Construction methods for adjusted orthogonal row-column designs

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Abstract. Adjusted orthogonal row-column designs have certain desirable properties. In this paper we give a definition of adjusted orthogonal row-column designs, summarise the known designs, give some construction methods and indicate some open problems. We briefly consider the relationship between adjusted orthogonal row-column designs and orthogonal main effects block designs.

1. Introduction

We begin with some definitions, based on notation given in Preece (1976). Consider a (t, b, r, k, λ) BIBD. It has two *constraints*, namely the blocks and the treatments, which occur at b and t levels, respectively. A Latin square design has three constraints — the rows, the columns and the treatments — and a pair of mutually orthogonal Latin squares have 4 constraints — the rows, the columns, and the two sets of treatments. After ordering a design's constraints, we can define the *incidence matrix* of the x^{th} constraint with respect to the y^{th} by

$$N_{xy} = (n_{ij}),$$

where n_{ij} is the number of times that the i^{th} level of the x^{th} constraint occurs with the j^{th} level of the y^{th} constraint. N_{xy} is a $k_x \times k_y$ matrix, where there are k_i levels of the i^{th} constraint, and $N_{yx} = N_{xy}^T$.

For example, if we regard the blocks, of a BIBD, as the first constraint and the treatments as the second constraint, then N_{21} is the usual incidence matrix and

$$N_{21}N_{21}^T = (r - \lambda)I + \lambda J.$$

A row-column design is a design with 3 constraints, namely rows, with k levels, columns, with b levels, and treatments, with t levels. Each cell in the $k \times b$ array contains exactly one treatment. The matrices N_{13} and N_{23} are the incidence

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matrices of block designs. These designs are call the *component block designs* of the row-column design.

For example, a Latin square is a row-column design in which $k = b = t$ and in which $N_{13} = N_{23} = J_t$, where J_t is a $t \times t$ matrix of ones. Note that for all row-column designs, $N_{12} = J_{k,b}$ (where $J_{k,b}$ is a $k \times b$ matrix of ones).

A Youden square is a row-column design in which $N_{13} = J_{k,t}$ and N_{32} is the incidence matrix of a BIBD. A complete BIBD is often called a *randomised complete block design* (RCBD). So the component block designs for the Latin square are both randomised complete block designs and for a Youden square are a BIBD and a RCBD.

Example 1: Let $k = 3, b = t = 7$. Then the following array is a Youden square:

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 1 \\ 4 & 5 & 6 & 7 & 1 & 2 & 3 \end{array}.$$

We see that $N_{13} = J_{3,7}, N_{12} = J_{3,7}$ and

$$N_{23} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We will denote a row-column design with t treatments, each replicated r times, with k rows and b columns by RCD(t, b, k, r). Observe that then $tr = bk$.

Next we consider the motivation for the definition of the concept of adjusted-orthogonality.

Let Y_{ij} be the yield of the (i, j) cell in the array and let $d(i, j)$ be the treatment applied to cell (i, j) in the RCD, d. We assume that the yield Y_{ij} is the sum of a row effect (ρ_i), a column effect (κ_j), a treatment effect $(\tau_{d(i,j)})$ and an error term (E_{ij}). The yields are assumed to be independent of each other (so Corr$(E_{ij}, E_{gw}) = 0$ for all pairs $(i, j) \neq (g, w)$) and the variance of the error terms is assumed to be constant (so var$(E_{ij}) = \sigma^2$ for all pairs (i, j)). Thus we may write this model as

$$Y_{ij} = \rho_i + \kappa_j + \tau_{d(i,j)} + E_{ij},$$
$$i = 1, 2, \ldots, k; j = 1, 2, \ldots, b; \text{Var}(E_{ij}) = \sigma^2.$$

This is an example of a linear model.

To facilitate further discussion of the linear model, we write it in matrix notation. Let $\boldsymbol{\theta} = (\rho_1, \ldots, \rho_k, \kappa_1, \ldots, \kappa_n, \tau_1, \ldots, \tau_t)^T$, $\mathbf{Y} = (Y_{11}, Y_{12}, \ldots, Y_{1b}, Y_{21}, \ldots,$

$Y_{2b}, \ldots, Y_{k1}, \ldots, Y_{kb})^T, \mathbf{E} = (E_{11}, E_{12}, E_{1b}, E_{21}, \ldots, E_{2b}, \ldots, E_{k1}, \ldots, E_{kb})^T$ and write $\mathbf{Y} = X_d\boldsymbol{\theta} + \mathbf{E}$. X_d is a $(0, 1)$ matrix and is called the *design matrix*.

We can write $\mathbf{Y} = X_d\boldsymbol{\theta} + \mathbf{E} = \boldsymbol{\eta} + \mathbf{E}$, say. Our eventual aim is to estimate the elements in $\boldsymbol{\theta}$, but we begin by estimating the elements in $\boldsymbol{\eta}$. To do this we have available the data vector $\mathbf{Y}$ which differs from $\boldsymbol{\eta}$ by $\mathbf{E}$. Note that $\mathbf{Y}, \boldsymbol{\eta}$ and $\mathbf{E}$ are all vectors in R^{rp}, $\mathbf{E}$ has no preferred direction in R^{rp} since its elements are independent of each other and have constant variance. Hence the most natural estimate of $\boldsymbol{\eta}$ is that vector in the the range of X_d which is closest to $\mathbf{Y}$ in the usual Euclidean sense. Thus our estimate of $\boldsymbol{\eta}, \widehat{\boldsymbol{\eta}}$ say, minimises $(\mathbf{Y} - \boldsymbol{\eta})^T(\mathbf{Y} - \boldsymbol{\eta})$.

Suppose we choose $\mathbf{b}$ such that $X_d^T(\mathbf{Y} - X_d\mathbf{b}) = 0$, that is, $X_d^TX_d\mathbf{b} = X_d^T\mathbf{Y}$. Clearly $X_d\mathbf{b}$ is in the range of X_d, $\mathbf{Y} - X_d\mathbf{b}$ is in the orthogonal complement of the range of X_d and $\mathbf{Y} = X_db + \mathbf{Y} - X_d\mathbf{b}$. These facts, together with the fact that $\boldsymbol{\eta}$ is in the range of X_d, give

$$\begin{aligned}(\mathbf{Y} - \boldsymbol{\eta})^T(\mathbf{Y} - \boldsymbol{\eta}) &= (\mathbf{Y} - X_d\mathbf{b} + X_d\mathbf{b} - \boldsymbol{\eta})^T(\mathbf{Y} - X_d\mathbf{b} + X_d\mathbf{b} - \boldsymbol{\eta}) \\ &= (\mathbf{Y} - X_d\mathbf{b})^T(\mathbf{Y} - X_d\mathbf{b}) + (X_d\mathbf{b} - \boldsymbol{\eta})^T(X_d\mathbf{b} - \boldsymbol{\eta}).\end{aligned}$$

Since $(\mathbf{Y} - X_d\mathbf{b})^T(\mathbf{Y} - X_d\mathbf{b})$ is a constant we see that $(\mathbf{Y} - \boldsymbol{\eta})^T(\mathbf{Y} - \boldsymbol{\eta})$ is minimised if $\widehat{\boldsymbol{\eta}} = X_d\mathbf{b}$. Any vector $\widehat{\boldsymbol{\theta}}$ such that

$$X_d\widehat{\boldsymbol{\theta}} = \widehat{\boldsymbol{\eta}} = X_d\mathbf{b}$$

produces the same vector $\widehat{\boldsymbol{\eta}}$ and is a least squares estimator of $\boldsymbol{\theta}$. Thus we have

$$X_d^TX_d\widehat{\boldsymbol{\theta}} = X_d^TX_d\mathbf{b} = X_d^T\mathbf{Y},$$

and

$$\widehat{\boldsymbol{\theta}} = (X_d^TX_d)^-X_d^T\mathbf{Y}$$

where $(X_d^TX_d)^-$ is the Moore-Penrose generalised inverse of $X_d^TX_d$ (see Searle (1971)). $X_d^TX_d$ is called the *information matrix* (of the design *d*) for estimating $\boldsymbol{\theta}$.

For a RCD(t, b, k, r) we have that

$$X_d^TX_d = \begin{bmatrix} bI_k & N_{12} & N_{13} \\ N_{21} & kI_b & N_{23} \\ N_{31} & N_{32} & rI_t \end{bmatrix} = \begin{bmatrix} bI_k & J_{k,b} & N_{13} \\ J_{b,k} & kI_b & N_{23} \\ N_{31} & N_{32} & rI_t \end{bmatrix}$$

Thus

$$\begin{aligned} bI_k\widehat{\boldsymbol{\rho}} + J_{k,b}\widehat{\boldsymbol{\kappa}} + N_{13}\widehat{\boldsymbol{\tau}} &= Z_1 \\ J_{b,k}\widehat{\boldsymbol{\rho}} + kI_b\widehat{\boldsymbol{\kappa}} + N_{23}\widehat{\boldsymbol{\tau}} &= Z_2 \\ N_{31}\widehat{\boldsymbol{\rho}} + N_{32}\widehat{\boldsymbol{\kappa}} + rI_t\widehat{\boldsymbol{\tau}} &= Z_3 \end{aligned}$$

and so

$$(bkI_t - bN_{32}N_{23} - kN_{31}N_{13})\widehat{\boldsymbol{\tau}} - krJ_{t,b}\widehat{\boldsymbol{\kappa}} - brJ_{t,k}\widehat{\boldsymbol{\rho}} = bkZ_3 - bN_{32}Z_2 - kN_{31}Z_1.$$

From the first equation we have

$$rJ_t\widehat{\boldsymbol{\tau}} - J_{t,k}Z_1 = -(kJ_{t,b}\widehat{\boldsymbol{\kappa}} + bJ_{t,k}\widehat{\boldsymbol{\rho}}).$$

Substituting we get

$$(bkrI_t - bN_{32}N_{23} - kN_{31}N_{13} + r^2J_t)\widehat{\boldsymbol{\tau}} = bkZ_3 - bN_{32}Z_2 - kN_{31}Z_1 + J_{tk}Z_1.$$

Let

$$C_{12} = rI_t - \frac{1}{b}N_{31}N_{13} - \frac{1}{k}N_{32}N_{23} + \frac{r}{t}J_t.$$

Then C_{12} is called the *coefficient matrix for treatments* in the RCD.
The information matrices for the component block designs are

$$\begin{bmatrix} bI_k & N_{13} \\ N_{31} & rI_t \end{bmatrix} \text{ and } \begin{bmatrix} kI_b & N_{23} \\ N_{32} & rI_t \end{bmatrix}.$$

Hence the coefficient matrices for treatments in the component designs are

$$C_1 = rI_t - \frac{1}{b}N_{31}N_{13}$$

and

$$C_2 = rI_t - \frac{1}{k}N_{32}N_{23}.$$

We see that

$$C_{12} = C_1 + C_2 - r\left(I_t - \frac{1}{t}J_t\right).$$

Example 2:

(i) In a Latin square design, $N_{13} = N_{23} = N_{12} = J_t$ so $C_1 = C_2 = C_{12} = tI_t - J_t$.

(ii) In a Youden design, $N_{13} = J_{k,t}$ and $N_{32}N_{23} = (k-\lambda)I_t + \lambda J_t$. Thus $C_1 = k(I_t - \frac{1}{t}J_t), C_2 = kI_t - \frac{1}{k}(k-\lambda)I_t + \lambda J_t = \frac{1}{k}(k^2-k+\lambda)I_t - \frac{\lambda}{k}J_t = \frac{\lambda t}{k}\left(I_t - \frac{1}{t}J\right)$ and $C_{12} = \left(I_t - \frac{1}{t}J_t\right)\left(k + \frac{\lambda t}{k} - k\right) = \frac{\lambda t}{k}\left(I_t - \frac{1}{t}J_t\right)$.

Adjusted Orthogonality

Eccleston and Russell (1975, 1977) introduced the concept of adjusted orthogonality.

We will say that an equi-replicate RCD is *adjusted-orthogonal* if and only if

$$N_{13}N_{32} = rJ_{k,b}.$$

Thus, any row-column pair in the design have r treatments in common.

Example 3: The RCD(9, 6, 3, 2) given below is adjusted-orthogonal.

$$\begin{matrix} 1 & 4 & 7 & 6 & 7 & 2 \\ 2 & 5 & 8 & 8 & 3 & 4 \\ 3 & 6 & 9 & 1 & 5 & 9 \end{matrix}$$

A Latin square is adjusted orthogonal since each row and each column contain all t treatments. A Youden design is adjusted orthogonal since each row contains the t treatments and each column contains k treatments. Generalised Youden designs were introduced by Kiefer (1975) and are RCDs in which each component block design is a balanced block design. Such designs are not usually adjusted-orthogonal, as the next example shows.

Example 4: Consider the following generalised Youden design which is an RCD(4, 6, 6, 9).

$$\begin{matrix} 1 & 4 & 2 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 & 3 & 4 \\ 2 & 3 & 1 & 3 & 4 & 2 \\ 1 & 3 & 3 & 1 & 2 & 4 \\ 4 & 1 & 4 & 2 & 1 & 3 \\ 3 & 2 & 1 & 4 & 2 & 1 \end{matrix}.$$

Then

$$N_{31} = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 2 & 1 & 2 & 1 & 1 & 2 \\ 1 & 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 & 2 & 1 \end{bmatrix}, \quad N_{23} = \begin{bmatrix} 2 & 2 & 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 & 1 & 2 \end{bmatrix}$$

and

$$N_{13}N_{32} = \begin{bmatrix} 9 & 8 & 9 & 9 & 9 & 10 \\ 8 & 9 & 9 & 10 & 9 & 9 \\ 9 & 9 & 8 & 9 & 10 & 9 \\ 9 & 10 & 9 & 9 & 9 & 8 \\ 9 & 9 & 10 & 9 & 8 & 9 \\ 10 & 9 & 9 & 8 & 9 & 9 \end{bmatrix}$$

It can be show (see Eccleston and Kiefer (1981)) that if a RCD is adjusted orthogonal then

$$C_{12} = \frac{1}{r} C_1 C_2 .$$

Thus there is a matrix Q, say, which simultaneously diagonalises C_{12}, C_1 and C_2, such that the eigenvalues of C_{12} are those of C_1 or C_2 or zero, and further that C_{12}, C_1 and C_2 have a common set of eigenvectors.

The eigenvalues of the coefficient matrices are a guide to how well the treatment effects, τ_i, are estimated by the design. Thus for an adjusted orthogonal RCD the optimality properties of the RCD can be calculated from those of the component block designs.

There exist very few methods for constructing adjusted orthogonal designs, particular classes of designs are given by Raghavarao and Shah (1980), Anderson and Eccleston (1985) and John and Eccleston (1986).

The concept of orthogonality of Latin square designs has been generalised to sets of mutually orthogonal Latin squares (MOLS). In a similar way adjusted orthogonality can be generalised to sets of RCD(t, b, k, r) as follows.

Definition 5: Two RCD(t, b, k, r) are defined to be *mutually adjusted orthogonal* if and only if any row and column of either design have r elements in common.

Definition 6: A set of RCD(t, b, k, r) is said to be a set of *mutually adjusted orthogonal row-column designs* if and only if any two RCDs in the set are mutually adjusted orthogonal.

Example 7: The following two RCD(8,4,4,2) are mutually adjusted orthogonal.

1	5	8	4	2	6	1	5
2	6	3	7	4	8	2	6
3	7	6	2	5	1	7	3
4	8	5	1	7	3	8	4

Consider a set of p RCD(t, b, k, r) and let N_{12i} denote N_{12} for the i^{th} RCD in the set and so on. Then, if the p RCDs are mutually adjusted orthogonal,

$$N_{13i} N_{32j} = r J_{k,b}, \quad i, j = 1, 2, \ldots, p.$$

Eccleston and John (1988) discussed adjusted-orthogonality in relation to nested RCDs, which is related to the above concept of mutually adjusted orthogonal RCDs.

Lemma 8.

(i) *If* $N_{13} = J_{k,t}$ *then the RCD is adjusted orthogonal if and only if* $k = r$.

(ii) *If* $N_{23} = J_{b,t}$ *then the RCD is adjusted orthogonal if and only if* $b = r$.

Proof: The (i, j) entry of $N_{13} N_{32}$ is the number of treatments in column j, which is k. The proof of (ii) is similar.

Corollary 8.1.

(i) *If* $N_{13} = \alpha J_{k,t}$ *then the RCD is adjusted orthogonal if and only if* $\alpha k = r$.
(ii) *If* $N_{23} = \alpha J_{b,t}$ *then the RCD is adjusted orthogonal if and only if* $\alpha b = r$.

Hence once again we see that Latin squares and Youden squares are adjusted-orthogonal.

Construction Methods

The results are examples of patchwork constructions.

Construction 9. *Let* $D_1, D_2, \ldots, D_p$ *be a set* p *mutually adjusted orthogonal* RCD(t, b_i, k, r), $i = 1, 2, \ldots, p$. *Then*

$$(D_1\ D_2\ \cdots\ D_p)$$

is an adjusted orthogonal RCD$(t, \sum_i b_i, k, pr)$.

Proof: We know that $N_{13i}N_{32j} = rJ_{k,b_j}, i, j = 1, 2, \ldots, p$. For the array

$$(D_1\ D_2\ \cdots\ D_p)$$

we have that

$$N_{13} = \sum_i N_{13i}, \quad N_{32} = (N_{321} \ldots N_{32p}).$$

Hence

$$\begin{aligned} N_{13}N_{32} &= \left(\sum_i N_{13i}\right)(N_{321} \cdots N_{32p}) \\ &= \left(\sum_i N_{13i}N_{321} \cdots \sum_i N_{13i}N_{32p}\right) \\ &= \left(\sum_i rJ_{k,b_1} \ldots \sum_i rJ_{k,b_p}\right) = rpJ_{k\sum b_i}, \end{aligned}$$

as required.

Then next two results are proved similarly.

Construction 10. *Let* $D_1, D_2, \ldots, D_p$ *be a set of* p *mutually adjusted orthogonal* RCD(t, b, k_i, r), $i = 1, 2, \ldots, p$. *Then*

$$(D_1^T\ D_2^T\ \cdots\ D_p^T)^T$$

is an adjusted orthogonal RCD$(t, b, \sum_i k_i, pr)$.

Construction 11. *Let* D_1, D_2, D_3, D_4 *be 4 mutually adjusted orthogonal* RCD(t, b, k, r). *Then*

$$\begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}$$

is an adjusted-orthogonal RCD$(t, 2b, 2k, 4r)$.

Construction 11 can be generalised in an obvious way to give much larger adjusted orthogonal RCD's.

Example 12: Using the designs in Example 7 we get the following adjusted orthogonal RCD$(8, 8, 8, 8)$. (Note how different this design is from an 8×8 Latin square which is also an adjusted orthogonal RCD.)

1	5	8	4	2	6	1	5
2	6	3	7	4	8	2	6
3	7	6	2	5	1	7	3
4	8	5	1	7	3	8	4
2	6	1	5	1	5	8	4
4	8	2	6	2	6	3	7
5	1	7	3	3	7	6	2
7	3	8	4	4	8	5	1

Construction 13. *Let* L *and* M *be two MOLS of order* k. *Then there exists an adjusted-orthogonal* RCD$(k^2, k, 2k, 2)$.

Proof: Let A be a $k \times k$ array containing the numbers $1, 2, \ldots, k^2$. Construct a $k \times k$ array, B say, by setting

$$b_{ij} = a_{l_{ij} m_{ij}}.$$

Then $(A\ B)$ is the required RCD. The values of t, k, b and r are obvious. To check for adjusted-orthogonality we note that A and B are each adjusted-orthogonal and that each row (column) of B contains one entry from each row and column of A, since L and M are orthogonal Latin squares.

The design of Example 3 is an example of the construction with $k = 3$.

Dual Designs

We will define the *dual* of a RCD to be a block design with two sets of treatments in the following way. The rows of the RCD will correspond to the first set of treatments, the columns will correspond to the second set of treatments and the treatments will correspond to the blocks of the dual design.

Example 14: The following array is RCD(8, 4, 4, 2).

$$\begin{matrix} 1 & 2 & 5 & 6 \\ 3 & 4 & 7 & 8 \\ 5 & 6 & 1 & 4 \\ 7 & 8 & 3 & 2 \end{matrix}.$$

The dual design is given below, where the treatments of the first set correspond to rows, those of the second set correspond to columns.

Blocks

1	2	3	4	5	6	7	8
11	12	21	22	13	14	23	24
33	44	43	34	31	32	41	42

The dual design of an adjusted orthogonal RCD is a Graeco-Latin design. These have been studied by Preece (1966, 1976), Seberry (1979) and Street (1981). Using the results in these papers we can say that there are adjusted-orthogonal RCD($2p, p+1, p, (p+1)/2$) for p a prime or a prime power.

We could, however, view the entries in the dual design as the levels of two treatment sets which may interact. In this case the treatment sets are usually referred to as *factors*. The aims of a factorial experiment are to estimate the interaction effect of the two factors and the main effect of each of the factors. Since each possible level of the first factor appears with each level of the second factor exactly once in the design, the design is a single replicate factorial design. The dual design is a $k \times b$ factorial design, since the first factor has k levels and the second factor b levels, in t blocks each of size r.

The coefficient matrix for treatment effects for the dual of a RCD is

$$\begin{bmatrix} kI_k - \frac{1}{r}N_{13}N_{31} & J_{k,b} - \frac{1}{r}N_{13}N_{32} \\ J_{b,k} - \frac{1}{r}N_{23}N_{31} & bI_b - \frac{1}{r}N_{23}N_{32} \end{bmatrix}.$$

If the RCD is adjusted orthogonal then $N_{13}N_{32} = rJ_{k,b}$ and the coefficient matrix is block diagonal. This means that the main effect of each of the factors can be estimated independently of the other and we say that the dual design is an *orthogonal main effects plan* for a $k \times b$ factorial design in t blocks of size r.

The next example gives an orthogonal main effects plan for a 5×6 factorial design in 10 blocks of size 3.

Example 15: The following array is an adjusted orthogonal RCD(10, 6, 5, 3).

$$\begin{matrix} 1 & 2 & 4 & 5 & 6 & 10 \\ 4 & 1 & 6 & 3 & 7 & 8 \\ 8 & 5 & 2 & 4 & 9 & 7 \\ 9 & 8 & 3 & 10 & 5 & 6 \\ 10 & 3 & 9 & 7 & 1 & 2 \end{matrix}$$

The dual design is

				Blocks					
1	2	3	4	5	6	7	8	9	10
11	12	24	13	14	15	25	26	35	16
22	33	43	21	32	23	36	31	41	44
55	56	52	34	45	46	54	42	53	51

Unequal Replication

We denote a RCD in which treatment i is replicated r_i times by RCD$(t, b, k, (r_1, \ldots, r_t))$. Let $R = \operatorname{diag}(r_1, r_2, \ldots, r_t)$. Then an RCD$((t, b, k, (r_1, \ldots, r_t))$ is said to be adjusted orthogonal if

$$N_{13} R^{-1} N_{32} = J_{k,b}.$$

Construction 16. *Let D_1 be a Latin square of order k and let D_2 be a Youden design with t treatments arranged in blocks of size k. Then $(D_1\ D_2)$ is an adjusted orthogonal* RCD$(t, t+k, k, \mathbf{r})$, *where* $\mathbf{r}$ *is the replication tuple.*

Proof: For D_1, $N_{131} = N_{121} = N_{231} = J_k$. For D_2, $N_{122} = J_{k,t} = N_{132}$ and N_{232} satisfies $N_{322} N_{232} = (k - \lambda) I_t + \lambda J_t$. For the design $(D_1\ D_2)$

$$N_{12} = J_{k,t+k},\ N_{13} = (2 J_{k,k}, J_{k,t-k}) \text{ and } N_{32} = \begin{bmatrix} J_k \\ 0 \end{bmatrix} + N_{322}.$$

Then, since without loss of generality,

$$R = \begin{bmatrix} 2kI_k & 0 \\ 0 & kI_{t-k} \end{bmatrix},$$

we see that

$$\begin{aligned} N_{13} R^{-1} N_{32} &= (2 J_k\ J_{k,t-k}) \begin{bmatrix} \frac{1}{2k} I_k & 0 \\ 0 & \frac{1}{k} I_{t-k} \end{bmatrix} \left\{ \begin{bmatrix} J_k \\ 0 \end{bmatrix} + N_{322} \right\} \\ &= \left(\frac{1}{k} J_k\ \frac{1}{k} J_{k,t-k} \right) \left\{ \begin{bmatrix} J_k \\ 0 \end{bmatrix} + N_{322} \right\} \\ &= \frac{1}{k} J_{k,t} \left\{ \begin{bmatrix} J_k \\ 0 \end{bmatrix} + N_{322} \right\} \\ &= J_{k,t+k} \end{aligned}$$

as required.

Example 17: Let

$$D_1 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \quad \text{and} \quad D_2 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{bmatrix}$$

Then

$$D_3 = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 2 & 1 & 4 & 3 \\ 2 & 3 & 1 & 3 & 4 & 1 & 2 \end{bmatrix}$$

is an adjusted orthogonal RCD$(4,7,4,(6,6,6,3))$. The dual of D_3 is given by

Blocks			
1	2	3	4
11	12	13	17
14	15	16	26
22	23	21	35
25	24	27	
33	31	32	
36	37	34	

In general the dual of an adjusted orthogonal RCD$(t, b, k, \mathbf{r})$ will be a $k \times b$ orthogonal main effects plan with blocks of unequal size.

Conclusion

Adjusted-orthogonal RCDs form a useful class of designs but few are available. Known sets of mutually adjusted-orthogonal RCDs seem to be limited to trivial examples such as any set of Latin squares, any set of Latin squares together with a Youden square, and sets consisting of the repetition of one design. Sets of mutually adjusted-orthogonal RCDs are useful in the construction of larger adjusted-orthogonal RCDs and non-trivial examples of such sets would lead to non-isomorphic adjusted-orthogonal RCDs.

These problems merit investigation by combinatorialists.

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