in G$. A *semiframe* is a group-divisible design whose block set B can be written as a disjoint union $P \cup Q$, where P can be partitioned into parallel classes and Q can be partitioned into holey parallel classes. Note that when $P = \emptyset$ we have a frame, while at the other end of the spectrum when $Q = \emptyset$ we have a resolvable GDD.

We are concerned herein with semiframes having block size two, in which all the groups have the same size. We begin by remarking that the spectra for frames and resolvable GDDs with block size two are well-known (short proofs can be found, e.g. in [6]):

Theorem 1.0. *A resolvable* 2-GDD *with* u *groups of size* g *exists if and only if* $u \geq 2$ *and* $gu \equiv 0 \pmod 2$. *A 2-frame with* u *groups of size* g *exists if and only if* $u \geq 3$ *and* $g(u-1) \equiv 0 \pmod 2$.

We will assume then that there are both parallel classes and holey parallel classes present (such a semiframe will be called *proper*), whence we have the following (our notation is as in [7]: a 2-SF$(p, d; g^u)$ denotes a semiframe with u groups of size g in which there are p parallel classes and in which for each group G_j there are d holey parallel classes of pairs that partition $X - G_j$):

Lemma 1.1. [7, Corollary 2.2] *If there is a proper* 2-SF$(p, d; g^u)$ *then* g *is even,* $u \geq 3$, $p \equiv 0 \pmod{u-1}$ *and* $d = g - \frac{p}{u-1}$.

We will show that the conditions in lemma 1.1 are sufficient. Our main tools will be the following special cases of constructions 3.1 and 3.2 from [7]:

Construction 1.2. *Let* (X, G, B) *be a frame with block size two containing* t_i *groups of size* s_i, $i = 1, \ldots, j$. *Suppose that for each* $i = 1, \ldots, j$ *there is a* 2-SF$(p_i, d; g^{1+\frac{s_i}{g}})$. *Then there is a* 2-SF$(\sum_i p_i t_i, d; g^{1+\frac{|X|}{g}})$.

Construction 1.3. *Let* (X, G, B) *be a* 2-SF$(p, d; g^u)$ *and suppose that there are* $u - 1$ *parallel classes of blocks whose union can be partitioned into* u *holey parallel classes. Then for each positive integer* n *and each* $i = 0, 1, \ldots, n$ *there is a* 2-SF$(np - i(u-1), nd + i; (ng)^u)$.

We will also make use of the following construction.

Construction 1.4. *Suppose that there is a transversal design* TD$(u+1, n)$ *and that for each* $i = 1, 2, \ldots, n$ *there is a* 2-SF$(p_i, d_i; g^u)$. *Then there is a* 2-SF$(\sum p_i, \sum d_i; (ng)^u)$.

Proof: Take one of the groups and label its points $1, 2, \ldots, n$, and let B_i denote the pencil of blocks containing point i. Now for each $i = 1, 2, \ldots, n$ and each $B \in B_i$ build a 2-SF$(p_i, d_i; g^u)$ on $B - \{i\}$. (Each point in $B - \{i\}$ is, of course, being replaced by g new points.). ▮

2. Semiframes with block size two

We will use as our principal building block a design called a partitioned balanced tournament design. A *balanced tournament design* BTD(n) is an arrangement of the $\binom{2n}{2}$ distinct unordered pairs of a $2n$-set X into an $n \times 2n-1$ array such that

(i) each element of X appears exactly once in each column of the array, and
(ii) each element of X appears at most twice in each row of the array.

Schellenberg, van Rees and Vanstone [8] have established the existence of BTD(n) for every positive integer $n \neq 2$. Note that BTD(n) gives a schedule for a $2n$-player round-robin tournament with n playing surfaces in which the players are distributed over the playing surfaces in as fair a manner as possible (the columns of the array correspond to rounds and the rows of the array correspond to playing surfaces).

A balanced tournament design is said to be *partitionable* (and is denoted PBTD(n)) if its columns can be partitioned into three sets C_1, C_2, C_3, of sizes 1, $n-1$ and $n-1$, so that each subarray $C_1 \cup C_2$ and $C_1 \cup C_3$ satisfies:

(ii)' each element of X appears exactly once in each row of the subarray.

These designs have been studied by Stinson [9], Lamken and Vanstone [2, 3, 4] and Lamken [1] and their spectrum has been almost completely determined:

Theorem 2.1. *There exists a* PBTD(n) *for every positive integer* $n \geq 5$, *except possibly for* $n \in \{9, 11, 15, 26, 28, 34, 44\}$.

Now we make an observation: a PBTD(n) implies the existence of a 2-SF($2n-2, 0; 2^n$) whose blocks can be partitioned into two sets, each of which can be viewed alternatively either as forming $n-1$ parallel classes of pairs or n holey parallel classes of pairs. The following is now an immediate consequence of theorem 2.1 and construction 1.3.

Theorem 2.2. *Let* $u \geq 5$, $u \notin \{9, 11, 15, 26, 28, 34, 44\}$, *and let* $g \equiv 0 \pmod 2$. *Then for each* $p \equiv 0 \pmod{u-1}$, $0 \leq p \leq g(u-1)$, *there is a* 2-SF($p, d; g^u$) *where* $d = g - \frac{p}{u-1}$.

Proof: If $\frac{g}{2}(u-1) \leq p \leq g(u-1)$ apply construction 1.3 to a PBTD(u) viewed as a 2-SF($2u-2, 0; 2^u$). If $0 \leq p \leq \frac{g}{2}(u-1)$ apply construction 1.3 to a PBTD(u), viewed instead as a 2-SF($u-1, 1; 2^u$). ∎

Lemma 2.3. *The conditions in lemma 1.1 are sufficient when* $u \in \{9, 11, 15\}$.

Proof: Apply construction 1.3 to the following semiframes. In each case the relevant parallel classes are indicated by an asterisk.

<u>2-SF($8, 1, 2^9$)</u> Points $\mathbb{Z}_9 \times \{1, 2\}$

Groups $\{\{i_1, i_2\} : i \in \mathbb{Z}_9\}$

Holey parallel classes:

$$1_j8_j \quad 2_j7_j \quad 3_j6_j \quad 4_j5_j \quad , j = 1, 2 \qquad \text{mod } 9$$

(*) parallel classes: develop each of the pairs

$$1_12_2 \quad 2_14_2 \quad 3_16_2 \quad 4_18_2 \quad 5_11_2 \quad 6_13_2 \quad 7_15_2 \quad 8_17_2 \qquad \text{mod } 9.$$

<u>2-SF($16, 0; 2^9$)</u> Points $(\mathbb{Z}_8 \times \{1, 2\}) \cup \{\infty_1, \infty_2\}$

Groups $\{\{i_1, i_2\} : i \in \mathbb{Z}_8\} \cup \{\{\infty_1, \infty_2\}\}$

Parallel classes:

$i_1(i+1)_2$	$(i+5)_1(i+7)_2$		$i_1(i+5)_2$	$(i+7)_1(i+3)_2$		
$(i+1)_1(i+3)_2$	$(i+7)_1(i+2)_2$		$(i+1)_1(i+5)_1$	$(i+2)_2(i+6)_2$		
$(i+2)_1i_2$	$(i+6)_1\infty_2$	;	$(i+3)_1(i+7)_2$	$(i+2)_1\infty_2$		$i = 0, 1, 2, 3$
$(i+3)_1(i+6)_2$	$(i+4)_2\infty_1$		$(i+4)_1(i+1)_2$	$i_2\infty_1$		
$(i+4)_1(i+5)_2$			$(i+6)_1(i+4)_2$			

(*) parallel classes: columns in the 9×8 array whose first row is

$$2_11_2 \quad 3_12_2 \quad 4_13_2 \quad \ldots \quad 1_10_2$$

and whose remaining rows are obtained by cyclic shifts of

$$\infty_10_1 \quad \infty_20_2 \quad 5_16_1 \quad 5_26_2 \quad 1_13_1 \quad 1_23_2 \quad 4_17_1 \quad 4_27_2 \qquad \text{mod } 8$$

(e.g. The third row is $5_2 0_2 \quad \infty_1 1_1 \quad \infty_2 1_2 \quad 6_1 7_1 \quad 6_2 7_2 \quad 2_1 4_1 \quad 2_2 4_2 \quad 5_1 0_1$, etc.).

<u>2-SF$(10, 1, 2^{11})$</u> Points $\mathbb{Z}_{11} \times \{1, 2\}$

Groups $\{\{i_1, i_2\} : i \in \mathbb{Z}_{11}\}$

Holey parallel classes:

$$1_j 10_j \quad 2_j 9_j \quad 3_j 8_j \quad 4_j 7_j \quad 5_j 6_j, \quad j = 1, 2 \qquad \text{mod } 11$$

(*) parallel classes: develop each of the pairs

$$\begin{array}{llllll} 1_1 2_2 & 2_1 4_2 & 3_1 6_2 & 4_1 8_2 & 5_1 10_2 & \\ 6_1 1_2 & 7_1 3_2 & 8_1 5_2 & 9_1 7_2 & 10_1 9_2 & \text{mod } 11. \end{array}$$

<u>2-SF$(20, 0, 2^{11})$</u> Points $(\mathbb{Z}_{10} \times \{1, 2\}) \cup \{\infty_1, \infty_2\}$

Groups $\{\{i_1, i_2\} : i \in \mathbb{Z}_{10}\} \cup \{\{\infty_1 \infty_2\}\}$

Parallel classes:

$$\begin{array}{llcllc} i_1(i+1)_2 & (i+7)_1 i_2 & & i_1(i+4)_2 & (i+6)_1(i+3)_2 & \\ (i+1)_1(i+3)_2 & (i+8)_1(i+4)_2 & & (i+1)_1(i+8)_2 & (i+7)_1(i+6)_2 & \\ (i+2)_1(i+5)_2 & (i+9)_1(i+7)_2 & ; & (i+2)_1(i+1)_2 & i_2(i+5)_2 & i = 0, 1, \ldots, 4 \\ (i+3)_1(i+9)_2 & (i+4)_1\infty_2 & & (i+3)_1(i+8)_1 & (i+9)_1\infty_2 & \\ (i+5)_1(i+6)_2 & (i+2)_2\infty_1 & & (i+4)_1(i+2)_2 & (i+7)_2\infty_1 & \\ (i+6)_1(i+8)_2 & & & (i+5)_1(i+9)_2 & & \end{array}$$

(*) parallel classes: columns in the 11×10 array whose first row is

$$3_1 8_2 \quad 4_1 9_2 \quad 5_1 0_2 \quad \ldots \quad 2_1 7_2$$

and whose remaining rows are obtained by cyclic shifts of

$$\begin{array}{llllll} \infty_1 0_1 & 2_2 4_2 & 6_2 9_2 & 7_1 8_1 & 1_1 5_1 & \\ \infty_2 0_2 & 2_1 4_1 & 6_1 9_1 & 7_2 8_2 & 1_2 5_2 & \text{mod } 10. \end{array}$$

<u>2-SF$(14, 1; 2^{15})$</u> Points $\mathbb{Z}_{15} \times \{1, 2\}$

Groups $\{\{i_1, i_2\} : i \in \mathbb{Z}_{15}\}$

Holey parallel classes:

$$1_j 14_j \quad 2_j 13_j \quad 3_j 12_j \quad 4_j 11_j \quad 5_j 10_j \quad 6_j 9_j \quad 7_j 8_j, \quad j = 1, 2 \qquad \text{mod } 15$$

(*) parallel classes: develop each of the pairs

$$1_1 2_2 \quad 2_1 4_2 \quad 3_1 6_2 \quad 4_1 8_2 \quad 5_1 10_2 \quad 6_1 12_2 \quad 7_1 14_2 \quad 8_1 1_2 \quad 9_1 3_2 \quad 10_1 5_2$$
$$11_1 7_2 \quad 12_1 9_2 \quad 13_1 11_2 \quad 14_1 13_2 \qquad \text{mod } 15.$$

$\underline{2\text{-SF}(28,0;2^{15})}$ Points $(\mathbb{Z}_{14} \times \{1,2\}) \cup \{\infty_1\infty_2\}$

Groups $\{\{i_1, i_2\} : i \in \mathbb{Z}_{14}\} \cup \{\{\infty_1\infty_2\}\}$

Parallel classes:

$$\begin{array}{ll} i_1(i+1)_2 & (i+9)_1(i+12)_2 \\ (i+1)_1(i+3)_2 & (i+10)_1 i_2 \\ (i+2)_1(i+5)_2 & (i+11)_1(i+2)_2 \\ (i+3)_1(i+7)_2 & (i+12)_1(i+6)_2 \\ (i+4)_1(i+9)_2 & (i+13)_1(i+11)_2 \\ (i+6)_1(i+4)_2 & (i+5)_1\infty_2 \\ (i+7)_1(i+8)_2 & (i+13)_2\infty_1 \\ (i+8)_1(i+10)_2 & \end{array} ; \quad \begin{array}{ll} i_1(i+11)_2 & (i+8)_1 i_2 \\ (i+1)_1(i+7)_2 & (i+9)_1(i+5)_2 \\ (i+2)_1(i+12)_2 & (i+10)_1(i+9)_2 \\ (i+3)_1(i+2)_2 & (i+13)_1(i+8)_2 \\ (i+4)_1(i+11)_1 & (i+3)_2(i+10)_2 \\ (i+5)_1(i+13)_2 & (i+12)_1\infty_2 \\ (i+6)_1(i+1)_2 & (i+6)_2\infty_1 \\ (i+7)_1(i+4)_2 & \end{array} \quad i = 0, 1, \dots, 6$$

(*) Parallel classes: columns in the 15×14 array whose first row is

$$4_1 11_2 \quad 5_1 12_2 \quad 6_1 13_2 \quad \dots \quad 3_1 10_2$$

and whose remaining rows are obtained by cyclic shifts of

$$\infty_1 0_1 \quad 3_2 5_2 \quad 1_1 7_1 \quad 10_1 11_1 \quad 2_1 6_1 \quad 8_2 13_2 \quad 9_1 12_1 \quad \infty_2 0_2 \quad 3_1 5_1 \quad 1_2 7_2$$
$$10_2 11_2 \quad 2_2 6_2 \quad 8_1 13_1 \quad 9_2 12_2 \qquad \text{mod } 14.$$

This completes the proof of lemma 2.3. ∎

Remark: The designs 2-SF$(n-1,1;2^n)$ constructed in lemma 2.3 are special cases of a more general construction that appears in [**8**, lemma 2.3].

Lemma 2.4. *The conditions in lemma 1.1 are sufficient when* $u \in \{26, 28, 34, 44\}$.

Proof: Here we will use construction 1.2. Consider first the case $u = 26$. Let g and d be given. Take a transversal design TD$(4,7)$ and remove three points from one of the groups; replace each point by g new points and each block by a frame (theorem 1.0) and so construct a frame with block size two having three groups of size $7g$ and one group of size $4g$. Now apply construction 1.2 (the required 'input' semiframes exist by theorem 2.2).

The cases $u = 28$, 34 and 44 are handled analogously, using respectively, TD$(4,7)$, TD$(4,9)$, TD$(4,11)$. Note that in the last case we require, as input, semiframes with 11 groups; these exist by lemma 2.3. ∎

Finally, consider the cases $u = 3, 4$:

Lemma 2.5. *The conditions in lemma 1.1 are sufficient when* $u = 3$ *and also when* $u = 4$ *and* $g \equiv 0 \mod 4$.

Proof: First suppose that $u = 3$. Let g and d be given. Take $2g - 2d$ 1-factors on the complete tripartite graph $K_{g,g,g}$ as parallel classes (see theorem 1.0) and then partition the remaining pairs on $K_{g,g,g}$ into holey parallel classes.

Now let $u = 4$. Apply construction 1.3 to the following semiframes, according to the value of d (as in lemma 2.3 the relevant parallel classes are indicated by an asterisk):

$0 \leq d \leq \frac{1}{2}g$

2-SF$(6,0;2^4)$ Points $\{1,1',\ldots,4,4'\}$

Groups $\{\{1,1'\},\ldots,\{4,4'\}\}$

Parallel classes:

1,2	1,3	1,4
3,4	2,4	2,3
1′,2′	1′,3′	1′,4′
3′,4′	2′,4′	2′,3′

(*) Parallel classes:

1,3′	3,2′	2,1′
2,4′	4,1′	1,2′
3,1′	1,4′	4,3′
4,2′	2,3′	3,4′

$\frac{1}{2}g \leq d \leq g$

2-SF$(6,2;4^4)$ Points $\mathbb{Z}_8 \times \{1,2\}$

Groups $\{\{i_1(i+4)_1 i_2(i+4)_2\} : 0 \leq i \leq 3\}$

Holey parallel classes:

$1_1 2_1 \quad 1_2 2_2 \quad 3_1 6_1 \quad 3_2 6_2 \quad 5_1 7_1 \quad 5_2 7_2 \qquad \text{mod } 8$

(*) parallel classes: develop the following six pairs of blocks mod 8:

$1_1 2_2 \quad 5_1 6_2 \qquad 2_1 7_2 \quad 6_1 3_2 \qquad 3_1 5_2 \quad 7_1 1_2$

$2_1 1_2 \quad 6_1 5_2 \qquad 7_1 2_2 \quad 3_1 6_2 \qquad 5_1 3_2 \quad 1_1 7_2$

This completes the proof of lemma 2.5. ∎

Lemma 2.6. *The conditions in lemma 1.1 are sufficient when* $u = 4$ *and* $g \equiv 2 \bmod 4$.

Proof: We consider first the cases $g = 2, 6$.

2-SF$(3,1;2^4)$ Points $\{1,1',2,2',3,3',4,4'\}$

Groups $\{\{i,i'\} : 1 \leq i \leq 4\}$

Parallel classes:

1,2	1,3	1,4
3,4	2,4	2,3
1′,2′	1′,3′	1′,4′
3′,4′	2′,4′	2′,3′

Holey parallel classes:

$$\begin{array}{cccc} 4,2' & 1,4' & 2,4' & 3,2' \\ 2,3' & 3,1' & 4,1' & 1,3' \\ 3,4' & 4,3' & 1,2' & 2,1' \end{array}$$

2-SF$(18-3d, d; 6^4)$, $1 \le d \le 3$. Apply construction 1.3 to the 2-SF$(6,0;2^4)$ given in the proof of lemma 2.5.

2-SF$(6,4;6^4)$ Points $\mathbb{Z}_6 \times \{1,2,3,4\}$

Groups $\{\mathbb{Z}_6 \times \{j\} : 1 \le j \le 4\}$

Parallel classes:

$$\begin{array}{ccc} \begin{array}{c} 0_1 0_2 \\ 0_3 0_4 \end{array} ; & \begin{array}{c} 0_1 0_3 \\ 0_2 0_4 \end{array} ; & \begin{array}{c} 0_1 0_4 \\ 0_2 0_3 \end{array} ; \\ \begin{array}{c} 0_1 3_2 \\ 0_3 3_4 \end{array} ; & \begin{array}{c} 0_1 3_3 \\ 0_2 3_4 \end{array} ; & \begin{array}{c} 0_1 3_4 \\ 0_2 3_3 \end{array} \end{array} \quad \text{mod } 6$$

Holey parallel classes:

The holey parallel classes with respect to $\mathbb{Z}_6 \times \{4\}$ are

$$\begin{array}{c} i_1(i+2)_2 \\ (i+1)_2(i+2)_3 \\ (i+1)_3(i+3)_1 \end{array} ; \begin{array}{c} (i+1)_1(i+2)_2 \\ (i+1)_2 i_3 \\ (i+1)_3 i_1 \end{array} ; \begin{array}{c} (i+1)_1(i+3)_2 \\ i_2(i+2)_3 \\ (i+1)_3(i+2)_1 \end{array} ; \begin{array}{c} (i+1)_1 i_2 \\ (i+1)_2(i+3)_3 \\ i_3(i+2)_1 \end{array} \quad i = 0,2,4.$$

The holey parallel classes with respect to $\mathbb{Z}_6 \times \{j\}$ $(j = 1,2,3)$ are obtained by letting the permutation α_j act on the subscripts of the above holey parallel classes, where $\alpha_1 = (124)$, $\alpha_2 = (142)$ and $\alpha_3 = (12)(34)$.

2-SF$(3,5;6^4)$. Use the 2-SF$(6,4;6^4)$ constructed above, changing the last three parallel classes to the four holey parallel classes

$$\begin{array}{llll} \text{(i)} & 0_i 3_j \quad 1_i 4_j \quad 2_i 5_j & (i,j) \in \{(1,2),(2,3),(3,1)\}, \\ \text{(ii)} & 0_i 3_j \quad 1_i 4_j \quad 2_i 5_j & (i,j) \in \{(2,4),(4,3),(3,2)\}, \\ \text{(iii)} & 0_i 3_j \quad 1_i 4_j \quad 2_i 5_j & (i,j) \in \{(4,1),(1,3),(3,4)\}, \\ \text{(iv)} & 0_i 3_j \quad 1_i 4_j \quad 2_i 5_j & (i,j) \in \{(2,1),(1,4),(4,2)\}. \end{array}$$

Now suppose that $g \ge 10$, and let $g = 2n$. Let p and d be given, and let $p_1, p_2 \ldots, p_n$ be any sequence of 0s, 3s and 6s such that $\sum p_i = p$ (this can be done since, from lemma 1.1, $p \equiv 0 \mod 3$ and $0 \le p \le 3g$). Apply construction 1.4 to a TD$(5,n)$, using as input designs 2-SF$(0,2;2^4)$, 2-SF$(3,1;2^4)$ and 2-SF$(6,0;2^4)$ according to the values p_i. (A 2-SF$(3,1;2^4)$ is constructed earlier in this lemma, and the other two semiframes exist by theorem 1.0.) Note that for each i, $d_i = 2 - \frac{1}{3}p_i$, whence

$$\sum_i d_i = \sum_i \left(2 - \frac{1}{3}p_i\right) = g - \frac{1}{3}p = d.$$

This completes the proof of lemma 2.6. ∎

Collecting the results in lemma 1.1, theorem 2.2 and lemmas 2.3 through 2.6 we get our main theorem.

Theorem 2.7. *There exists a proper* 2-SF$(p, d; g^u)$ *if and only if* g *is even,* $n \geq 3$, $p \equiv 0 \pmod{u-1}$ *and* $d = g - \frac{p}{u-1}$.

Theorems 1.0 and 2.7 complete the spectrum for semiframes with block size two.

3. Applications

In section 4 of [7] two particular applications of semiframes were discussed. The first involves resolvable designs admitting a spanning set of resolvable subdesigns. If F is a one-factorization of the complete graph K_v then a one-factorization F' of some $K_g \subseteq K_v$ is said to be a *sub-one-factorization* of F provided that for each one-factor $f' \in F'$ there is a one-factor $f \in F$ such that $f' \subseteq f$. It is well known (see for example [5]) that the complete graph K_v has a one-factorization admitting a sub-one-factorization of K_g if and only if $v \equiv g \equiv 0 \pmod 2$ and $v \geq 2g$. A collection $F_1', F_2', \ldots, F_u'$ of sub-one-factorizations of F will be called an *orthogonal disjoint set* provided that

(i) the u vertex sets covered by $F_1', F_2', \ldots, F_u'$ form a partition of the vertex set of K_v, and
(ii) if e_i is an edge in a one-factor of F_i' and e_j is an edge in a one-factor of F_j', and $i \neq j$, then e_i and e_j occur in different one-factors of F.

Note that if each F_i' consists of a single edge then the above reduces to the usual definition of an orthogonal one-factor with respect to F.

Theorem 3.1. *The complete graph* K_{gu} *admits a one-factorization with an orthogonal set of disjoint sub-one-factorizations of* K_g *if and only if* g *is even and* $u \geq 3$.

Proof: The required one-factorization is equivalent to a 2-SF$(u-1, g-1; g^u)$ whose 'holes' have been filled with one-factorizations of K_g. The required semiframes exist by Theorem 2.7. ∎

The second application of our semiframes is in the construction of a certain class of incomplete group-divisible designs. The following is a special case of lemma 4.3 in [7]:

Lemma 3.2. *A* 2-SF$(p, d; g^u)$ *is equivalent to a* 3-IGDD of type $(g+d, d)^u(p, p)^1$ *in which every block intersects the hole.*

Proof: Adjoin d new points to each 'hole' in the semiframe, each new point completing a holey parallel class. Then adjoin p more points, each one completing a parallel classs in the semiframe. The construction is reversible, since we would be starting with a 3-IGDD in which every block intersects the hole. ∎

The spectrum of 3-IGDDs of the type indicated in lemma 3.2 now follows as a direct corollary to Theorems 1.0 and 2.7.

4. Acknowledgments

This research was supported in part by the Natural Sciences and Engineering Research Council under grant OGP 36507.

The author acknowledges with thanks many helpful suggestions by the referee, who supplied construction 1.4 and the bulk of the proof of lemma 2.6.

References

1. E.R. Lamken, *A Note on Partitioned Balanced Tournament Designs*, Ars Comb. **24** (1987), 5–16.
2. E.R. Lamken and S.A. Vanstone, *Partitioned Balanced Tournament Designs of Side* $4n+1$, Ars Comb. **20** (1985), 29–44.
3. E.R. Lamken and S.A. Vanstone, *The Existence of Partitioned Balanced Tournament Designs of Side* $4n+3$, Annals of Discrete Math. **34** (1987), 319–338.
4. E.R. Lamken and S.A. Vanstone, *The Existence of Partitioned Balanced Tournament Designs*, Annals of Discrete Math. **34** (1987), 339–352.
5. E. Mendelsohn and A. Rosa, *One-Factorizations of the Complete Graph–a Survey*, J. Graph Theory **9** (1985), 43–65.
6. R. Rees, *A Simple Construction for 2-frames*, Research Report M/CS **87-3** (Mount Allison, 1987).
7. R. Rees, *Semiframes and Nearframes*, "Proceedings of 'Combinatorics 88'", Universita degli Studi di Napoli (to appear).
8. P.J. Schellenberg, G.H.J. van Rees and S.A. Vanstone, *The Existence of Balanced Tournament Designs*, Ars Comb. **3** (1977), 303–318.
9. D.R. Stinson, *Room Squares with Maximum Empty Subarrays*, Ars Comb. **20** (1985), 159–166.