Homogeneity Conditions for Finite Partially Ordered Sets

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Abstract. We classify the finite partially ordered sets which satisfy certain homogeneity conditions. One of the conditions considered is that the automorphism group of the partially ordered set acts multiply transitively on the set of elements of the same height.

1. Introduction.

Homogeneous partially ordered sets have been widely investigated (see, for example, [**4**] and the references given there). It is clear that whenever a partially ordered set $(X, \leq)$ has a transitive automorphism group then X is infinite or the order is trivial. This suggests the consideration of somewhat weaker conditions on homogeneity for finite sets.

Let $(X, \leq)$ be a finite partially ordered set (in short, a poset). For $x \in X$ the *height* $h(x)$ of x is defined to be the maximum cardinality of a chain in $\{y \in X \mid y < x\}$. Let $h(X)$ be one less than the maximum cardinality of a chain in $(X, \leq)$. Note that $h(X)$ is the maximum of all $h(x)$ for $x \in X$. For $0 \leq i \leq h(X)$ let $H_i(X) = \{x \in X \mid h(x) = i\}$. Clearly, each $H_i(X)$ is an antichain in $(X, \leq)$. If i, j are such that $0 \leq i < j \leq h(X)$ then let $H_{ij}(X) = H_i(X) \cup H_j(X)$. Note that the automorphism group $\mathrm{Aut}(X, \leq)$ of $(X, \leq)$ leaves each of the sets $H_i(X), H_{ij}(X)$ invariant.

Let $k \geq 1$. We say that $(X, \leq)$ is *k-height-homogeneous* if whenever S, T are subsets of X with $|S| = |T| \leq k$ and $f\colon S \to T$ is an order-isomorphism such that $h(x) = h(xf)$ for all $x \in S$ then there exists $g \in \mathrm{Aut}(X, \leq)$ such that $xf = xg$ for all $x \in S$. We say that $(X, \leq)$ is *height-homogeneous* if it is k-height-homogeneous for all $k \geq 1$. The following posets play a vital role in our results. For $n \geq 2$ define the poset $(A_n, \leq)$ by $A_n = \{0, 1\} \times \{0, 1, \ldots, n-1\}$ with $(i, j) \leq (i', j')$ if and only if $j = j'$ and $i \leq i'$. For $m \geq 3$ let $(B_m, \leq)$ be the poset of 1-element and $(m-1)$-element subsets of $\{0, 1, \ldots, m-1\}$ ordered by set-theoretic inclusion. Define the poset $(C_{pq}, \leq)$ for $p, q \geq 1$ by $C_{pq} = \{(0, i) \mid 0 \leq i < p\} \cup \{(1, i) \mid 0 \leq j < q\}$ where $(i, j) \leq (i', j')$ if and only if $i < i'$ or $(i, j) = (i', j')$. The diagrams of some posets of this form are given in Figure 1. Note that $(A_1, \leq)$ is isomorphic to $(C_{11}, \leq)$ and this is the only isomorphism between those posets as we define B_m only for $m \geq 3$ (otherwise we would also have $(A_2, \leq)$ isomorphic to $(B_2, \leq)$). Note that if $(X, \leq)$ is a poset isomorphic to some $(A_n, \leq)$ respectively $(B_m, \leq)$ then there exists a bijection $b\colon H_0(X) \to H_1(X)$ such that for $x \in H_0(X)$ and $y \in H_1(X)$ we have $x < y$

if and only if $y = xb$ (for $(A_n, \leq)$), respectively $y \neq xb$ (for $(B_m, \leq)$). Finally, we note that there is a direct decomposition of the automorphism group $\mathrm{Aut}(X, \leq)$ of a poset in which $(H_{i,i+1}(X), \leq)$ is isomorphic to some $(C_{pq}, \leq)$.

Lemma 1.1. *Let $(X, \leq)$ be a finite poset such that $(H_{i,i+1}(X), \leq)$ is isomorphic to $(C_{pq}, \leq)$ for some $p, q \geq 1$. Let $Y = \{x \in X \mid h(x) \leq i\}$ and $Z = \{x \in X \mid h(x) \geq i + 1\}$. Then* $\mathrm{Aut}(X, \leq) \cong \mathrm{Aut}(Y, \leq) \times \mathrm{Aut}(Z, \leq)$.

Proof: Note that we have $y < z$ whenever $y \in Y$ and $z \in Z$. It follows that if $g \in \mathrm{Aut}(Y, \leq)$ and $h \in \mathrm{Aut}(Z, \leq)$ we can construct g', $h' \in \mathrm{Aut}(Y, \leq)$ by $xg' = xg$ and $xh' = x$ if $x \in Y$ and $xg' = x$ and $xh' = xh$ if $x \in Z$. As g' and h' commute, it follows that $\mathrm{Aut}(X, \leq)$ contains a subgroup isomorphic to $\mathrm{Aut}(Y, \leq) \times \mathrm{Aut}(Z, \leq)$, and it is easy to see that this is, in fact, the whole of $\mathrm{Aut}(X, \leq)$. ∎

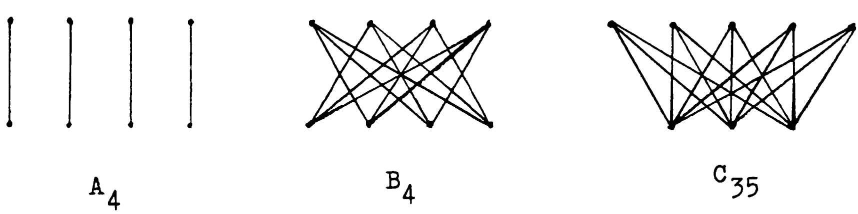

Figure 1

2. Multiple transitivity on elements of the same height.

Before considering height-homogeneous posets we shall deal with the weaker condition that the automorphism group is as transitive as possible on elements of the same height. In order to do this, it is useful to consider incidence structures whose automorphism group is multiply transitive on the sets of points and blocks.

Lemma 2.1. *Let (P, B, I) be an incidence structure (with $|P|, |B| > 1$) whose automorphism group G is $\min(3, |P|)$-transitive on P and $\min(3, |B|)$-transitive on B. Then either every point is incident with every block, or there exists a bijection b: $P \rightarrow B$ such that either $x \; I \; y$ if and only if $y = xb$, or $x \; I \; y$ if and only if $y \neq xb$ for $x \in P$, $y \in B$.*

Proof: As G is transitive on B, it follows that there exists $k \geq 1$ such that every block is incident with k points, and as G is transitive on P, it follows that there exists $r \geq 1$ such that every point is incident with r blocks. Furthermore, as G is 2-transitive on P, there exists $\lambda \geq 0$ such that any two distinct points are incident with λ blocks, and as G is 2-transitive on B, there exists $\overline{\lambda} \geq 0$ such that any two distinct blocks are incident with $\overline{\lambda}$ points. ∎

Let $\lambda = 0$, and note that we then must have $k = 1$. If $r > 1$ then there exist two distinct points y_1, y_2 and 4 distinct blocks $x_{11}, x_{12}, x_{21}, x_{22}$ such that $x_{ij}\ I\ y_i$. But as G is 2-transitive on P, there exists $g \in G$ such that $x_{11}g = x_{11}$ and $x_{12}g = x_{21}$, which gives a contradiction, as from the first equality it follows that $y_1 g = y_1$, and from the second it follows that $y_1 g = y_2$. Thus $r = 1$, and hence there exists a bijection $b\colon P \to B$ such that $x\ I\ y$ if and only if $y = xb$ for $x \in P$, $y \in B$. The same follows if $\overline{\lambda} = 0$.

We now can assume that $\lambda \geq 1$ and $\overline{\lambda} \geq 1$, thus both the incidence structure (P, B, I) and its dual structure (B, P, I^*) are block designs. Furthermore, if $|P| = k$ then also $|B| = r$, and every point is incident with every block. Thus, we can also assume that $k < |P|$ (and also $r < |B|$).

Application of Fisher's inequality (see, for example, I.8.6 in [1]) for (P, B, I) gives $|P| \leq |B|$, and for (B, P, I^*) gives $|B| \leq |P|$, thus we have $|P| = |B|$, and also $r = k$, that is, the design is symmetric. Now let $k + 1 = |P|$, and thus also $r + 1 = |B|$. It is then easy to see that there is a bijection $b\colon P \to B$ such that $x\ I\ y$ if and only if $y \neq xb$ for $x \in P$, $y \in B$.

We now assume that $|P| > k + 1$. If $\lambda = 1$ then it is easy to see that (P, B, I) is a projective plane. However, there is clearly no isomorphism carrying three collinear points onto three points which form a triangle, and thus this case cannot occur. If $\lambda > 1$ then by the Dembowski-Wagner theorem (see XII.2.13 in [1]) it follows that (P, B, I) is isomorphic to a projective space, but by a similar argument as above, this case cannot occur either. ∎

Theorem 1. *Let $(X, \leq)$ be a finite poset. Then the following are equivalent.*

(1) $\mathrm{Aut}(X, \leq)$ *induces* $\mathrm{Sym}(H_i(X))$ *on* $H_i(X)$ *for* $0 \leq i \leq h(X)$.

(2) *If* $t = \min(3, |H_i(X)|)$ *then* $\mathrm{Aut}(X, \leq)$ *acts* t*-transitively on* $H_i(X)$ *for* $0 \leq i \leq h(X)$.

(3) *Whenever* i, j *are such that* $0 \leq i < j \leq h(X)$ *then* $H_{ij}(X)$ *is isomorphic to one of the posets* A_n, B_m *or* C_{pq} *for some* $n \geq 2$, $m \geq 3$ *or* $p, q \geq 1$. *Furthermore, if* $H_{ij}(X)$ *and* $H_{jk}(X)$ *are both isomorphic to* A_n *for* $n \geq 3$ *then* $H_{ik}(X)$ *is not isomorphic to* B_n.

Proof: Clearly, (1) implies (2). Assume (2), let i, j be such that $0 \leq i < j \leq h(X)$ and consider $H_{ij}(X)$. Let G be the group of automorphisms induced by $\mathrm{Aut}(X, \leq)$ on $(H_{ij}(X), \leq)$. If $|H_i(X)| = 1$ or $|H_j(X)| = 1$ then, as G is transitive on both $H_i(X)$ and $H_j(X)$ it is clear that $H_{ij}(X)$ is isomorphic to a poset C_{1q} respectively C_{p1}. Thus we can assume that $|H_i(X)| > 1$ and $|H_j(X)| > 1$. Consider the incidence structure (P, B, I) with point set $P = H_i(X)$, block set $B = H_j(X)$, and where $x\ I\ y$ if and only if $x < y$. Note that every automorphism of this incidence structure is induced by an order-automorphism of $(H_{ij}(X), \leq)$. By Lemma 2.1 it follows that $H_{ij}(X)$ is isomorphic to a poset A_n, B_m or $C_{p,q}$.

Suppose there exist i, j, k with $0 \leq i < j < k \leq h(X)$ such that $H_{ij}(X)$ and $H_{jk}(X)$ are both isomorphic to A_n for $n \geq 3$ and $H_{ik}(X)$ is isomorphic to B_n.

Without loss of generality, let $H_u(X) = \{(u,1),\ldots,(u,n)\}$ for $u \in \{i,j,k\}$ such that $(i,v) < (j,v)$ and $(j,v) < (k,v)$ for $1 \le v \le n$. There exists a fixed-point free permutation b of $\{1,\ldots,n\}$ such that $(i,v) < (k,w)$ if and only if $w \neq vb$. It follows that there exist distinct $v_1, v_2, v_3 \in \{1,\ldots,n\}$ such that $v_1 b = v_3$. Then there exists $g \in \mathrm{Aut}(X,\le)$ such that $(i,v_1)g = (i,v_2)$, $(i,v_2)g = (i,v_1)$ and $(i,v_3)g = (i,v_3)$. As $H_{ij}(X)$ and $H_{jk}(X)$ are both isomorphic to A_n, it follows that $(j,v_3)g = (j,v_3)$ and $(k,v_3)g = (k,v_3)$. As (i,v_1) and (k,v_3) are incomparable, it thus follows that $(i,v_1)g = (i,v_2)$ and $(k,v_3)g = (k,v_3)$ are also incomparable. As $H_{ik}(X)$ is isomorphic to B_n, by definition of b we get $v_2 b = v_3 = v_1 b$, and thus $v_1 = v_2$, which is a contradiction. This shows that (3) holds.

Now suppose that (3) holds. Let $i \in \{0,\ldots,n\}$ and $f \in \mathrm{Sym}(H_i(X))$. Define $g: X \to X$ in the following way. Let $xg = xf$ for all $x \in H_i(X)$. Suppose we have defined g on $H_j(X)$ with $j_0 \le j \le j_1$ such that $x < y$ if and only if $xg < yg$ for all $x, y \in \{z \in X \mid j_0 \le h(z) \le j_1\}$. If $j_0 > 0$ then define g on $H_{j_0-1}(X)$ in the following way. If $H_{j_0-1\,j_0}(X)$ is isomorphic to some C_{pq} then $xg = x$ for all $x \in H_{j_0-1}(X)$. If $H_{j_0-1\,j_0}(X)$ is isomorphic to some A_n (respectively B_m) then there exists a bijection $b: H_{j_0-1}(X) \to H_{j_0}(X)$ such that for $x \in H_{j_0-1}(X)$, $y \in H_{j_0}(X)$ we have $x < y$ if and only if $y = xb$ (respectively $y \neq xb$). Now define $xg = xbgb^{-1}$ for all $x \in H_{j_0-1}(X)$.

We claim that $x < y$ if and only if $xg < yg$ for all $x, y \in \{z \in X \mid j_0 - 1 \le h(z) \le j_1\}$. By construction, it is sufficient to take $x \in H_{j_0-1}(X)$ and $y \in H_j(X)$ for some j with $j_0 \le j \le j_1$. The claim is trivial if $H_{j_0-1\,j}(X)$ is isomorphic to some $C_{p,q}$. If $H_{j_0-1\,j_0}(X)$ is isomorphic to A_n, then $x < y$ if and only if $xb^{-1} \le y$, hence, by induction the claim follows. Finally, if both $H_{j_0-1\,j_0}(X)$ and $H_{j_0-1\,j}(X)$ are isomorphic to B_m then either $j = j_0$ or $H_{j_0\,j}(X)$ is isomorphic to A_m, and thus $x < y$ if and only if $xb^{-1} \not\le y$, and again the claim follows. If $j_1 < h(X)$ then define g on $H_{j_1+1}(X)$ analogously. It is then clear that $g \in \mathrm{Aut}(X)$, hence (1) holds, which concludes the proof of the theorem. ∎

3. Height-homogeneous posets.

In this section, we consider posets which satisfy the stronger condition of height-homogeneity, and we give a classification similar to that in Theorem 1.

Theorem 2. *Let $(X,\le)$ be a finite poset. Then the following are equivalent.*

(1) *$(X,\le)$ is height-homogeneous.*
(2) *$(X,\le)$ is 3-height-homogeneous.*
(3) *Whenever i,j are such that $0 \le i < j \le h(X)$ then $H_{ij}(X)$ is isomorphic to one of the posets A_n, B_m or C_{pq} for some $n \ge 2$, $m \ge 3$ or $p,q \ge 1$, and the following conditions hold.*
   (a) *Whenever i,j,k are such that $0 \le i < j < k \le h(X)$ and $H_{ij}(X)$ respectively $H_{jk}(X)$ is isomorphic to A_n with $n \ge 2$ then $H_{ik}(X)$*

is isomorphic to $H_{jk}(X)$ respectively $H_{ij}(X)$.

(b) *There do not exist i, j, k with $0 \leq i < j < k \leq h(X)$ such that $H_{ij}(X)$ and $H_{jk}(X)$ are both isomorphic to B_m with $m \geq 3$.*

Proof: Clearly (1) implies (2). Assume that (2) holds. Then $(X, \leq)$ satisfies condition (2) of Theorem 1. By Theorem 1, every $H_{ij}(X)$ is isomorphic to some A_n, B_m or C_{pq}.

Let i, j, k be such that $0 \leq i < j < k \leq h(X)$, and suppose $H_{ij}(X)$ is isomorphic to A_n. There exists a bijection $b\colon H_i(X) \to H_j(X)$ such that $x < y$ if and only if $xb = y$ for $x \in H_i(X)$, $y \in H_j(X)$. We claim that the mapping $c\colon H_{ik}(X) \to H_{jk}(X)$ given by $xc = xb$ for $x \in H_i(X)$ and $xc = x$ for $x \in H_k(X)$ is an isomorphism. Clearly, it is bijective and its inverse is order-preserving. On the other hand, let $x \in H_i(X)$ and $y \in H_k(X)$ such that $x < y$. Let $y' \in H_k(X)$ such that $xb < y'$. By 3-height-homogenity, there exists $g \in \mathrm{Aut}(X, \leq)$ such that $xg = x$ and $yg = y'$. Hence, also $xbg = xb$, and we have $xb = xbg^{-1} < y'g^{-1} = y$. Therefore c is order-preserving, and thus $H_{ik}(X)$ is isomorphic to $H_{jk}(X)$. If $H_{jk}(X)$ is isomorphic to A_n then the result follows similarly. Thus (a) holds.

Finally, suppose that both $H_{ij}(X)$ and $H_{jk}(X)$ are isomorphic to B_m $(m \geq 3)$, then clearly $H_{ik}(X)$ is isomorphic to C_{mm}. Let $b_1\colon H_i(X) \to H_j(X)$ and $b_2\colon H_j(X) \to H_k(X)$ be bijections such that $x < y$, respectively $y < z$, if and only if $y \neq xb_1$, respectively $z \neq yb_2$, for $x \in H_i(X), y \in H_j(X)$, $z \in H_k(X)$. Let $x \in H_i(X)$ and $y \in H_k(X) \setminus \{xb_1 b_2\}$. As $x < xb_1 b_2$ and $x < y$, there exists $g \in \mathrm{Aut}(X, \leq)$ with $xg = x$ and $xb_1 b_2\, g = y$. But xb_1 is incomparable with both x and $xb_1 b_2$, whereas each element of $H_j(X)$ is comparable with x or y, which gives a contradiction. Therefore (b) holds, which concludes the proof of (3).

Now suppose that (3) holds. We prove (1) by induction on $h(X)$. It is trivial if $h(X) = 0$. Suppose that $n = h(X) > 0$. Let $S, T \subseteq X$ and $f\colon S \to T$ be an order-isomorphism with $h(x) = h(xf)$ for all $x \in S$. If there exists i with $0 \leq i < h(X)$ such that $H_{i,i+1}(X)$ is isomorphic to C_{pq} then let $X_1 = \{x \in X \mid h(x) \leq i\}$ and $X_2 = \{x \in X \mid h(x) \geq i + 1\}$. Note that f induces order-isomorphisms $f_i\colon S \cap X_i \to T \cap X_i$ for $i \in \{1, 2\}$. By the induction hypothesis, there exists $g_i \in \mathrm{Aut}(X_i, \leq)$ with $xg_i = xf_i = xf$ for $x \in S \cap X_i$ for $i \in \{1, 2\}$. If we define g by $xg = xg_i$ for $x \in X_i$ $(i \in \{1, 2\})$ then it is clear that $g \in \mathrm{Aut}(X, \leq)$ and $xg = xf$ for all $x \in S$. Next suppose that $H_{n-1,n}(X)$ is isomorphic to A_m $(m \geq 2)$, and let $b\colon H_n(X) \to H_{n-1}(X)$ be a bijection such that $y < x$ if and only if $y = xb$ for $x \in H_n(X)$, $y \in H_{n-1}(X)$. Let $X_0 = \{x \in X \mid h(x) < n\}$, and let $S_0 = X_0 \cap (S \cup \{xb \mid x \in S \cap H_n(X)\})$ and $T_0 = X_0 \cap (T \cup \{xb \mid x \in T \cap H_n(X)\})$. Define f_0 by $xf_0 = xf$ for $x \in S \cap X_0$ and $xf_0 = xb^{-1}fb$ for $x \in \{xb \mid x \in S \cap H_n(X)\}$. It is easy to see that $f_0\colon S_0 \to T_0$ is well-defined and an order-isomorphism. By the induction hypothesis, there exists $g_0 \in \mathrm{Aut}(X_0, \leq)$ such that $xg_0 = xf_0$ for $x \in S_0$. Define

g by $xg = xg_0$ for $x \in X_0$ and $xg = xbg_0b^{-1}$ for $x \in H_n(X)$. It is easy to see that $g \in \mathrm{Aut}(X, \leq)$ and $xg = xf$ for all $x \in S$. A dual construction works if $H_{01}(X)$ is isomorphic to some A_m. Thus we are left with the case that $h(X) = 1$ and X is isomorphic to B_m ($m \geq 3$). But it is easy to see that B_m is height-homogeneous. This concludes the proof of the theorem. ∎

4. Further remarks.

An easy consequence of Theorem 2 is the fact that the full automorphism group of a finite height-homogeneous poset is a direct product of symmetric groups. This result even holds for the posets classified in Theorem 1. In fact, if $Z = \{i \mid$ there exist p, q such that $H_{i-1,i}(X)$ is isomorphic to $C_{pq}\}$ then $\mathrm{Aut}(X, \leq) \cong \mathrm{Sym}(H_0(X)) \times \Pi_{i \in Z}\ \mathrm{Sym}(H_i(X))$. This follows from first considering the decomposition according to Lemma 1.1, and then noting that in a poset $(X, \leq)$ which has the homogeneity property of Theorem 1 and in which no subposet $(H_{i,i+1}(X), \leq)$ is isomorphic to C_{pq} every permutation of $H_i(X)$ can be uniquely extended to an automorphism of $(X, \leq)$.

It is natural to ask whether 3 can be replaced by 2 in statement (2) of Theorem 1 and Theorem 2. This is not the case, as can be seen from the example of a poset consisting of the points and lines of a Desarguesian projective plane, where a point is smaller than a line if it is incident with it. This is 2-height-homogeneous, but clearly the automorphism group is not 3-transitive on the set of points (which is $H_0(X)$). There are other similar constructions [3, 5]. Finally, we note that there are many finite posets which are 1-height-homogeneous, and the construction given in [2] shows that every finite group is isomorphic to the full automorphism group of such a poset.

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