(r, s)-Domination in Graphs and Directed Graphs

Zhuguo Mo
Computer and Engineering Services, Inc.
6964 Crooks Road
Troy, Michigan 48098

Kenneth Williams
Computer Science Department
Western Michigan University
Kalamazoo, Michigan 49008
U.S.A.

Abstract. Let $G = (V, E)$ be a graph or digraph, and let r and s be two positive integers. A subset U of V is called an (r, s)-dominating set if for any $v \in V - U$, there exists $u \in U$ such that $d(u, v) \leq r$ and for any $u \in U$ there exists $u' \in U (u' \neq u)$ for which $d(u', u) \leq s$. For graphs, a $(1, 1)$-dominating set is the same as a total dominating set. The (r, s)-domination number $\delta_{r,s}(G)$ of a graph or digraph G is the cardinality of a smallest (r, s)-dominating set of G. Various bounds on $\delta_{r,s}(G)$ are established including that, for an arbitrary connected graph of order $n \geq 2$, if $s \leq r + 1$ then $\delta_{r,s}(G) \leq \max(2n/(r + s + 1), 2)$, and if $s \geq r + 1$ then $\delta_{r,s}(G) \leq \max(n/(r + 1), 2)$. Both bounds are sharp.

1. Introduction.

The study of total dominating sets in graphs was initiated by Cockayne, Dawes and Hedetniemi [1] in 1980. Several of the results in this paper are generalizations of those in [1].

The maximum degree and minimum degree of graph G are denoted by $\Delta(G)$ and $\delta(G)$ respectively. A digraph D is *strongly connected* or *strong* if for every two distinct vertices of D, each vertex is reachable from the other. For a connected graph G, or a strong digraph G, we denote the *distance* $d_G(u, v)$ between two vertices u and v as the minimum of the lengths of the $u - v$ paths of G. The *eccentricity* $e(v)$ of a vertex v of a connected graph or strong digraph G is the number max $d_G(u, v)$, where the max is taken over all the vertices $u \in V(G)$. The *radius,* rad G, is defined as $\min_{v \in V} e(v)$ while the *diameter*, diam G, is $\max_{u,v \in V} d(u, v)$. A *total dominating set of a graph* $G = (V, E)$ is a subset U of V such that each vertex in V is adjacent to some vertex in U. Let $G = (V, E)$ be a graph, and r and s be two positive integers. A subset U of V is called an (r, s)*-dominating set of* G if for any $v \in V - U$ there exists $u \in U$ such that $d_G(u, v) \leq r$, and for any $u_1 \in U$ there exists $u_2 \in U$ $(u_2 \neq u_1)$ such that $d_G(u_1, u_2) \leq s$. Similarly, let $D = (V, A)$ be a digraph, and r and s be two positive integers. A subset U of V is called an (r, s)*-dominating set of* D if for any $v \in V - U$ there exists $u \in U$ such that $d_D(u, v) \leq r$, and for any $u_1 \in U$ there exists $u_2 \in U$ $(u_2 \neq u_1)$ such that $d_D(u_2, u_1) \leq s$. Clearly, an

(r, s)-dominating set is a dominating set of radius r. Note that if a digraph D has an (r, s)-dominating set, then no vertex of D has in-degree 0. Also, a total dominating set is the same as a $(1, 1)$-dominating set for graphs.

The cardinality of a smallest (r, s)-dominating set in a graph G is called the (r, s)*-domination number* and is denoted by $\delta_{r,s}(G)$. We note that this parameter is only defined for graphs without isolated vertices and with $\delta_{r,s}(G) \geq 2$. In the case that $r = s = 1$, $\delta_{r,s}(G)$ is the same as $\delta_t(G)$ which is the total domination number for graphs.

2. Bounds on (r, s)-domination number.

Let $G = (V, E)$ be a graph and r be a nonnegative integer. Define $\text{End}_r(G) = \{v \in V \mid \exists$ an end-vertex $u \in V$ (of a path) such that $d(u, v) < r\}$. Note that $\text{End}_1(G)$ is the set of end-vertices in G.

Theorem 2.1. *Let G be a connected graph of order $n \geq 2$, and r and s be two positive integers. Then*

$$\delta_{r,s}(G) \leq \max\{2, \min\{n - |\text{End}_r(T)|\}\}$$

where the minimum is taken over all spanning trees T of G.

Proof: Let T be a spanning tree of $G = (V, E)$ and $U = V - \text{End}_r(T)$. Then $|U| = n - |\text{End}_r(T)|$. Define set U' as follows:

if $|U| \geq 2$ then $U' = U$;
if $|U| = 1$ then U' is the union of U and some vertex adjacent to U;
if $|U| = 0$ then U' is any two adjacent vertices where at least one of the vertices has maximal eccentricity.

Clearly, U' is an (r, s)-dominating set of T. Therefore,

$$\delta_{r,s}(T) \leq |U'| \leq \max\{2, n - |\text{End}_r(T)|\}.$$

Thus

$$\begin{aligned}\delta_{r,s}(G) &\leq \min \delta_{r,s}(T) \\ &\leq \min \max\{2, n - |\text{End}_r(T)|\} \\ &= \max\{2, \min\{n - |\text{End}_r(T)|\}\},\end{aligned}$$

where the minimum is taken over all spanning trees T of G. ∎

Theorem 2.2. *Let G be a nontrivial connected graph, and r and s be two positive integers. Then $\delta_{r,s}(G) = \min \delta_{r,s}(T)$, where the minimum is taken over all spanning trees T of G.*

Proof: Let G be a nontrivial connected graph and T be a spanning tree of G. Then any (r,s)-dominating set of T is also an (r,s)-dominating set of G. Therefore $\delta_{r,s}(G) \leq \delta_{r,s}(T)$.

It follows that, $\delta_{r,s}(G) \leq \min \delta_{r,s}(T)$, where the minimum is taken over all spanning trees T of G.

Now we show the reverse inequality. If G is a tree, the theorem holds trivially. So we may assume that G is a connected non-acyclic graph. Let U be a minimum (r,s)-dominating set of G and C be a smallest cycle in G. If we can show that U is an (r,s)-dominating set of $G-e$ for some cycle edge e, then $\delta_{r,s}(G-e) \leq |U| = \delta_{r,s}(G)$. By applying this result a finite number of times, we have $\delta_{r,s}(T) \leq \delta_{r,s}(G)$ for some spanning tree T of G. Thus

$$\delta_{r,s}(G) \geq \min \delta_{r,s}(T),$$

where the minimum is taken over all spanning trees T of G.

Select two adjacent vertices x and y in $V(C)$ such that $d_G(x,U) + d_G(y,U) = \max\{d_G(u,U) + d_G(v,U) | uv \in E(C)\}$. We will show that U is an (r,s)-dominating set of $G-e$, where $e = xy$.

Note that for any two adjacent vertices u and v in G, the difference of $d_G(u,U)$ and $d_G(v,U)$ is at most one. This implies that for $t = x$ or y, $d_G(t,U) = \max\{d_G(v,U) \mid v \in V(C)\}$. Without loss of generality, suppose that $d_G(x,U) = \max\{d_G(v,U) \mid v \in V(C)\}$.

Let z be the vertex in $V(C)$ such that $zx \in E(C)$ and $z \neq y$. By the way in which x and y were chosen, $d_G(z,U) \leq d_G(y,U)$. Since an (r,s)-dominating set is a dominating set of radius r, by the proof of Theorem 2.1, U is a dominating set of radius r of $G-e$. In addition $d_{G-e}(v,U) = d_G(v,U)$, for all vertices v in $V(G)$. This equality will be used frequently in the rest of the proof.

Now it only remains to show that for any $u_1 \in U$, there exists $u_2 \in U$ $(u_2 \neq u_1)$ such that $d_{G-e}(u', u_2) \leq s$. Suppose, to the contrary, that there exists $u_1 \in U$ such that $d_{G-e}(u_1, U-u_1) > 2$. Let x' and y' be vertices in U such that $d_{G-e}(x,x') = d_{G-e}(x,U)$ and $d_{G-e}(y,y') = d_{G-e}(y,U)$. Since U is an (r,s)-dominating set of G, there exists $u_2 \in U$ $(u_2 \neq u_1)$ for which $d_G(u_1,u_2) = d_G(u_1, U-u_1) \leq s$. Let P be a $u_1 - u_2$ path of length $d_G(u_1,u_2)$ in G. Clearly, $e \in E(P)$. Observe that either the $u_1 - x$ subpath of P or the $u_1 - y$ subpath of P is in $G-e$. Thus we consider two cases:

Case 1: The $u_1 - y$ subpath P_1 of P is in $G-e$.

In this case, we may choose u_2 to be x'. For simplicity, we assume that $u_2 = x'$.

Let n and n_1 be the lengths of the paths P and P_1 respectively. Then $n = n_1 + 1 + d_{G-e}(x, U)$. If $u_1 \neq y'$, then

$$\begin{aligned} d_{G-e}(u_1, U - u_1) &\leq d_{G-e}(u_1, y') \\ &\leq d_{G-e}(u_1, y) + d_{G-e}(y, y') \\ &= d_{G-e}(u_1, y) + d_{G-e}(y, U) \\ &= n_1 + d_G(y, U) \\ &\leq n_1 + d_G(x, U) \\ &= n_1 + d_{G-e}(x, U) \\ &< n \\ &\leq s, \end{aligned}$$

which is a contradiction.

So we may assume that $u_1 = y'$ (see Figure 2.1). Let P_e be the path obtained from C by removing the edge e and let w be the vertex in $V(P_e)$ such that $d_{G-e}(w, U) = d_{G-e}(w, w')$, for some $w' \in U$, $w \neq y$, $w' \neq y'$, and $d_P(w, y)$ is the smallest. The existence of the vertex w is provided by the fact that $x \in V(P_e)$ and $d_{G-e}(x, U) = d_{G-e}(x, x')$, where $x' \in U$ and $x' = u_2 \neq u_1 = y'$. Let w_1 be the vertex in $V(P_e)$ such that $d_{P_e}(w_1, y) = d_{P_e}(w, y) - 1$. Then w and w' are adjacent and $d_{G-e}(w_1, U) = d_{G-e}(w_1, y')$.

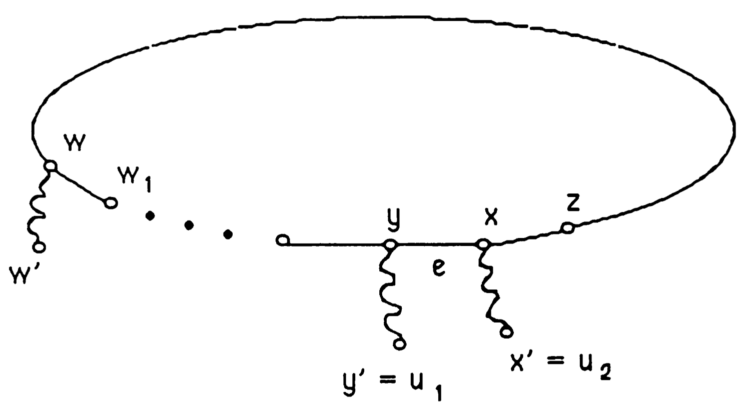

Figure 2.1

By the way in which x and y were chosen,

$$
\begin{aligned}
d_{G-e}(u_1, w') &= d_{G-e}(y', w') \\
&\le d_{G-e}(w_1, y') + d_{G-e}(w, w') + 1 \\
&= d_{G-e}(w_1, U) + d_{G-e}(w, U) + 1 \\
&= d_G(w_1, U) + d_G(w, U) + 1 \\
&\le d_G(x, U) + d_G(y, U) + 1 \\
&= d_{G-e}(x, U) + d_{G-e}(y, U) + 1 \\
&= d_{G-e}(x, U) + d_{G-e}(y, u_1) + 1 \\
&= d_{G-e}(x, U) + n_1 + 1 \\
&= n \\
&\le s,
\end{aligned}
$$

which contradicts $d_{G-e}(u_1, U - u_1) > s$.

Case 2: The $u_1 - x$ subpath P_2 of P is in $G - e$.

The proof of this case is similar to Case 1. Without loss of generality, suppose that $u_2 = y'$. Let n and n_2 be the lengths of the paths P and P_2 respectively. Then $n = n_2 + 1 + d_{G-e}(y, U)$. If $u_1 \neq x'$, then

$$
\begin{aligned}
d_{G-e}(u_1, U - u_1) &\le d_{G-e}(u_1, x') \\
&\le d_{G-e}(u_1, x) + d_{G-e}(x, x') \\
&\le d_{G-e}(u_1, x) + d_{G-e}(x, U) \\
&\le n_2 + d_{G-e}(y, U) + 1 \\
&= n \\
&\le s,
\end{aligned}
$$

which is a contradiction. So we may assume that $u_1 = x'$.

The rest of the proof is exactly the same as the second part of Case 1 where $u_1 = y'$, except we replace x, x', n_1 by y, y', n_2 respectively and vice versa.

A contradiction also arises for Case 2.

Therefore in either case, a contradiction arises. Thus for any $u_1 \in U$, there exists $u_2 \in U$ ($u_2 \neq u_1$) such that $d_{G-e}(u_1, u_2) \le s$. In addition we have established that U is a dominating set of radius r of $G - e$. Therefore, U is an (r, s)-dominating set of $G - e$. This completes the proof. ∎

Lemma 2.1. *Let $G = (V, E)$ be a nontrivial connected graph, and r and s be two positive integers. If $\operatorname{rad} G \le r$, then $\delta_{r,s}(G) = 2$.*

Proof: Let v be a vertex in the center of G and u be a vertex adjacent to v. Since $\operatorname{rad} G \le r$, $\{u, v\}$ is an (r, s)-dominating set of G. So $\delta_{r,s}(G) = 2$. ∎

Lemma 2.1 is useful when establishing certain upper bounds on $\delta_{r,s}(G)$.

Lemma 2.2. *Let G be a graph without isolated vertices, r_1, s_1, r_2, and s_2 be positive integers such that $r_1 \le r_2$ and $s_1 \le s_2$. Then*

$$\delta_{r_2,s_2}(G) \le \delta_{r_1,s_1}(G).$$

Proof: Lemma 2.2 follows from the fact that an (r_1, s_1)-dominating set of G is also an (r_2, s_2)-dominating set of G, where r_1, s_1, r_2, and s_2 are positive integers such that $r_1 \le r_2$ and $s_1 \le s_2$. ∎

Lemma 2.3. *Let $G = (V, E)$ be a graph, and r and s be two positive integers such that $s \ge 2r + 1$. A subset U of V is an (r, s)-dominating set of G if and only if U is an $(r, 2r + 1)$-dominating set of G.*

Proof: It is clear that an $(r, 2r + 1)$-dominating set of G is an (r, s)-dominating set of G for $s \ge 2r + 1$. Now suppose that U is an (r, s)-dominating set of a graph G, where $s \ge 2r + 1$. Then U is a dominating set of radius r of G. For any vertex $u_1 \in U$, there exists $u_2 \in U$ such that $d_G(U - \{u_1\}, u_1) = d_G(u_2, u_1)$. Denote $d_G(u_2, u_1)$ by n. Let P be a $u_2 - u_1$ path of length n in G and let $v \in V(P)$ such that $d_G(v, u_1) = \lfloor n/2 \rfloor$. If $d_G(u_2, u_1) > 2r + 1$, then $d_G(U, v) > r$, contradicting that U is a dominating set of radius r of G. Therefore $d_G(u_2, u_1) \le 2r + 1$. Thus U is an $(r, 2r + 1)$-dominating set of G. ∎

By Lemma 2.3, for graphs we need only consider (r, s)-dominating sets and (r, s)-domination numbers for $s \le 2r + 1$.

The next algorithm will be used by Theorem 2.4.

Algorithm 2.1 SUBTREE-RS-DOMINATION(T, v, r, s, P, U, j)

/* This algorithm finds a minimum (r, s)-dominating set for some subtree of T, where $s \le r + 1$.*/

INPUT

> T is a tree with root v such that rad $T > r$.
> r and s are positive integers such that $s \le r + 1$.
> P is a longest path in T with end-vertices u and v.
> x and y are the vertices on P such that $d(x, u) = r$ and $d(y, u) = r + s$.
> The $x - y$ subpath of P is: $x = v_0, v_1, \ldots, v_s = y$.

OUTPUT

> j is the index of vertex v_j.
> U is a minimum (r, s)-dominating set for the subtree of T with root v_j.

```
begin
        U ← {x}
        For i = 1 to s loop
              For each child w(≠ v_{i-1}) of v_i loop
                    Let T_w be the subtree of T having root w
                          and let w' be a vertex in T_w such that e(w) = d(w, w'),
                          where e(w) is the eccentricity of w in T_w.
                    if e(w) ≥ r then
                          Let w'' be a vertex in T_w such that
                          d(w', w'') = r and d(w'', w) = e(v) − r.
                          U ← U ∪ {w''}
                    else
                          if e(w) = r − 1 then
                          U ← U ∪ {v_i}
                          endif
                    endif
              end loop
              Let T_{v_i} be the subtree of T having root v_i.
              if (∃z ∈ V(T_{v_i}) such that d(U, z) > r) or (v_i ∈ U) then
                    U ← U ∪ {v_i};  j ← i
                    exit loop
              endif
              if (i = s) then
                    if (|U| = 1) or (∃z ∈ U such that d(U − {z}, z) > s)
                          then U ← U ∪ {v_i}
                    endif
                    j ← i
              endif
        end loop
end Algorithm 2.1
```

Theorem 2.3. *If T is a tree and r and s are two positive integers such that $s \leq r+1$, Algorithm 2.1 finds a minimum (r, s)-dominating set for some subtree of T.*

Proof: Note that each vertex u in $U - \{v_j\}$ is required to be in U by an end-vertex descendant of u. If $v_j \in U$, then v_j is required to be in U to insure that U is an (r, s)-dominating set of the subtree T_{v_j} of T with root v_j. ∎

The (r, s)-domination number of a disconnected graph can be very large, for example, $\delta_{r,s}(G) = |V(G)|$ for $G = mK_2$, $m \geq 1$. It is easy to see that mK_2 is the only graph with this property. Cockayne *et al* [1] have shown that for a connected graph of order $n \geq 3$, $\delta_t(G) \leq 2n/3$.

Before presenting a generalization of this result, we define an r-star. An *r-star* is a graph which can be obtained from a set of disjoint paths of length r by identifying one end-vertex of each path to some fixed end-vertex of a path in the set. Thus each star is a 1-star.

Theorem 2.4. *Let G be a connected graph of order $n \geq 2$, and r and s be two positive integers such that $s \leq r + 1$. Then*

$$\delta_{r,s}(G) \leq \max\{2n/(r+s+1), 2\}.$$

Furthermore, this bound is sharp.

Proof: By Theorem 2.2, we need only show that for any tree T of order $n \geq 2$ and $s \leq r+1$, $\delta_{r,s}(T) \leq \max\{2n/(r+s+1), 2\}$.

The proof is by induction on n. Let $T = (V, E)$ be a tree of order $n \geq 2$. If rad $T \leq r$, then by Lemma 2.1, $\delta_{r,s}(T) = 2 \leq \max\{2n/(r+s+1), 2\}$. Consequently, $\delta_{r,s}(T) = 2$, for any nontrivial tree of order at most $2r+1$.

Now suppose that for any tree T' of order m, $2 \leq m < n$,

$$\delta_{r,s}(T') \leq \max\{2m/(r+s+1), 2\},$$

and T is a tree of order n such that rad $T > r$. Let P be a longest path in T, u and v be the end-vertices of P, and k be the length of P. Since rad $T > r$, $k \geq 2r+1$.

Let x and y be the vertices of P such that $d(x,u) = r$ and $d(y,u) = r+s$, and the $x-y$ subpath of P is: $x = v_0, v_1, \ldots, v_s = y$.

In the following, the tree T is treated as a rooted tree with root v. Use Algorithm 2.1 to find a minimum (r,s)-dominating set U of some subtree T_{v_j} of T with root v_j, where j is the integer returned from Algorithm 2.1. For each vertex v in U, there is a set S_v of vertices such that $|S_v| \geq r+1$ for $v \neq v_s$, and $|S_v| \geq s$ for $v = v_s$ if $v_s \in U$. Each vertex in S_v is within distance r and v, and $S_v \cap S_{v'} = \phi$ if the corresponding vertices v and v' in U are different. Let $t = d_T(U, v_j)$ and $d = |U| = \delta_{r,s}(T_{v_j})$. If $t = 0$, then $|V(T_{v_j})| \geq (d-1)\cdot(r+1)+s$; otherwise $|V(T_{v_j})| \geq d(r+1)+t$.

Let T' be the subtree of T obtained from T by removing the subtree rooted at v_j (including vertex v_j) from T, and let n' be the order of T'. Then $n' < n$, by the inductive hypotheses, $\delta_{r,s}(T') \leq \max\{2n'/(r+s+1), 2\}$.

We consider three cases:

Case 1: $2n'/(r+s+1) < 1$.

Since $s \leq r+1$, $n' < (r+s+1)/2 \leq r+1$, it follows that $n' \leq r$.

If $t + n' \leq r$, then U is an (r,s)-dominating set of T. Since U is a minimum (r,s)-dominating set of T_{v_j}, U is necessarily a minimum (r,s)-dominating set of T. By the inductive hypotheses,

$$\begin{aligned}\delta_{r,s}(T) &= \delta_{r,s}(T_{v_j})\\ &\leq 2(n-n')/(r+s+1)\\ &< 2n/(r+s+1).\end{aligned}$$

Otherwise, $t + n' \geq r + 1$. Since $n' \leq r, t \geq 1$. This implies that $|V(T_{v_j})| \geq d \cdot (r + 1) + t$. Therefore $n = |V(T)| = |V(T_{v_j})| + n' \geq (d + 1) \cdot (r + 1)$. Note that $U' = U \cup \{v_j\}$ is an (r, s)-dominating set of T. Since $s \leq r + 1$, we have

$$\begin{aligned}\delta_{r,s}(T) &\leq |U'| \\ &= d + 1 \\ &= (d + 1)(r + s + 1)/(r + s + 1) \\ &\leq 2(d + 1)(r + 1)/(r + s + 1) \\ &\leq 2n/(r + s + 1).\end{aligned}$$

Case 2: $1 \leq 2n'/(r + s + 1) < 2$.

In this case $n' < r + s + 1$. Since n' is a positive integer, $n' \leq r + s$.

If $t + n' \leq r + s$, let $S' = \{v \in V(T') | d_T(U, v) = s\}$. If $S' \neq \phi$, then let u' be a vertex in S', otherwise let u' be any fixed vertex in T'. Then $U \cup \{u'\}$ is an (r, s)-dominating set of T, thus

$$\begin{aligned}\delta_{r,s}(T) &\leq 1 + |U| \\ &\leq 2n'/(r + s + 1) + |U| \\ &\leq 2n'/(r + s + 1) + 2(n - n')/(r + s + 1) \\ &= 2n/(r + s + 1).\end{aligned}$$

Otherwise, $t + n' \geq r + s + 1$. Since $n' \leq r + s$, $t \geq 1$. So $|V(T_{v_j})| \geq d(r + 1) + t$. It follows that $n = |V(T)| = |V(T_{v_j})| + n' \geq (d + 1) \cdot (r + 1) + s$. Let $S' = \{v \in V(T') | d_T(v_j, v) = s\}$. If $S' \neq \phi$, then let u' be a vertex in S', otherwise let u' be a fixed vertex in the center of T'. Then $U \cup \{v_j, u'\}$ is an (r, s)-dominating set of T, thus

$$\begin{aligned}\delta_{r,s}(T) &\leq |U| + 2 \\ &= d + 2 \\ &= d(r + s + 1)/(r + s + 1) + 2 \\ &\leq 2d \cdot (r + 1)/(r + s + 1) + 2 \\ &= 2\{(d + 1) \cdot (r + 1) + s\}/(r + s + 1) \\ &\leq 2n/(r + s + 1),\end{aligned}$$

where $s \leq r + 1$.

Case 3: $2n'/(r + s + 1) \geq 2$.

In this case $\delta_{r,s}(T') \leq 2n'/(r + s + 1)$. Note that the union of an (r, s)-dominating set of T' and an (r, s)-dominating set of T_{v_j} is an (r, s)-dominating set of T. Thus, by induction

$$\begin{aligned}\delta_{r,s}(T) &\leq \delta_{r,s}(T') + \delta_{r,s}(T_{v_j}) \\ &\leq 2n'/(r+s+1) + 2(n-n')/(r+s+1) \\ &= 2n/(r+s+1).\end{aligned}$$

By mathematical induction,

$$\delta_{r,s}(T) \leq \max\{2n/(r+s+1), 2\},$$

for all trees T of order $n \geq 2$ and $s \leq r+1$. Thus

$$\delta_{r,s}(G) \leq \max\{2n/(r+s+1), 2\},$$

for all connected graphs G of order $n \geq 2$ and positive integers r and s such that $s \leq r+1$.

Now we show that this bound is sharp. We need only show that the bound $2n/(r+s+1)$ is obtainable under the assumption that $n \geq r+s+1$. Let β_{r+s+2} be the set of graphs each of which can be obtained by taking an end-vertex from an $(r+s+2)$-star graph. By observation, for $n \geq r+s+1$, the upper bound $2n/(r+s+1)$ is obtainable by all the graphs in β_{r+s+2}. Thus the bound is sharp. ∎

The graph G represented by Figure 2.2 is a graph in β_{r+s+2}, where $r = 3$ and $s = 2$. G has order $n = 24$. The set of solid vertices is an (r,s)-dominating set of G of cardinality $\delta_{r,s}(G)$ which is equal to $2n/(r+s+1)$.

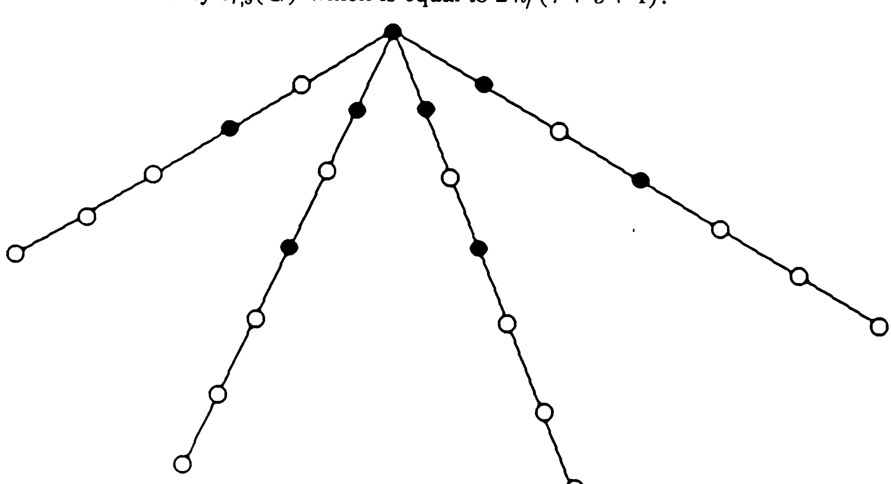

Figure 2.2

As a consequence, we have that for any $(r+s+2)$-star G of order n and positive integers r and s such that $s \leq r+1$,

$$\delta_{r,s}(G) = 2(n-1)/(r+s+1).$$

Theorem 2.5. *Let G be a connected graph of order $n \geq 2$, and r and s be two positive integers such that $s \geq r+1$. Then*

$$\delta_{r,s}(G) \leq \max\{n/(r+1), 2\}.$$

Furthermore, this bound is sharp.

Proof: Since $s \geq r+1$, by Lemma 2.2 and Theorem 2.4, we have

$$\begin{aligned}\delta_{r,s}(G) &\leq \delta_{r,r+1}(G)\\ &\leq \max\{n/(r+1), 2\}.\end{aligned}$$

To show this bound is sharp, we need only show that the bound $n/(r+1)$ is obtainable under the assumption that $n \geq 2(r+1)$. Let β_{r+1} be the set of graphs each of which can be obtained by taking an end-vertex from an $(r+1)$-star graph. By observation, for $n \geq 2(r+1)$, the upper bound $n/(r+1)$ is obtainable by all the graphs in β_{r+1}. Thus the bound is sharp. ∎

The following result has been obtained by Cockayne *et al* [1]: If G is a connected graph of order n such that $\Delta(G) < n-1$, then $\delta_t(G) \leq n-\Delta$. This result is generalized by Theorem 2.6.

Denote the set of end-vertices of a tree T by End(T). By observation, $|\text{End}(T)| \geq \Delta(T)$. Theorem 2.6 relates the (r,s)-domination number and the maximum degree of a graph.

Theorem 2.6. *Let G be a connected graph of order $n \geq 2$ with maximum degree $\Delta = \Delta(G)$, and r and s be two positive integers such that $s \leq r+1$. Then*

$$\delta_{r,s}(G) \leq \max\{2, n-(r+s+\Delta)+2\}.$$

Proof: Let r and s be two positive integers such that $s \leq r+1$. By Theorem 2.2, it is sufficient to show that $\delta_{r,s}(T) \leq \max\{2, n-(r+s+\Delta)+2\}$, for any tree T of order $n \geq 2$ with maximum degree $\Delta = \Delta(T)$.

If rad $T \leq r$, then by Lemma 2.1, $\delta_{r,s}(T) = 2$. So we may assume that rad $T > r$. Let P be a longest path in T with end-vertices u and v. Then there exist vertices x and y of P such that $d(x,u) = r$ and $d(y,u) = r+s$. Let P' be the $u-y$ subpath of P, $V' = V(P') - \{x,y\}$, and $U = V(T) - (V' \cup \text{End}(T))$. Then $\{x,y\} \subseteq U$, it follows that $|U| \geq 2$. Thus U is an (r,s)-dominating set of

T. Since $u \in V' \cap \text{End}(T)$ and $|\text{End}(T)| \geq \Delta(T)$, where End$(T)$ is the set of end-vertices of T, we have

$$\begin{aligned}\delta_{r,s}(T) &\leq |V(T)| - |V' \cup \text{End}(T)| \\ &\leq |V(T)| - |V'| - |\text{End}(T)| + 1 \\ &\leq n - (r + s - 1) - \Delta(T) + 1 \\ &= n - (r + s + \Delta(T)) + 2. \end{aligned}$$

∎

Corollary 2.6. *Let G be a graph of order n which contains no isolated vertices and there exists a component C of G such that $\Delta(C) = \Delta(G)$ and $|V(C)| \geq r + s + \Delta(G)$, where r and s are positive integers such that $s \leq r + 1$. Then*

$$\delta_{r,s}(G) \leq n - (r + s + \Delta) + 2.$$

Proof: Let U be a minimum (r,s)-dominating set in a component C of G such that $\Delta(C) = \Delta(G)$ and $n' = |V(C)| \geq r + s + \Delta(G)$. Since G contains no isolated vertices, $(V(G) - V(C)) \cup U$ is an (r,s)-dominating set of G. By Theorem 2.1,

$$\begin{aligned}\delta_{r,s}(G) &\leq |V(G) - V(C)| + |U| \\ &= n - n' + \delta_{r,s}(C) \\ &= n - n' + n' - (r + s + \Delta(C)) + 2 \\ &= n - (r + s + \Delta(G)) + 2. \end{aligned}$$

∎

The following theorem gives a lower bound for $\delta_{r,s}(G)$ in terms of the diameter of a graph G and r, s.

Theorem 2.7. *Let G be a graph which contains no isolated vertices, and r and s be two positive integers. Then $\delta_{r,s}(G) \geq 2\lfloor(\text{diam}(G) + 1)/(2r + s + 1)\rfloor$.*

Proof: Let u be a vertex in $G = (V, E)$ such that $e(u) = \text{diam}(G)$ and U be a minimum (r,s)-dominating set of G. Then $\delta_{r,s}(G) = |U|$. Define $L_i = \{v \in V \mid d(u,v) = i\}$, $0 \leq i \leq \text{diam}(G)$, and $L_j = \phi$, $j > \text{diam}(G)$. A set L_i is said to be dominated by some set S if each element in L_i is dominated by S. Observe that any two vertices in U alone can dominate at most $2r + s + 1$ L_i's in G within distance r. Therefore, $|U \cap (L_k \cup L_{k+1} \cup \ldots \cup L_{k+2r+s})| \geq 2$, for $k = 0, 2r + s + 1, \ldots, (\lfloor(\text{diam}(G) + 1)/(2r + s + 1)\rfloor - 1) \cdot (2r + s + 1)$. It follows that

$$\delta_{r,s}(G) = |U| \geq 2\lfloor(\text{diam}(G) + 1)/(2r + s + 1)\rfloor.$$

∎

3. Summary.

This paper has extended the definition of total dominating sets to (r, s)-dominating sets in graphs and digraphs. Various bounds on the (r, s)-domination number of a graph have been investigated.

Acknowledgements.

The authors want to thank an anonymous referee for some valuable suggestions including a change in the proof of Theorem 2.1.

References

1. E.J. Cockayne, R.M. Dawes and S.T. Hedetniemi, *Total domination in graphs*, Networks **10** (1980), 211-219.