A note on indecomposable Kirkman squares

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Abstract. A $KS_2(v;1,\lambda)$ is called indecomposable if it is not isomorphic to the direct sum of a $KS_2(v;1,\lambda_1)$ with a $KS_2(v;1,\lambda_2)$ for some λ_1 and λ_2 which add to λ. In this note, we show that there exists an indecomposable $KS_2(v;1,\lambda)$ for $v \equiv 0 \pmod 2$, $v \geq 4$ and $\lambda \geq 2$.

1. Introduction

A Kirkman square with index λ, latinicity μ, block size k and v points, $KS_k(v;\mu,\lambda)$, is a $t \times t$ array defined on a v-set V such that

(1) each point of V is contained in precisely μ cells of each row and column,
(2) each cell of the array is either empty or contains a k subset of V, and
(3) the collection of blocks obtained from the nonempty cells of the array is a (v,k,λ)-BIBD.

Necessary conditions for the existence of a $KS_k(v;\mu,\lambda)$ are established in [4]. These are shown to be sufficient for the existence of $KS_2(v;\mu,\lambda)$ with $\mu \geq 2$ in [4, 5]. A $KS_2(v;1,1)$ is a Room square of side $v-1$ or a $RS(v-1)$. The existence of Room squares was completed in 1975, [6]; there exists a $RS(v-1)$ if and only if $v \equiv 0 \pmod 2$ and $v \neq 4$ or 6. It is easy to construct a $KS_2(v;1,\lambda)$ for $\lambda \geq 2$ and $v \geq 8$ by taking the direct sum of λ copies of a $RS(v-1)$. (λ copies of a $RS(v-1)$ are placed along the diagonal of the array and the other cells are left empty.) The remaining two small cases for $KS_2(v;1,\lambda)s$, $v=4$ and $v=6$, were done in [9].

A $KS_2(v;1,\lambda)$ is called indecomposable if it is not isomorphic to the direct sum of a $KS_2(v;1,\lambda_1)$ with a $KS_2(v;1,\lambda_2)$ for some λ_1 and λ_2 which add to λ. W.D. Wallis conjectured in [9] that there exists an indecomposable $KS_2(v;1,\lambda)$ for all $v \equiv 0 \pmod 2$. In this note, we show that there exists an indecomposable $KS_2(v;1,\lambda)$ for $v \equiv 0 \pmod 2$, $v \geq 4$ and $\lambda \geq 2$.

2. Preliminary definitions and results

Let $S = \{S_1, S_2, \dots, S_n\}$ be a partition of a set V into n pairs. For $m \geq n$, we define a circulant $m \times m$ matrix on S, $M_m(S_1, S_2, \dots, S_n)$, as follows. Place the pair S_i in cell $(j, i+j-1)$ for $j = 1, 2, \dots, m$ and $i = 1, 2, \dots, n$ where the second argument is taken modulo m. The remaining cells of $M_m(S_1, S_2, \dots, S_n)$ are left empty.

The following lemma on subarrays of $M_m(S_1, S_2, \dots, S_n)$ will be important in our constructions.

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Lemma 2.1. *Let l be a positive integer, $n \leq l < m$. Suppose L is an $l \times l$ array with the following properties. (1) L contains l copies of each pair S_i, $i = 1, 2, \ldots, n$. (2) Every element of V occurs once in each row and column of L. Then $M_m(S_1, S_2, \ldots, S_n)$ does not contain L as a subarray.*

Proof: Let $M = M_m(S_1, S_2, \ldots, S_n)$. We try to construct a subarray L of M with the required properties. Since every element of V must occur once in each row and column of L, row i of L must contain the n nonempty cells of row i of M. Without loss of generality, we may suppose that the first n cells of row 1 of M are in L. Since the nonempty cells in column 2 of M must now be in L, $S_1, S_2, \ldots, S_n$ in row 2 must also be placed in L. Continuing this process, we see that L must equal M. ∎

We will also use Howell designs to construct indecomposable Kirkman squares. A Howell design of side s and order $2n$, or more briefly an $H(s, 2n)$, is an $s \times s$ array in which each cell is either empty or contains an unordered pair of elements from some $2n$-set, say X, such that

(1) each row and column is Latin (that is, every element of X is in precisely one cell of each row and column) and
(2) every unordered pair of elements from X is in at most one cell of the array.

It follows immediately from the definition an $H(s, 2n)$ that $n \leq s \leq 2n - 1$. We note that an $H(2n-1, 2n)$ is a $RS(2n-1)$. The spectrum of $H(s, n)$ was completed by D.R. Stinson in [8]. We will use the following existence result for $H(2n, 2n+2)$.

Theorem 2.2. *(Schellenberg and Vanstone, [7]). For any positive integer $2n$, $2n > 2$, there is an $H(2n, 2n+2)$; furthermore, there is no $H(2, 4)$.*

We will also use the existence of some (s, t)-incomplete Room squares ([2, 3]). An (s, t)-incomplete Room square is a Room square of side s which is missing a sub-Room square of side t. (The sub-Room square of side t need not exist.)

Theorem 2.3. *(Dinitz and Stinson, [2], Dinitz, Stinson and Wallis, [3]). For all $t = 3, 5, 7$, and for all odd $s \geq 3t + 2$, there is an (s, t)-incomplete Room square.*

3. Indecomposable $KS_2(4; 1, \lambda)$ and $KS_2(6; 1, \lambda)$ for $\lambda \geq 2$

In this section, we construct indecomposable $KS_2(v; 1, \lambda)$ for the two smallest cases, $v = 4$ and $v = 6$.

Lemma 3.1. *For $\lambda \geq 2$, there exists an indecomposable $KS_2(4; 1, \lambda)$.*

Proof: We use circulant matrices to generalize the constructions for indecomposable $KS_2(4; 1, 2)$ and $KS_2(4; 1, 3)$ in [3]. Define 3 $\lambda \times \lambda$ arrays.

$$K_1 = M_\lambda(01, 23)$$
$$K_2 = M_\lambda(02, 13)$$
$$K_3 = M_\lambda(03, 12)$$

The direct sum of K_1, K_2 and K_3 is a $KS_2(4;1,\lambda)$. We use Lemma 2.1 to see that it is indecomposable. ▮

Lemma 3.2. *For* $\lambda \geq 2$, *there exists an indecomposable* $KS_2(k;1,\lambda)$.

Proof: An indecomposable $KS_2(6;1,2)$ is displayed in [3]. We generalize the construction for the indecomposable $KS_2(6;1,\lambda)$ for $\lambda \geq 3$. Let H be an $H(4,6)$ defined on the set $\{0,1,2,3,\alpha,\infty\}$ where the missing pairs of the design are $\{0,1\}$, $\{2,3\}$ and $\{\alpha,\infty\}$. The direct sum of λ copies of H and $M_\lambda(01, 23, \alpha\infty)$ is an indecomposable $KS_2(6;1,\lambda)$. (Again we use Lemma 2.1 to see that the $KS_2(6;1,\lambda)$ is indecomposable.) ▮

4. Indecomposable $KS_2(v;1,\lambda)$ for $\lambda \geq 2$

We first generalize the Howell design construction used in Lemma 3.2.

Theorem 4.1. *If* $v \geq 6$ *and* $v \equiv 0 \pmod 2$, $v \leq 2\lambda$, *then there exists an indecomposable* $KS_2(v;1,\lambda)$.

Proof: Let V be a set of v elements, $v \equiv 0 \pmod 2$, $v \leq 2\lambda$ and $v \geq 6$. Let H be an $H(v-2,v)$ defined on V (Theorem 2.2). Suppose the $\frac{v}{2}$ pairs which do not occur in H are $S_1, S_2, \ldots, S_{\frac{v}{2}}$. The direct sum of λ copies of H and $M_\lambda(S_1, S_2, \ldots, S_{\frac{v}{2}})$ is an indecomposable $KS_2(v;1,\lambda)$. ▮

Lemma 4.2. *There exist indecomposable* $KS_2(v;1,\lambda)$ *for* $\lambda \geq 2$ *and* $v = 8$ *and* $v = 10$.

Proof (i) $v = 8$: By Theorem 4.1, there exists an indecomposable $KS_2(8;1,\lambda)$ for $\lambda \geq 4$. An indecomposable $KS_2(8;1,2)$ is displayed in Figure 1 and an indecomposable $KS_2(8;1,3)$ in Figure 2. (These designs were constructed using starters and adders ([5]).)

(ii) $v = 10$: By Theorem 4.1, there exists an indecomposable $KS_2(10;1,\lambda)$ for $\lambda \geq 5$. An indecomposable $KS_2(10;1,2)$ is displayed in Figure 3. Starters and adders ([5]) for an indecomposable $KS_2(10;1,3)$ and an indecomposable $KS_2(10;1,4)$ are listed in Table 1. ▮

Theorem 4.3. *There exists an indecomposable* $KS_2(v;1,\lambda)$ *for* $\lambda \geq 2$ *and* $v \equiv 0 \pmod 2$, $v \geq 4$.

Proof: The cases $v = 4, 6, 8$ and 10 have been taken care of in Lemmas 3.1, 3.2 and 4.2 respctively.

Let $v \equiv 0 \pmod 2$ and $v \geq 12$. For $v \geq 12$, there exists a $(v-1,3)$-incomplete Room square (Theorem 2.3). Let R denote this incomplete $RS(v-1)$. Suppose the missing 3×3 subarray "$RS(3)$" is defined on the set $\{0,1,2,3\}$. R can be written in the following form.

$$R = \begin{array}{|cc|} A & B \\ C & E \end{array}$$ where E denotes the missing $RS(3)$.

∞0			15		24		36						
	∞1			26		35		40					
46		∞2			30				51				
	50		∞3			41				62			
52		61		∞4							03		
	63		02		∞5							14	
		04		13		∞6							25
	24							56		∞3			01
		35					12		60		∞4		
			46					23		01		∞5	
				50					34		12		∞6
					61		∞0			45		23	
						02		∞1			56		34
13							45		∞2			60	

Figure 1
An indecomposable $KS_2(8; 1, 2)$

∞_0						12							35	46						
23	∞_1						46								50					
	34	∞_2						50								61				
		45	∞_3						61								02			
			56	∞_4						02								13		
				60	∞_5						13								24	
					01	∞_6						24								35
						03	∞_1						46	25						
14							50	∞_2							36					
	25							61	∞_3							40				
		36							02	∞_4							51			
			40							13	∞_5							62		
				51							24	∞_6							03	
					62							35	∞_0							14
56							23								14					∞_0
	60							34						∞_1		25				
		01							45						∞_2		36			
			12							56						∞_3		40		
				23							60						∞_4		51	
					34							01						∞_5		62
						45							12	03					∞_6	

Figure 2
An indecomposable $KS_2(8; 1, 3)$

Let K_1 be the $\lambda(v-1) \times \lambda(v-1)$ array constructed by taking the direct sum of λ copies of R. K_1 contains a $3\lambda \times 3\lambda$ empty array K_2; the diagonal of K_2 contains the λ copies of E. Place an indecomposable $KS_2(4; 1, \lambda)$ defined on $\{0, 1, 2, 3\}$ in K_2. The resulting array K is a $KS_2(v; 1, \lambda)$. Since the $KS_2(4; 1, \lambda)$ is indecomposable, K is indecomposable. ∎

∞0						35		12		67				48			
23	∞1						46				78				50		
	34	∞2						57				80				61	
68		45	∞3										01				72
	70		56	∞4					83					12			
		81		67	∞5					04					23		
			02		78	∞6					15					34	
				13		80	∞7					26					45
					24		01	∞8	56				37				
47	26								∞0							58	13
	58	37							24	∞1							60
		60	48						71	35	∞2						
			71	50						82	46	∞3					
				82	61						03	57	∞4				
					03	72						14	68	∞5			
						14	83						25	70	∞6		
							25	04						36	81	∞7	
15								36							47	02	∞8

Figure 3
An indecomposable $KS_2(10; 1, 2)$

We conclude with some remarks about the one-factorizations of λK_v ([9, 1]) formed by the rows and columns of a $KS_2(v; 1, \lambda)$. The indecomposable $KS_2(6; 1, 2)$ in [3] has the property that the one-factorization of $2K_6$ formed by the columns is indecomposable ([9]). We can construct an indecomposable $KS_2(v; 1, 2)$ with the property that the one factorization of $2K_v$ formed by its columns is also indecomposable for $v \geq 18$. We take the direct sums of 2 copies of a $(v-1, 5)$-incomplete Room square (Theorem 2.3); then we fill in the empty 10×10 array with the indecomposable $KS_2(6; 1, 2)$ mentioned above.

A one factorization F of λK_v is called simple if it has no repeated one factors. Recently, J. Dinitz has constructed indecomposable $KS_2(2n; 1, 2)$ for $n \geq 6$ where the one factorization of $2K_{2n}$ formed by the columns is indecomposable and simple and the one factorization of $2K_{2n}$ formed by the columns is also indecomposable and simple, [1].

Table 1

Starters and adders for Indecomposable $KS_2(10; 1, \lambda)$ for $\lambda = 3$ and $\lambda = 4$

1. An indecomposable $KS_2(10; 1, 3)$ is generated by 9 9×9 arrays. The arrays are arranged as follows:

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}$$

A	$\infty 0$	12	B	57	38	C	46	
	0	1		0	1		7	
	$\infty 0$	23		57	40		24	
D	30		E	$\infty 7$	25	F	16	48
	4			3	1		0	1
	74			$\infty 1$	36		16	50
G	45	68	H	17		I	$\infty 0$	23
	1	2		1			3	5
	56	81		28			$\infty 3$	78

2. An indecomposable $KS_2(10; 1, 4)$ is generated by 16 9×9 arrays. The arrays are arranged as follows.

$$\begin{bmatrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ M & N & O & P \end{bmatrix}$$

A	$\infty 4$	12	B	35		C	67		D	80	
	0	1		1			2			2	
	$\infty 4$	23		46			80			12	
E	24		F	$\infty 1$	07	G	58		H	36	
	4			7	0		7			0	
	68			$\infty 8$	07		36			36	
I	48		J	01		K	$\infty 6$	25	L	37	
	1			2			5	8		2	
	50			23			$\infty 2$	14		50	
M	36		N	15		O	07		P	$\infty 2$	48
	4			0			7			5	0
	71			15			57			$\infty 7$	48

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