The minimum size of critically m-neighbour-connected graphs

Shu-Shih Y. Wu and Margaret B. Cozzens

Department of Mathematics
Northeastern University
Boston, MA 02115

Abstract. A graph G is said to be m-neighbour-connected if the neighbour-connectivity of the graph, $K(G) = m$. A graph G is said to be critically m-neighbour-connected if it is m-neighbour-connected and the removal of the closed neighbourhood of any one vertex yields an $(m-1)$-neighbour-connected subgraph. In this paper, we give some upper bounds of the minimum size of the critically m-neighbour-connected graphs of any fixed order ν, and show that the number of edges in a minimum critically m-neighbour-connected graph with order ν (a multiple of m) is $\lceil \frac{1}{2} m\nu \rceil$.

1. Introduction

In 1978 Gunther and Hartnell [4] introduced, and in 1985-86 Gunther [6] [7] further developed the idea of modeling a spy network by a graph whose vertices represent the stations and whose edges represent lines of communication. If a station is destroyed, the adjacent stations will be betrayed so that the betrayed stations become useless to the network as a whole. Therefore instead of removing only vertices from a communication graph, we want to consider removing vertices and all of their adjacent vertices.

Suppose that G is a graph. Let u be any vertex in G. $N(u) = \{v \in V(G) | v \neq u, v \text{ and } u \text{ are adjacent}\}$ is the **open neighbourhood** of u, and $N[u] = \{u\} \cup N(u)$ denotes the **closed neighbourhood** of u. A vertex u in G is said to be **subverted** when the closed neighbourhood $N[u]$ is deleted from G. A set of vertices $S = \{u_1, u_2, u_3, \cdots, u_m\}$ is called a **subversion strategy** if each of the vertices in S has been subverted. Let G/S be the survival-subgraph left after each vertex of S has been subverted from G. S is called a **cut-strategy** of G if the survival-subgraph G/S is disconnected, or is a clique, or is $\emptyset$. We define the **neighbour-connectivity**, $K(G)$, of G to be the minimum size of all cut-strategies S of G. A graph G is **m-neighbour-connected** if $K(G) = m$.

A graph G is called **critically m-neighbour-connected** if $K(G) = m$, and for any vertex u in G, $K(G/\{u\}) = m - 1$. Reliability of a spy network may be determined by the neighbour-connectivity. In a critically m-neighbour-connected graph, each communication station is so important that any subversion reduces the reliability of the corresponding spy communication network. A graph G is a **minimum** critically m-neighbour-connected graph if no critically m-neighbour-connected graph with the same number of vertices has fewer edges than G. In this paper, we give some upper bounds of the minimum size of the critically m-neighbour-connected graphs of any fixed order ν, and we show that if m is a positive integer then the number of edges in a minimum critically m-neighbour-connected graph with order ν (a multiple of m) is $\lceil \frac{1}{2} m\nu \rceil$; hence a minimum critically m-neighbour-connected graph with order ν (a multiple of m) is m-regular.

$\lfloor x \rfloor$ is the greatest integer less than or equal to x, and $\lceil x \rceil$ is the smallest integer greater than or equal to x.

2. A Class of Critically m-Neighbour-Connected Graphs

Now we consider the following operation, say E, on a graph G to create a collection of graphs, say GE.

A new graph $Ge \in GE$ is created by the following:

(i) Each vertex v of G is replaced by a clique C_v of order $\geq$ deg v.

(ii) C_{v_1} and C_{v_2} are joined by, at most, one edge and they are joined by an edge if, and only if, vertices v_1 and v_2 are joined in G.

(iii) Each vertex in C_v is incident with, at most, one edge not entirely contained in C_v.

Example 1

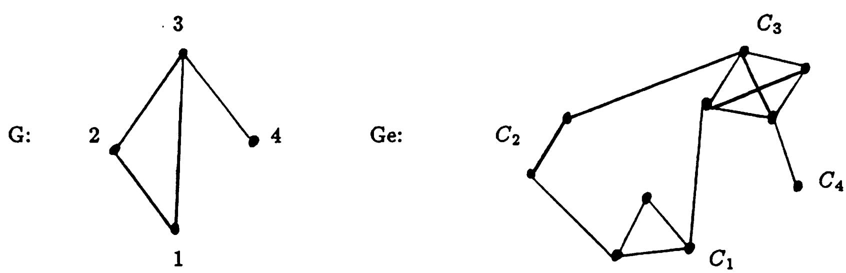

Figure 1

The **connectivity**, $\kappa(G)$, of a graph G is the smallest number of vertices whose removal disconnects G or leaves a single vertex. The graph G is m-**connected**, if the connectivity of G, $\kappa(G) = m$. Thus we apply operation E to an m-connected graph G to obtain an m-neighbour-connected graph.

Theorem 2.1. *Let G be an m-connected graph. Apply operation E to G to obtain Ge. Then Ge is an m-neighbour-connected graph.*

Proof: Observe that deleting any $m - 1$ neighbourhoods in Ge is equivalent to deleting the $m - 1$ corresponding vertices in G. Therefore we obtain the theorem. ∎

Theorem 2.2. *For any positive integers m,n such that $m > 1, n \geq m+1$, there exists a class of critically m-neighbour-connected graphs each of which has n cliques.*

Proof: For any positive integers n, m such that $m > 1, n \geq m+1$, we may construct a Harary graph $H_{m,n}$ which is a m-connected graph [2] . By Theorem 2.1, we apply operation E to $H_{m,n}$, and to $H_{m,n} - \{u\}$, for any vertex u in $H_{m,n}$, to obtain a class of critically m-neighbour-connected graphs $H_{m,n}$E each of which has n cliques. ∎

Example 2

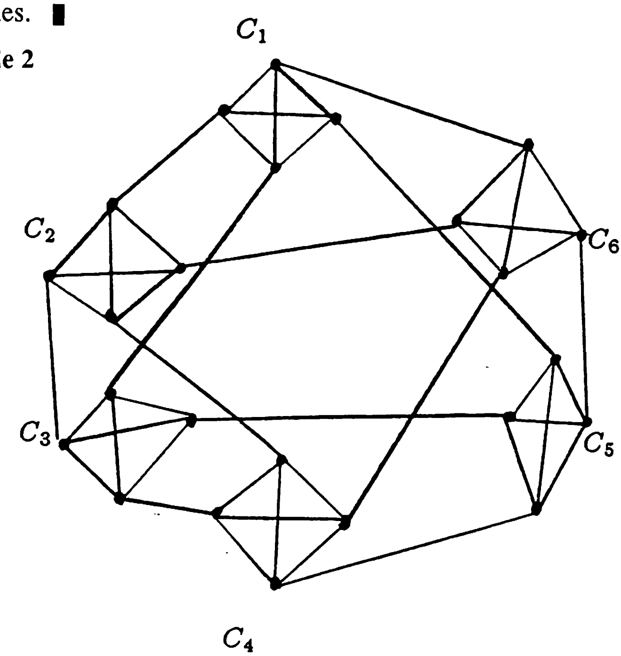

Figure 2 $H_{4,6}e$

3. The Upper Bounds of the Minimum Size of Critically m-Neighbour-Connected Graphs

For any given positive integers ν, m, and n, with $n \geq m+1$, we may construct a class of graphs, $H_{m,n}E$, each of which is critically m-neighbour-connected with order ν. For convenience, we call this class of graphs $G(m, n)$.

Let the vertices in $H_{m,n}$ (Harary graph) be $v_0, v_1, v_2, \cdots, v_{n-1}$, and the corresponding cliques in each of $G(m,n)$ be $C_0, C_1, C_2, \cdots, C_{n-1}$. Let the number of vertices in the cliques $C_0, C_1, C_2, \cdots, C_{n-1}$ be $x_0, x_1, x_2, \cdots, x_{n-1}$, respectively, where $x_i \geq \deg v_i = m$, for all $i = 1, 2, 3, \cdots, n-1$, $x_i \geq deg v_0 = m$ if at least one of n, m is even, and $x_0 \geq \deg v_0 = m+1$ if both of m and n are odd. Hence

$$\sum_{i=0}^{n-1} x_i = \nu$$

and the number of edges in each of $G(m,n)$ is

$$\begin{cases} \frac{1}{2}\left(\sum_{i=0}^{n-1} x_i(x_i-1)+mn\right), & \text{if at least one of } m,n \text{ is even;} \\ \frac{1}{2}\left(\sum_{i=0}^{n-1} x_i(x_i-1)+mn+1\right), & \text{if both of } m \text{ and } n \text{ are odd.} \end{cases}$$

To discuss the minimum size of critically m-neighbour-connected graphs, we minimize $|E(G(m,n))|$ under the condition $\sum_{i=0}^{n-1} x_i = \nu$, and we let $\widetilde{G}(m,n)$ be a subclass of $G(m,n)$ having the smallest number of edges, which is denoted as $\widetilde{g}(m,n)$.

Case 1: At least one of m,n is even $(n \geq m+1)$.

$$\min_{x_i}\left(\sum_{i=0}^{n-1} x_i^2 + mn - \nu\right)$$

$$\textit{subject to}\begin{cases} \sum_{i=0}^{n-1} x_i & = \nu; \\ x_i & \geq m, \text{ for all } i=0,1,2,\cdots,n-1; \\ x_i & \in Z^+, \text{ for all } i=0,1,2,\cdots,n-1. \end{cases}$$

Since $x_i \geq m$, $\nu = \sum_{i=0}^{n-1} x_i \geq \sum_{i=0}^{n-1} m = mn$, we have $n \leq \frac{\nu}{m}$. n is an integer, so $n \leq \lfloor \frac{\nu}{m} \rfloor$.

Case 2: Both of m and n are odd. $(n \geq m+1)$.

$$\min_{x_i}\left(\sum_{i=0}^{n-1} x_i^2 + mn - \nu + 1\right)$$

$$\textit{subject to}\begin{cases} \sum_{i=0}^{n-1} x_i & = \nu; \\ x_0 \geq m+1; & \\ x_i \geq m, & \text{for all } i=1,2,3,\cdots,n-1; \\ x_i \in Z^+, & \text{for all } i=0,1,2,3,\cdots,n-1. \end{cases}$$

Since $\nu = \sum_{i=0}^{n-1} x_i \geq m+1+(n-1)m = nm+1, \nu-1 \geq nm$, we have $n \leq \lfloor \frac{\nu-1}{m} \rfloor \leq \lfloor \frac{\nu}{m} \rfloor$.

By Lagrange's method, we obtain

$$x_i = Q \text{ or } Q+1, \text{ for all } i=0,1,2,3,\cdots,n-1 \text{ where } Q = \left\lfloor \frac{\nu}{n} \right\rfloor.$$

We may rearrange the subscripts of x_i, such that

$$x_i = \begin{cases} Q+1, & \text{if } 0 \le i \le R-1; \\ Q, & \text{if } R \le i \le n-1. \end{cases}$$
where $R = \nu - nQ$.

Therefore, $\tilde{g}(m,n)$

$$= \begin{cases} \frac{1}{2}((Q+1)^2 R + Q^2(n-R) \\ \quad -\nu + mn), & \text{if at least one of } m, n \text{ is even;} \\ \frac{1}{2}((Q+1)^2 R + Q^2(n-R) \\ \quad -\nu + mn + 1), & \text{if both of } m \text{ and } n \text{ are odd.} \end{cases}$$

$$= \begin{cases} \frac{1}{2}(2QR + R + Q^2 n - \nu + nm), & \text{if at least one of } m, n \text{ is even;} \\ \frac{1}{2}(2QR + R + Q^2 n - \nu + nm + 1), & \text{if both of } m \text{ and } n \text{ are odd.} \end{cases}$$

Example 3 $\nu = 32, m = 4, n = 6$ are given. Then we may construct $\tilde{G}(4,6)$-graphs, each of which is a critically 4-neighbour-connected graph with order 32. $Q = \lfloor \frac{32}{6} \rfloor = 5$. $R = \nu - nQ = 32 - 6 \text{x} 5 = 2$.

$\tilde{G}(4,6)$:

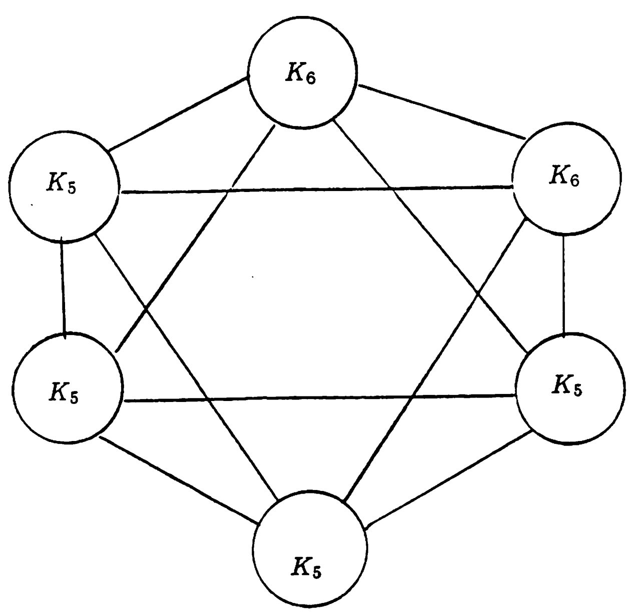

Figure 3 $\tilde{G}(4,6)$

$\tilde{g}(4,6) = \frac{1}{2}(2QR + R + Q^2 n - \nu + nm) = 82$.

Example 4 $\nu = 55, m = 5, n = 9$ are given. Then we may construct $\widetilde{G}(5,9)$-graphs, each of which is a critically 5-neighbour-connected graph with order 55. $Q = \lfloor \frac{55}{9} \rfloor = 6, R = \nu - nQ = 55 - 54 = 1.$

$\tilde{G}(5,9)$:

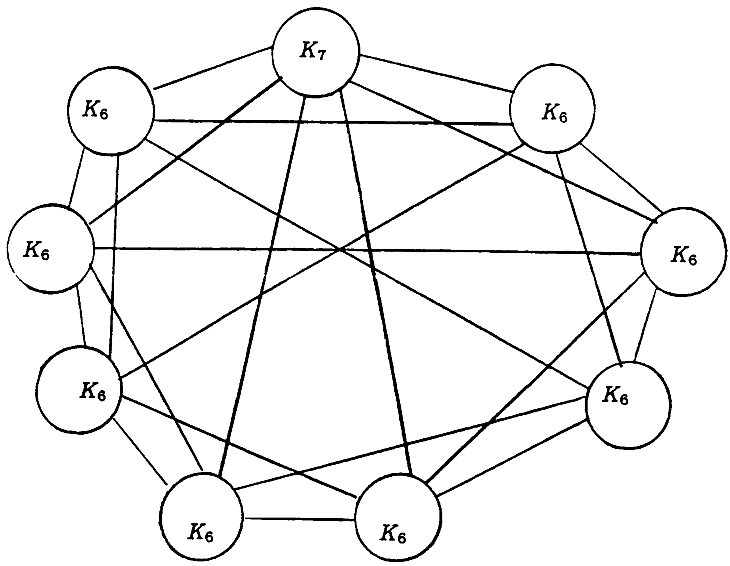

Figure 4 $\widetilde{G}(5,9)$

$\tilde{g}(5,9) = \frac{1}{2}(2QR + R + Q^2 n - \nu + nm + 1) = 164.$

Next we find an upper bound of the minimum size of critically m-neighbour-connected graphs with order ν. We regard n as a variable, ν, m as fixed integers, and

$$\min_n f(n)$$

$$\textit{subject to} \begin{cases} n \geq m + 1; \\ n \leq \lfloor \frac{\nu}{m} \rfloor. \end{cases}$$

where

$$f(n) = \begin{cases} 2QR + R + Q^2 n - \nu + nm, & \text{if at least one of } m, n \text{ is even;} \\ 2QR + R + Q^2 n - \nu + nm + 1, & \text{if both of } m \text{ and } n \text{ are odd.} \end{cases}$$

We shall show that the objective function $f(n)$ is decreasing when $m + 1 \leq n \leq \lfloor \frac{\nu}{m} \rfloor$.

Lemma 3.1. *For any fixed positive integers m, ν, if $m + 1 \leq n \leq \lfloor \frac{\nu}{m} \rfloor, Q = \lfloor \frac{\nu}{n} \rfloor, R = \nu - nQ$, then the function $f(n)$ is decreasing.*

Proof: Evaluate the values of f when $n = k$, and $n = k + 1$. Since $m + 1 \leq n \leq \lfloor \frac{\nu}{m} \rfloor$,

$$m + 1 \leq k \leq \lfloor \frac{\nu}{m} \rfloor \text{ and } m + 1 \leq k + 1 \leq \lfloor \frac{\nu}{m} \rfloor,$$

then

$$m + 1 \leq k \leq \lfloor \frac{\nu}{m} \rfloor - 1.$$

Let

$$q_1 = \lfloor \frac{\nu}{k} \rfloor, r_1 = \nu - kq_1, \text{ so } 0 \leq r_1 < k.$$

$$q_2 = \lfloor \frac{\nu}{k+1} \rfloor, r_2 = \nu - (k+1)q_2, \text{ so } 0 \leq r_2 < k + 1.$$

Then

$$q_1 \geq q_2 \geq m > 0.$$

We discuss two cases:

Case 1: $q_1 = q_2$
$\nu = kq_1 + r_1 = (k+1)q_2 + r_2$. Hence, $r_1 = q_2 + r_2$ or $r_1 = q_1 + r_2$, then $r_2 = r_1 - q_1$.

$$
\begin{aligned}
& f(k+1) - f(k) \\
&= \begin{cases}
[2q_2r_2 + r_2 + q_2^2(k+1) - \nu + (k+1)m] \\
\quad -[2q_1r_1 + r_1 + q_1^2k - \nu + km], & \text{if } m \text{ is even;} \\
[2q_2r_2 + r_2 + q_2^2(k+1) - \nu + (k+1)m + 1] \\
\quad -[2q_1r_1 + r_1 + q_1^2k - \nu + km], & \text{if } m \text{ is odd and } k \text{ is even;} \\
[2q_2r_2 + r_2 + q_2^2(k+1) - \nu + (k+1)m] \\
\quad -[2q_1r_1 + r_1 + q_1^2k - \nu + km + 1], & \text{if both of } m \text{ and } k \text{ are odd.}
\end{cases} \\
&= \begin{cases}
m - q_1 - q_1^2, & \text{if } m \text{ is even;} \\
m - q_1 - q_1^2 + 1, & \text{if } m \text{ is odd and } k \text{ is even;} \\
m - q_1 - q_1^2 - 1, & \text{if both of } m \text{ and } k \text{ are odd.}
\end{cases} \\
&\leq 0.
\end{aligned}
$$

Hence $f(k+1) \leq f(k)$, for all $m + 1 \leq k \leq \lfloor \frac{\nu}{m} \rfloor - 1$

Case 2: $q_1 \neq q_2$. (i.e. $q_1 \geq q_2 + 1 > 1$)

$$\nu = kq_1 + r_1 = (k+1)q_2 + r_2, \text{ so } kq_1 - kq_2 = q_2 + r_2 - r_1$$

$$1 < q_2 + 1 \leq q_1, \text{ so } \nu(q_2 + 1) \leq \nu q_1$$

$$\nu(q_2+1) = ((k+1)q_2 + r_2)(q_2+1) = kq_2^2 + q_2^2 + r_2q_2 + kq_2 + q_2 + r_2$$

and

$$\nu q_1 = (kq_1 + r_1)q_1 = kq_1^2 + r_1q_1.$$

Hence

$$kq_2^2 + q_2^2 + r_2q_2 + kq_2 + q_2 + r_2 \le kq_1^2 + r_1q_1$$

$$kq_2^2 - kq_1^2 + q_2^2 + r_2q_2 - r_1q_1 \le -(kq_2 + q_2 + r_2) = -\nu.$$

Subcase 1: $r_1 \ge 1$. Then,

$$
\begin{aligned}
&f(k+1) - f(k)\\
&= \begin{cases}
[2q_2r_2 + r_2 + q_2^2(k+1) - \nu + (k+1)m] \\
\quad -[2q_1r_1 + r_1 + q_1^2k - \nu + km], & \text{if } m \text{ is even;}\\
[2q_2r_2 + r_2 + q_2^2(k+1) - \nu + (k+1)m + 1] \\
\quad -[2q_1r_1 + r_1 + q_1^2k - \nu + km], & \text{if } m \text{ is odd and } k \text{ is even;}\\
[2q_2r_2 + r_2 + q_2^2(k+1) - \nu + (k+1)m] \\
\quad -[2q_1r_1 + r_1 + q_1^2k - \nu + km + 1], & \text{if both of } m \text{ and } k \text{ are odd.}
\end{cases}\\
&= \begin{cases}
2q_2r_2 + q_2^2k + q_2^2 + m + r_2 - 2q_1r_1 \\
\quad -r_1 - q_1^2k, & \text{if } m \text{ is even;}\\
2q_2r_2 + q_2^2k + q_2^2 + m + r_2 - 2q_1r_1 \\
\quad -r_1 - q_1^2k + 1, & \text{if } m \text{ is odd and } k \text{ is even;}\\
2q_2r_2 + q_2^2k + q_2^2 + m + r_2 - 2q_1r_1 \\
\quad -r_1 - q_1^2k - 1, & \text{if both of } m \text{ and } k \text{ are odd.}
\end{cases}\\
&= \begin{cases}
2q_2r_2 + q_2^2k + q_2^2 + m + \nu - (k+1)q_2 \\
\quad -2q_1r_1 - \nu + kq_1 - q_1^2k, & \text{if } m \text{ is even;}\\
2q_2r_2 + q_2^2k + q_2^2 + m + \nu - (k+1)q_2 \\
\quad -2q_1r_1 - \nu + kq_1 - q_1^2k + 1, & \text{if } m \text{ is odd and } k \text{ is even;}\\
2q_2r_2 + q_2^2k + q_2^2 + m + \nu - (k+1)q_2 \\
\quad -2q_1r_1 - \nu + kq_1 - q_1^2k - 1, & \text{if both of } m \text{ and } k \text{ are odd.}
\end{cases}\\
&= \begin{cases}
(kq_2^2 - kq_1^2 + q_2^2 + r_2q_2 - r_1q_1) + (r_2q_2 \\
\quad -r_1q_1 + kq_1 - kq_2) + m - q_2, & \text{if } m \text{ is even;}\\
(kq_2^2 - kq_1^2 + q_2^2 + r_2q_2 - r_1q_1) + (r_2q_2 \\
\quad -r_1q_1 + kq_1 - kq_2) + m - q_2 + 1, & \text{if } m \text{ is odd and } k \text{ is even;}\\
(kq_2^2 - kq_1^2 + q_2^2 + r_2q_2 - r_1q_1) + (r_2q_2 \\
\quad -r_1q_1 + kq_1 - kq_2) + m - q_2 - 1, & \text{if both of } m \text{ and } k \text{ are odd.}
\end{cases}
\end{aligned}
$$

$$\leq \begin{cases} -\nu + r_2 q_2 - r_1 q_1 + r_2 - r_1 + q_2 \\ \quad +m - q_2\,, & \text{if } m \text{ is even;} \\ -\nu + r_2 q_2 - r_1 q_1 + r_2 - r_1 + q_2 \\ \quad +m - q_2 + 1\,, & \text{if } m \text{ is odd and } k \text{ is even;} \\ -\nu + r_2 q_2 - r_1 q_1 + r_2 - r_1 + q_2 \\ \quad +m - q_2 - 1\,, & \text{if both of } m \text{ and } k \text{ are odd.} \end{cases}$$

$$\begin{aligned} &\leq -\nu + q_2(k+1) - r_1 q_1 + r_2 - r_1 + m \\ &= -\nu + \nu - q_1 r_1 - r_1 + m \\ &= -q_1 r_1 - r_1 + m \\ &= -r_1(q_1 + 1) + m \\ &\leq -(q_1 + 1) + m \\ &< 0\,. \end{aligned}$$

Hence $f(k+1) < f(k)$, for all $m + 1 \leq k \leq \lfloor \frac{\nu}{m} \rfloor - 1$.

Subcase 2: $r_1 = 0$.

Then when m is odd and k is even, we have $r_2 \neq k$ or $q_2 \neq m$. Since if $r_2 = k$ and $q_2 = m$, then $\nu = kq_1 = kq_2 + q_2 + r_2 = mk + m + k, q_1 = m + 1 + \frac{m}{k} \in Z^+$. Therefore we obtain $k|m$, a contradiction, since m is odd and k is even.

$$f(k+1) - f(k)$$

$$= \begin{cases} [2q_2 r_2 + r_2 + q_2^2(k+1) - \nu + (k+1)m] \\ \quad -[2q_1 r_1 + r_1 + q_1^2 k - \nu + km]\,, & \text{if } m \text{ is even;} \\ [2q_2 r_2 + r_2 + q_2^2(k+1) - \nu + (k+1)m + 1] \\ \quad -[2q_1 r_1 + r_1 + q_1^2 k - \nu + km]\,, & \text{if } m \text{ is odd and } k \text{ is even;} \\ [2q_2 r_2 + r_2 + q_2^2(k+1) - \nu + (k+1)m] \\ \quad -[2q_1 r_1 + r_1 + q_1^2 k - \nu + km + 1]\,, & \text{if both of } m \text{ and } k \text{ are odd.} \end{cases}$$

$$= \begin{cases} (2q_2 r_2 + r_2 + q_2^2 k + q_2^2 + m - q_1^2 k\,, & \text{if } m \text{ is even;} \\ (2q_2 r_2 + r_2 + q_2^2 k + q_2^2 + m - q_1^2 k + 1\,, & \text{if } m \text{ is odd and } k \text{ is even;} \\ (2q_2 r_2 + r_2 + q_2^2 k + q_2^2 + m - q_1^2 k - 1\,, & \text{if both of } m \text{ and } k \text{ are odd.} \end{cases}$$

$$= \begin{cases} (q_2^2 k - q_1^2 k + q_2^2 + r_2 q_2 - r_1 q_1) \\ \quad +r_2 q_2 + r_2 + m, & \text{if } m \text{ is even;} \\ (q_2^2 k - q_1^2 k + q_2^2 + r_2 q_2 - r_1 q_1) \\ \quad +r_2 q_2 + r_2 + m + 1, & \text{if } m \text{ is odd and } k \text{ is even;} \\ (q_2^2 k - q_1^2 k + q_2^2 + r_2 q_2 - r_1 q_1) \\ \quad +r_2 q_2 + r_2 + m - 1, & \text{if both of } m \text{ and } k \text{ are odd.} \end{cases}$$

$$\leq \begin{cases} -\nu + r_2q_2 + \nu - (k+1)q_2 + m, & \text{if } m \text{ is even;} \\ -\nu + r_2q_2 + \nu - (k+1)q_2 + m + 1, & \text{if } m \text{ is odd and } k \text{ is even;} \\ -\nu + r_2q_2 + \nu - (k+1)q_2 + m - 1, & \text{if both of } m \text{ and } k \text{ are odd.} \end{cases}$$

$$= \begin{cases} r_2q_2 - kq_2 + (m - q_2), & \text{if } m \text{ is even;} \\ r_2q_2 - kq_2 + (m - q_2) + 1, & \text{if } m \text{ is odd and } k \text{ is even;} \\ r_2q_2 - kq_2 + (m - q_2) - 1, & \text{if both of } m \text{ and } k \text{ are odd.} \end{cases}$$

≤ 0, since $r_2 \leq k, q_2 \geq m > 0$ and ($r_2 \neq k$ or $q_2 \neq m$).

Hence $f(k+1) \leq f(k)$, for all $m + 1 \leq k \leq \lfloor \frac{\nu}{m} \rfloor - 1$.

Lemma 3.2. *Let ν, n, m be three integers, $n \geq m+1$. If $n = \lfloor \frac{\nu}{m} \rfloor$, then $m = \lfloor \frac{\nu}{n} \rfloor$.*

Proof: n is the quotient of ν divided by m. Let R be the remainder of ν divided by m. So $\nu = nm + R$, where $0 \leq R < m$.

Since $0 \leq R < m$ and $m + 1 \leq n, 0 \leq R < n$. $\nu = mn + R$ and $0 \leq R < n$, hence m is the quotient of ν divided by n. That is, $m = \lfloor \frac{\nu}{n} \rfloor$. ∎

By using Lemma 3.1, we can obtain an upper bound of the minimum size of critically m-neighbour-connected graphs.

Theorem 3.3. *Let m be a positive integer. If G is a minimum critically m-neighbour-connected graph with order ν, then $\lceil \frac{1}{2}m\nu \rceil \leq |E(G)| \leq \lceil \frac{1}{2}m\nu + \frac{1}{2}mR \rceil$. Where $R = \nu - \lfloor \frac{\nu}{m} \rfloor m$, the remainder of the order ν divided by m.*

Proof: Let n be an integer, such that $n \geq m + 1$.

Let the order of each of $\widetilde{G}(m,n)$-graphs be ν. Hence, $\widetilde{g}(m,n) = \lceil \frac{1}{2}(2Q_nR_n + R_n + Q_n^2 n - \nu + nm) \rceil$, where $Q_n = \lfloor \frac{\nu}{n} \rfloor$ and $R_n = \nu - nQ_n$. By Theorem 2.2, $\widetilde{G}(m,n)$ is a class of critically m-neighbour-connected graphs, hence $|E(G)| \leq \widetilde{g}(m,n)$.

If $n > \lfloor \frac{\nu}{m} \rfloor$, n is an integer, then $n > \frac{\nu}{m}$. We have $nm > \nu$. By the construction of $G(m,n)$-graph, $|C_i| \geq m$, for all i. Thus $\nu = \sum_{i=0}^{n-1} |C_i| \geq mn$, a contradiction. Therefore, $n \leq \lfloor \frac{\nu}{m} \rfloor$.

The function $f(n)$ is a decreasing function of n, for $m + 1 \leq n \leq \lfloor \frac{\nu}{m} \rfloor$. Hence $f(n)$ has the minimum value, when $n = \lfloor \frac{\nu}{m} \rfloor$.

By Lemma 3.2, $n = \lfloor \frac{\nu}{m} \rfloor$ and $n \geq m + 1$, we have $m = \lfloor \frac{\nu}{n} \rfloor$. Hence $Q_n =$

$\lfloor\frac{\nu}{n}\rfloor = m$ and $R_n = \nu - nQ_n = \nu - nm = \nu - \lfloor\frac{\nu}{m}\rfloor m$. The minimum value of

$$
\begin{aligned}
f(n) &= f(\lfloor\frac{\nu}{m}\rfloor) \\
&= \begin{cases} 2mR_n + R_n + m^2 n - \nu + nm, & \text{if at least one of } m, n \text{ is even;} \\ 2mR_n + R_n + m^2 n - \nu + nm + 1, & \text{if both of } m \text{ and } n \text{ are odd.} \end{cases} \\
&= \begin{cases} 2mR_n + R_n + m^2 n - R_n - nm \\ \quad +nm, & \text{if at least one of } m, n \text{ is even;} \\ 2mR_n + R_n + m^2 n - R_n - nm \\ \quad +nm + 1, & \text{if both of } m \text{ and } n \text{ are odd.} \end{cases} \\
&= \begin{cases} 2mR_n + m^2 n, & \text{if at least one of } m, n \text{ is even;} \\ 2mR_n + m^2 n + 1, & \text{if both of } m \text{ and } n \text{ are odd.} \end{cases} \\
&= \begin{cases} m(mn + R_n) + mR_n, & \text{if at least one of } m, n \text{ is even;} \\ m(mn + R_n) + mR_n + 1, & \text{if both of } m \text{ and } n \text{ are odd.} \end{cases} \\
&= \begin{cases} m\nu + mR_n, & \text{if at least one of } m, n \text{ is even;} \\ m\nu + mR_n + 1, & \text{if both of } m \text{ and } n \text{ are odd.} \end{cases}
\end{aligned}
$$

Therefore when $n = \lfloor\frac{\nu}{m}\rfloor, \tilde{g}(m,n) = \frac{1}{2}f(n) = \lceil\frac{1}{2}(m\nu + mR)\rceil$, where $R = R_n = \nu - \lfloor\frac{\nu}{m}\rfloor m$. $|E(G)| \leq \tilde{g}(m,n) = \lceil\frac{1}{2}m\nu + \frac{1}{2}mR\rceil$.

Since G is an m-neighbour-connected graph, $m = K(G) \leq \delta(G)$ [6], it is easy to show that $\lceil\frac{1}{2}m\nu\rceil \leq |E(G)|$. ∎

Since $0 \leq R \leq m-1$, $\frac{1}{2}m\nu + \frac{1}{2}mR \leq \frac{1}{2}m\nu + \frac{1}{2}m(m-1) = \frac{1}{2}m(\nu + m - 1)$. It follows that

Corollary 3.4. *Let m be a positive integer. If G is a minimum critically m-neighbour-connected graph, then $\lceil\frac{1}{2}m\nu\rceil \leq |E(G)| \leq \lceil\frac{1}{2}m(\nu + m - 1)\rceil$, where $\nu = |V(G)|$.*

Corollary 3.5. *If the order of G, ν, is a multiple of m, and G is a minimum critically m-neighbour-connected graph, then $|E(G)| = \lceil\frac{1}{2}m\nu\rceil$.*

Proof: Since $R = \nu - \lfloor\frac{\nu}{m}\rfloor m = 0$, by Theorem 3.3, we obtain the result. ∎

Example 5 $\nu = 72, m = 7, G$ is a minimum critically 7-neighbour-connected graph with order 72, then by Theorem 3.3, $252 = \lceil\frac{1}{2}m\nu\rceil \leq |E(G)| \leq \lceil\frac{1}{2}m\nu + \frac{1}{2}mR\rceil = 259$.

Example 6 $\nu = 32, m = 4, G$ is a minimum critically 4-neighbour-connected graph with order 32, then by Corollary 3.5, $|E(G)| = \lceil\frac{1}{2}m\nu\rceil = 64$.

Acknowledgement

We acknowledge an unusually cogent referee's report on an earlier version of this paper.

References

1. J. A. Bondy and U.S.R. Murty, "Graph Theory with Applications", The Macmillan Press Ltd, (1976).
2. M. B. Cozzens and S.S.Y. Wu, *On Minimum Critically n-Edge-Connected Graphs*, SIAM J. Alg. and Disc. Meth. **8(4)** (October 1987), 659–669.
3. S. Goodman and D. Shier, *On Designing a Reliable Hierarchical Structure*, SIAM J. Appl. Math. **32(2)** (March 1977), 418–430.
4. G. Gunther and B. L. Hartnell, *On Minimizing the Effects of Betrayals in a Resistance Movement*, Proc. Eighth Manitoba Conference on Numerical Mathematics and Computing (1978), 285–306.
5. G. Gunther and B. L. Hartnell, *Optimal K-secure Graphs*, Discrete Applied Math. **2** (1980), 225–231.
6. G. Gunther, *Neighbour-Connectivity in Regular Graphs*, Discrete Applied Math. **11** (1985), 233–243.
7. G. Gunther, *On the Existence of Neighbour-Connected Graphs*, Congressus Numerantium **54** (1986), 105–110.
8. B. L. Hartnell, *The Optimum Defence Against Random Subversions in a Network*, Proc. Tenth Southeastern Conference on Combinatorics, Graph Theory and Computing, Boca Raton (April 1979), 293–299.