Existence of nilpotent SQS-skeins of class n

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Abstract. We construct nilpotent SQS-skeins of class n, for any positive integer n. These SQS-skeins are all subdirectly irreducible algebras. The nilpotent SQS-skeins of class n, which are constructed in this paper, are also solvable of order $\leq \frac{n+1}{2}$ if n is odd, and of order $\leq 1 + \frac{1}{2}n$ if n is even.

1. Introduction

An SQS-*skein* is an algebra $\tau := (T; q)$ where T is the underlying set and q is a ternary operation on T satisfying:

$q(x, y, z) = q(x, z, y) = q(z, x, y)$ totally symmetric identity;

$q(x, x, y) = y$ general idempotent identity;

$q(x, y, q(x, y, z)) = z$ Steiner identity.

There is a one-to-one correspondence between the SQS-skeins and the Steiner quadruple systems "SQS" [6]. An SQS-skein $\tau = (T; q)$ or its associate SQS is said to have cardinality r if $|T| = r$. In [4] Hanani has shown that an SQS of cardinality r exists iff $r \equiv 2$ or $4 \pmod 6$. An SQS-skein is called *Boolean* if it satisfies the identity:

$$q(x, u, q(u, y, z)) = q(x, y, z).$$

The variety of all Boolean SQS-skeins A_0 is the smallest non-trivial subvariety of the variety of all SQS-skeins. The variety A_0 is generated by the 2-element SQS-skein [7]. The variety of all SQS-skeins is a Mal'cev variety, which implies that the congruences on SQS-skeins are permutable. Moreover the congruences of such algebras are regular and uniform, i.e. any congruence is uniquely determined by any of its congruence classes, and any two congruence classes of the same congruence have the same cardinality [1]. Centrality is defined in general in [8] for Mal'cev varieties. Here we rewrite the definition of centrality, and consequently of nilpotence of SQS-skeins as in [1]. A congruence β on τ is said to be central on τ iff there is a congruence relation θ on β (considering β as a subalgebra of τ^2) containing the diagonal $\widehat{\tau} := \{(a, a) : a \in T\}$ as a congruence class. The largest central congruence of τ is called the *centre congruence*, which is denoted by $\xi(T)$. An SQS-skein τ has a *central series* of congruences, if there is a series $1 = \theta_0 \supseteq \theta_1 \supseteq \ldots \supseteq \theta_n = 0$ of congruences on τ such that $\theta_i/\theta_{i+1} \subseteq \xi(\tau_i/\theta_{i+1})$; $i = 0, 1, \ldots, n-1$. Then τ is called a *nilpotent* SQS-skein of class n, if n is the smallest length of a central series of congruences in τ.

The nilpotent SQS-skeins of class 1 are the Boolean SQS-skeins. In [1] are given examples of nilpotent SQS-skeins of class 2. In this paper, we will construct nilpotent SQS-skeins of class n, for each positive integer n.

2. Properties of nilpotent SQS-skeins

From the defining conditions for a nilpotent SQS-skein $\tau = (T, q)$, one can see directly that the cardinality $|T|$ of such a τ is equal to 2^m, for some positive integer m.

In Theorem 3 we will give a necessary and sufficient condition for a unique atom in the congruence lattice of an SQS-skein τ to be the centre of τ. And then we give the construction of nilpotent SQS-skeins of arbitrary class n in the third section.

Definition [1]. *A sub-SQS-skein N of an SQS-skein $\tau = (T; q)$ is* normal *iff for all $x_i, y_i \in T; i = 1, 2, 3$ and $a \in N$ one has*

$$q(a, x_i, y_i) \in N; (i = 1, 2, 3) \Rightarrow q(a, q(x_1, x_2, x_3), q(y_1, y_2, y_3)) \in N.$$

Theorem 1 [1]. *Let N be a non-empty subset of T. Then N is a normal sub-SQS-skein of the SQS-skein $\tau = (T; q)$ iff N is a congruence class of a congruence relation θ_N of τ, where θ_N is given by:*

$$\theta_N = \{(x, y) \in T^2 : q(a, x, y) \in N \ \&\ a \in N\}.$$

Theorem 2 [1][2]. *Every sub-SQS-skein of $\tau = (T; q)$ with cardinality $\frac{1}{2}|T|$ is normal.*

Theorem 3. *Let τ be an SQS-skein of cardinality 2^n, with a unique atom θ in its congruence lattice $C(\tau)$. Then θ is the centre of τ iff the cardinality of the factor SQS-skein τ/θ is equal to 2^{n-1}.*

Proof: Assume θ is the centre of τ; then the diagonal $\hat{\tau}$ is a congruence class for some congruence relation ϕ on θ, i.e. $\hat{\tau} = [(x, x)]\phi$ for $x \in T$. Let $\delta := \hat{\tau} \cup [(x, y)]\phi$, for $x, y \in T \ \&\ x \neq y$. Then δ is a reflexive sub-SQS-skein of θ, so that δ is a congruence relation of τ [**8**, p. 19].

We have $|\delta| = 2^n + 2^n = 2^{n+1} = 2^{n-1} \cdot 2^2$, therefore $|T/\delta| = 2^{n-1}$, consequently δ is an atom of the lattice $C(\tau)$. But θ is the unique atom of $C(\tau)$, hence $\theta = \delta$. It follows that $|\tau/\theta| = 2^{n-1}$.

Conversely assume $|\tau/\theta| = 2^{n-1}$, so that $|\theta| = 2^{n+1}$. We have $|\hat{\tau}| = 2^n$ and $|\hat{\tau}| = \frac{1}{2}|\theta|$. Now $\hat{\tau}$ is a sub-SQS-skein of θ, because θ is reflexive. From Theorem 2 it follows that $\hat{\tau}$ is a normal sub-SQS-skein of θ. This means that $\hat{\tau}$ is a congruence class for some congruence on θ. To prove that θ is the centre of τ, suppose ϕ is another central congruence of τ containing θ. If $\phi \supset \theta$ and $\phi \neq \theta$, then there exists $(x, y) \in \phi$ and $(x, y) \notin \theta$, and there is $\psi \in C(\phi)$ such that $\hat{\tau} = [(x, x)]\psi$. Let $\delta := [(x, x)]\psi \cup [(x, y)]\psi$, so that δ is a reflexive sub-SQS-skein of τ^2. From [**8**, p. 19] δ is a congruence relation on τ. On the other hand $|\delta| = 2^{n+1}$, so that δ is another atom of $C(\tau)$, contradicting the uniqueness of the atom θ. This implies that θ is the largest central congruence of $C(\tau)$, and completes the proof of the theorem.

3. Construction of nilpotent SQS-skeins of class n

Let $\overline{\tau} := (\overline{T}, \overline{q})$ be an SQS-skein of cardinality 2^n, and suppose that the congruence lattice $C(\overline{\tau})$ has a unique atom $\overline{\theta}$ with cardinality 2^{n+1}. We construct a new SQS-skein $\tau = (T, q)$ with cardinality 2^{n+1}, for which $C(\tau)$ has a unique atom θ_1, such that $\tau/\theta_1 \cong \overline{\tau}$. We do this in two steps (i) & (ii). The first step (i) constructs $\tau := (T, q)$ and a homomorphism ψ from τ onto $\overline{\tau}$. The second step (ii) proves that $\ker \psi := \theta_1$ is the unique atom of the congruence lattice $C(\tau)$.

Step (i)

Let $\overline{T} := \{x_0, x_1, \ldots, x_{2^n-1}\}$ and

$$\overline{\theta} := \bigcup_{\text{even } i=0}^{2^n-2} \{x_i, x_{i+1}\}^2.$$

Consider a set $S := \{y_0, y_1, \ldots, y_{2^n-1}\}$ such that $S \cap \overline{T} = \emptyset$, and then let $T = \overline{T} \cup S$.

For any three distinct elements $x, y, z \in T$, we have the following four cases:

$$\left.\begin{array}{ll} (1) & \{x,y,z\} = \{x_i, x_j, x_k\} \\ (2) & \{x,y,z\} = \{x_i, x_j, y_k\} \\ (3) & \{x,y,z\} = \{x_i, y_j, y_k\} \\ (4) & \{x,y,z\} = \{y_i, y_j, y_k\} \end{array}\right\} , i, j, k \in \{0, 1, \ldots, 2^n - 1\}$$

Let $\overline{q}(x_i, x_j, x_k) = x_e$ in $\overline{\tau}$. We define the ternary operation q' on T as follows

$$q'(x,y,z) := \begin{cases} x_e & \text{in case (1)} \\ y_e & \text{in case (2)} \\ x_e & \text{in case (3)} \\ y_e & \text{in case (4)} \end{cases}$$

Furthermore let q' be totally symmetric and satisfy the general idempotent identity.

We can easily show that the Steiner identity $q'(x, y, q'(x, y, z)) = z$ holds for any $x, y, z \in T$. This implies that $\tau' := (T, q')$ is an SQS-skein of cardinality 2^{n+1}.

Now let (T, B) be the corresponding Steiner quadruple system of τ', where B is given by:

$$B := \{\{x, y, z, q'(x, y, z)\} : x, y, z \in T \ \& \ x \neq y \neq z \neq x\}.$$

From the definition of $\overline{\theta}$ we have that $\{x_0, x_1, x_2, x_3\}$ is an element of B, and from the definition of q' the following set R is a subset of B.

$$R := \{\{x_0, x_1, x_2, x_3\}, \{x_0, x_1, y_2, y_3\}, \{x_0, x_2, y_1, y_3\}, \{x_0, x_3, y_1, y_2\}$$
$$\{y_0, y_1, y_2, y_3\}, \{y_0, y_1, x_2, x_3\}, \{y_0, y_2, x_1, x_3\}, \{y_0, y_3, x_1, x_2\}\}.$$

We consider the following set H of 4-element subsets of the set T.

$$H := \{\{y_0, x_2, y_1, y_3\}, \{y_0, x_1, x_2, x_3\}, \{y_0, x_3, y_1, y_2\}, \{y_0, x_1, y_2, y_3\}$$
$$\{x_0, y_3, x_1, x_2\}, \{x_0, y_1, y_2, y_3\}, \{x_0, y_2, x_1, x_3\}, \{x_0, y_1, x_2, x_3\}\}.$$

The two sets R and H are formed from the same set

$$A = \{x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3\}.$$

For any three-element subset $\{x, y, z\} \subseteq A$ and for any $b \in R$ with $\{x, y, z\} \subseteq b$, there exists $b' \in H$ with $\{x, y, z\} \subseteq b'$. From this it follows that $(T, (B \backslash R) \cup H)$ is a Steiner quadruple system [6].

Let $\tau := (T, q)$ be the corresponding SQS-skein of the Steiner quadruple system $(T, (B \backslash R) \cup H)$.

Now we want to prove that the map $\psi : T \longrightarrow \overline{T}$ defined by

$$\psi(x_i) = x_i \;\&\; \psi(y_i) = x_i \quad \text{for} \quad i = 0, \dots, 2^n - 1.$$

is a homomorphism from τ onto $\overline{\tau}$.

It is clear that ψ is an onto map. We have to prove that

$$\forall x, y, z \in T, \psi(q(x, y, z)) = \overline{q}(\psi(x), \psi(y), \psi(z)).$$

The equality is true if any two of x, y, z are equal. We observe that

$$q(x, y, z) = q'(x, y, z) \text{ for any } \{x, y, z\} \not\subseteq A.$$

For this reason we divide the proof into two cases: (a) $\{x, y, z\} \subseteq A$ and (b) $\{x, y, z\} \not\subseteq A$. Now let x, y, z be three distinct elements defines as in (1), (2), (3), and (4). This means that two cases (a) and (b) will be equivalent to the following:

$$\text{(a)} \quad \{i, j, k, e\} = \{0, 1, 2, 3\}$$
$$\text{(b)} \quad \{i, j, k, e\} \neq \{0, 1, 2, 3\}.$$

We prove the truth of the equality in the two cases (a) and (b) for each case (1), (2), (3), and (4).

Case (a) (1): if $\{x, y, z\} = \{x_i, x_j, x_k\}$, then from H we have:

$$\psi(q(x_i, x_j, x_k)) = \psi(y_e) = x_e \quad \text{and}$$
$$\overline{q}(\psi(x_i), \psi(x_j), \psi(x_k)) = \overline{q}(x_i, x_j, x_k) = x_e;$$

(2): if $\{x, y, z\} = \{x_i, x_j, y_k\}$, then

$$\psi(q(x_i, x_j, y_k)) = \psi(x_e) = x_e \quad \text{and}$$
$$\overline{q}(\psi(x_i), \psi(x_j), \psi(y_k)) = \overline{q}(x_i, x_j, x_k) = x_e;$$

(3): if $\{x, y, z\} = \{x_i, y_j, y_k\}$, then

$$\psi(q(x_i, y_j, y_k)) = \psi(y_e) = x_e \quad \text{and}$$
$$\overline{q}(\psi(x_i), \psi(y_j), \psi(y_k)) = \overline{q}(x_i, x_j, x_k) = x_e;$$

(4): if $\{x, y, z\} = \{y_i, y_j, y_k\}$, then

$$\psi(q(y_i, y_j, y_k)) = \psi(x_e) = x_e \quad \text{and}$$
$$\overline{q}(\psi(y_i), \psi(y_j), \psi(y_k)) = \overline{q}(x_i, x_j, x_k) = x_e.$$

Case (b) (1): if $\{x, y, z\} = \{x_i, x_j, x_k\}$, then

$$\psi(q(x_i, x_j, x_k)) = \psi(q'(x_i, x_j, x_k))$$
$$= \psi(\overline{q}(x_i, x_j, x_k)) = \psi(x_e) = x_e \quad \text{and}$$
$$\overline{q}(\psi(x_i), \psi(x_j), \psi(x_k)) = \overline{q}(x_i, x_j, x_k) = x_e;$$

(2): if $\{x, y, z\} = \{x_i, x_j, y_k\}$, then

$$\psi(q(x_i, x_j, y_k)) = \psi(q'(x_i, x_j, y_k)) = \psi(y_e) = x_e \quad \text{and}$$
$$\overline{q}(\psi(x_i), \psi(x_j), \psi(y_k)) = \overline{q}(x_i, x_j, x_k) = x_e;$$

(3): if $\{x, y, z\} = \{x_i, y_j, y_k\}$, then

$$\psi(q(x_i, y_j, y_k)) = \psi(q'(x_i, y_j, y_k)) = \psi(x_e) = x_e \quad \text{and}$$
$$\overline{q}(\psi(x_i), \psi(y_j), \psi(y_k)) = \overline{q}(x_i, x_j, x_k) = x_e;$$

(4): if $\{x, y, z\} = \{y_i, y_j, y_k\}$, then

$$\psi(q(y_i, y_j, y_k)) = \psi(q(y_i, y_j, y_k)) = \psi(y_e) = x_e \quad \text{and}$$
$$\overline{q}(\psi(y_i), \psi(y_j), \psi(y_k)) = \overline{q}(x_i, x_j, x_k) = x_e.$$

Now we can say in each case (a) and (b) that the required equality is true. Therefore ψ is a homomorphism from τ onto $\overline{\tau}$.

Step (ii)

From the Isomorphism Theorem, we have $\tau/\ker\psi \cong \overline{\tau}$. We want to prove that $\theta_1 = \ker\psi$ is the unique atom of the congruence lattice $C(\tau)$.

$$\begin{aligned}\text{Now } \theta_1 &= \{(x,y) \in T^2 : \psi(x) = \psi(y)\} \\ &= \{(x_i, y_i) : i = 0, 1, \ldots, 2^n - 1\}\end{aligned}$$

And we know that $C(\overline{\tau}) \cong [\theta_1 : 1]_{C(\tau)}$ where $[\theta_1 : 1]_{C(\tau)}$ is the interval between θ_1 and the largest congruence 1 in $C(\tau)$. Let $\theta_2 \in C(\overline{\tau})$ be the corresponding congruence relation of $\overline{\theta} \in C(\overline{\tau})$. Then we have $\theta_2/\theta_1 \cong \overline{\theta}$, and therefore

$$\theta_2 = \bigcup_{\text{even } i=0}^{2^n-2} \{x_i, x_{i+1}, y_i, y_{i+1}\}^2 .$$

Now suppose that δ is another atom of $C(\tau)$. Then we have $\theta_1 \wedge \delta = 0$, and this implies that $\theta_1 \circ \delta$ covers both θ_1 and δ [5,p. 70]. But $\overline{\tau}$ is chosen with a unique atom $\overline{\theta}$, and from $[\theta_1 : 1]_{C(\tau)} \cong C(\overline{\tau})$ we can deduce that $\theta_1 \circ \delta = \theta_2$. Hence θ_2 covers any atom of $C(\tau)$. Since the congruences of SQS-skeins are regular and uniform, we have the following two cases for the class $[x_0]\delta$:

$$[x_0]\delta = \{x_0, x_1\} \text{ or } [x_0]\delta = \{x_0, y_1\}.$$

A potential third case $[x_0]\delta = \{x_0, y_0\}$ does not arise because

$$\psi(x_0) = \psi(y_0) \Rightarrow \{x_0, y_0\} = [x_0]\theta_1 .$$

Now we can assume $[x_k]\delta = \{x_k, x_{k+1}\}$ and $q(x_0, x_2, x_4) = x_k$.

From Theorem 1,

$$\delta := \{(a,b) \in T^2 : q(x,a,b) \in [x]\delta\}.$$

We consider the first case $[x_0]\delta = \{x_0, x_1\}$. From the definitions of $\overline{\theta}$ and q we have:

$$\begin{aligned} q(x_0, x_4, x_5) = x_1 &\Rightarrow (x_4, x_5) \in \delta \\ q(x_0, x_2, y_3) = x_1 &\Rightarrow (x_2, y_3) \in \delta; \end{aligned}$$

then we have

$$(x_0, x_1), (x_4, x_5), (x_2, y_3) \in \delta \Rightarrow (q(x_0, x_4, x_2), q(x_1, x_5, y_3)) \in \delta.$$

From the assumption $q(x_0, x_4, x_2) = x_k$, we obtain $q(x_1, x_5, x_3) = x_{k+1}$ or x_k, and (according to the definition of q' and q) hence $q(x_1, x_5, y_3) = y_{k+1}$ or y_k. This implies (x_k, y_{k+1}) or $(x_k, y_k) \in \delta$. But from $q(x_0, x_k, x_{k+1}) = x_1$ we also get $(x_k, x_{k+1}) \in \delta$. This implies that $[x_k]\delta$ contains at least three distinct elements, which contradicts the uniformity of δ. Thus the assumption $[x_0]\delta = \{x_0, x_1\}$ is false.

Now we discuss the second case $[x_0]\delta = \{x_0, y_1\}$.

From $q(x_0, x_2, x_3) = y_1$, then $[x_2]\delta = \{x_2, x_3\}$.Since δ is regular, it can be written

$$\delta := \{(a, b) \in T^2 : q(x_0, a, b) \in [x_0]\delta\}$$

or

$$\delta := \{(a, b) \in T^2 : q(x_2, a, b) \in [x_2]\delta\}.$$

Hence we have:

$$q(x_2, x_4, x_5) = x_3 \Rightarrow (x_4, x_5) \in \delta;$$
$$q(x_0, y_0, x_1) = y_1 \Rightarrow (y_0, x_1) \in \delta;$$
$$q(x_2, x_k, x_{k+1}) = x_3 \Rightarrow (x_k, x_{k+1}) \in \delta;$$

and

$$q(x_0, y_2, y_3) = y_1 \Rightarrow (y_2, y_3) \in \delta.$$

Therefore $(y_0, x_1), (x_2, x_3), (x_4, x_5) \in \delta \Rightarrow (q(y_0, x_2, x_4), q(x_1, x_3, x_5)) = (y_k, x_k)$ or (y_k, x_{k+1}). This implies $y_k \in [x_k]\delta$, so that $[x_k]\delta$ contains at least three distinct elements, contradicting the uniformity of δ.

Finally the second case $[x_0]\delta = \{x_0, y_1\}$ also leads to a contradiction. We deduce that there is no other atom δ, and that θ_1 is the unique atom of the lattice $C(\tau)$.

Corollary 1. *For any positive integer n, there is a nilpotent SQS-skein of class n.*

Proof: Each SQS-skein of cardinality $\leq 2^3$ is Boolean, i.e. nilpotent of class 1. From [1] we have that any associated SQS-skein $\overline{\tau}$ a Steiner quadruple system of cardinality 2^4 with 14 subsystems of cardinality 2^3 is nilpotent of class 2. The congruence lattice $C(\overline{\tau})$ has a unique atom $\overline{\theta}_1$ with $|\overline{\theta}_1| = 2^5$ & $|\overline{\tau}/\overline{\theta}_1| = 2^3$. The smallest central series in $\overline{\tau}$ is $0 = \overline{\theta}_0 \subseteq \overline{\theta}_1 \subseteq \overline{\theta}_2 = 1$. The interval $[\overline{\theta}_1, \overline{\theta}_2]$ in $C(\overline{\tau})$ is isomorphic to the subgroup lattice of the abelian group $\mathcal{Z}_2^3$.

According to the preceding construction we can construct an SQS-skein τ with cardinality 2^5 and with congruence lattice $C(\tau)$ having a unique atom θ_1, such that the homomorphic image $\tau/\theta_1 = \overline{\tau}$. Then from Theorem 3, θ_1 is the centre of τ. Therefore the smallest central series of conguence in τ is $0 = \theta_0 \subseteq \theta_1 \subseteq \theta_2 \subseteq \theta_3 = 1$, where θ_2 is the image of $\overline{\theta}_1$ under the homomorphism $C(\overline{\tau}) \rightarrow C(\tau)$ induced by the homomorphism $\overline{\tau} \rightarrow \tau$. Hence τ is a nilpotent SQS-skein of class 3.

Now $|T| = 2^5$ and θ_1 is a unique atom in $C(\tau)$, with cardinality 2^6. Then by repeating the same construction, we will get a new nilpotent SQS-skein of class 4, with a unique atom. Continuing thus, we can construct a nilpotent SQS-skein of class n, for any positive integer n.

Corollary 2. *Each SQS-skein as constructed above is subdirectly irreducible.*

A *Boolean series* of congruence on an SQS-skein τ is a series of congruences $1 := \phi_0 \supseteq \phi_1 \supseteq \ldots \supseteq \phi_n := 0$ such that $[a]\phi_i/\phi_{i+1}$ is a Boolean SQS-skein for all $a \in T$ and $i = 0, 1, \ldots, n-1$. If n is the smallest length of such a Boolean series, then τ is called *solvable* of length n. It is known that any nilpotent SQS-skein of class n is solvable of length $\leq n$ [1]. in [1] are given only examples of nilpotent SQS-skeins of class 2 that are simultaneously solvable of length 2. In the following corollary it will be shown that there is a nilpotent SQS-skein of class n simultaneously solvable of length not greater than n.

Corollary 3. *Let τ be a nilpotent SQS-skein of class $2n-1$ constructed as above. Then τ is solvable of length $\leq n$.*

Proof: Let $1 = \theta_0 \supseteq \theta_1 \supseteq \ldots \supseteq \theta_{2n-1} = 0$ be the central series of congruences on τ, constructed as in the proof of Corollary 1. This means that the above series has the smallest length of central series of congruences in τ, because each θ_i/θ_{i+1} is the centre of τ/θ_{i+1}, $i = 0, 1, \ldots, 2n-2$.

One can see easily that $|[a]\theta_i/\theta_{i+1}| = 2$, for all $a \in T$ and $i = 1, 2, \ldots, 2n-2$. Moreover $|[a]\theta_0/\theta_1| = 8$ for all $a \in T$. Then we can select the following subseries

$$1 = \theta_0 \supseteq \theta_1 \supseteq \theta_3 \supseteq \theta_5 \supseteq \ldots \supseteq \theta_{2n-3} \supseteq \theta_{2n-1} = 0.$$

This series is a Boolean series, because $|[a]\theta_i/\theta_{i+2}| = 4$, for all $a \in T$ and $i = 3, 5, \ldots, 2n-3$.

This implies that τ is solvable of length $\leq n$.

One observes that if τ is a nilpotent SQS-skein of class $2n$, constructed as in Corollary 1, then it can be proved by the same way as in the above corollary that τ is solvable of length $\leq n+1$.

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