Cordial labellings of the cartesian product and composition of graphs

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Abstract. Let G be a graph. A labelling $f : V(G) \to \{0,1\}$ is called a binary labelling of G. A binary labelling f of G induces an edge labelling λ of G as follows: $\lambda(u,v) = |f(u) - f(v)|$ for every edge $uv \in E(G)$. Let $v_f(0)$ and $v_f(1)$ be the number of vertices of G labelled with 0 and 1 under f, and $e_f(0)$ and $e_f(1)$ be the number of edges labelled with 0 and 1 under λ, respectively. Then the binary labelling f of G is said to be cordial if

$$|v_f(0) - v_f(1)| \leq 1 \quad \text{and} \quad |e_f(0) - e_f(1)| \leq 1.$$

A graph G is cordial if it admits a cordial labelling.

In this paper we shall give a sufficient condition for the Cartesian product $G \times H$ of two graphs G and H to be cordial. The Cartesian product of two cordial graphs of even sizes is then shown to be cordial. We show that the Cartesian products $P_n \times P_n$ for all $n \geq 2$ and $P_n \times C_{4m}$ for all m and all odd n are cordial. The Cartesian product of two even trees of equal order such that one of them has a 2-tail is shown to be cordial. We shall also prove that the composition $C_n[K_2]$ for $n \geq 4$ is coridal if and only if $n \not\equiv 2 \pmod 4$. The cordiality of compositions involving trees, unicyclic graphs and some other graphs are also investigated.

1. Introduction

In this paper all graphs are finite, simple and undirected. Let $V(G)$ and $E(G)$ be the vertex set and edge set of a graph G. Let P_n, C_n, and K_n denote a path, a cycle and a complete graph of order n respectively. We write $P_n = (v_1, v_2, \cdots, v_n)$ to indicate $V(P_n) = \{v_1, v_2, \cdots, v_n\}$, where $v_i v_{i+1} \in E(P_n)$, $i = 1, 2, \cdots, n-1$, and write $C_n = [v_1, v_2, \cdots, v_n]$ to indicate $V(C_n) = \{v_1, v_2, \cdots, v_n\}$, where $v_n v_1, v_i v_{i+1} \in E(C_n)$, $i = 1, 2, \cdots, n-1$.

Let G be a graph. A mapping $f : V(G) \to \{0,1\}$ is called a binary labelling of G. Let f be a binary labelling of a graph G. Then for each $v \in V(G)$, $f(v)$ is called the (vertex) label of the vertex v under f, and for each edge $x = uv$, the load (or label) on x under f is given by

$$\lambda(u,v) = |f(u) - f(v)|.$$

The number of vertices (resp. edges) of G labelled with 0 and 1 under f will be denoted by $v_f(0)$ (resp. $e_f(0)$) and $v_f(1)$ (resp. $e_f(1)$) respectively. Let f be a binary labelling of a graph G. Then the mapping f' such that for each $v \in V(G)$,

$$f'(v) = \begin{cases} 0, & \text{if } f(v) = 1, \\ 1, & \text{if } f(v) = 0, \end{cases}$$

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is called the dual labelling of f. Note that $e_f(0) = e_{f'}(0)$, $e_f(1) = e_{f'}(1)$, and $v_f(0) < v_f(1)$ if and only if $v_{f'}(0) > v_{f'}(1)$. A binary labelling f of a graph G is said to be cordial if

$$|v_f(0) - v_f(1)| \leq 1 \quad \text{and} \quad |e_f(0) - e_f(1)| \leq 1.$$

A graph G is cordial if it admits a cordial labelling. Note that if f is a cordial labelling of a graph G, so is the dual labelling f' of f.

Cordial graphs are first introduced by I. Cahit as a weaker version of both graceful graphs and harmonious graphs [2]. Cahit proved the following [2].

Theorem A.

(i) *Every tree is cordial.*
(ii) *The complete graph K_n is cordial if and only if $n \leq 3$.*
(iii) *If G is a eulerian graph with $|E(G)| \equiv 2 \pmod 4$, then G is not cordial*
(iv) *The cycle C_n is cordial if and only if $n \not\equiv 2 \pmod 4$.*

In this paper we shall give a sufficient condition for the Cartesian product $G \times H$ of two graphs G and H to be cordial. The Cartesian product of two cordial graphs of even sizes is then shown to be cordial. We show that the Cartesian products $P_n \times P_n$ for all $n \geq 2$ and $P_n \times C_{4m}$ for all m and all odd n are cordial. The Cartesian product of two even trees of equal order such that one of them has a 2-tail is shown to be cordial. We shall also prove that the composition $C_n[K_2]$ for $n \geq 4$ is cordial if and only if $n \not\equiv 2 \pmod 4$. Some sufficient conditions for the composition of two graphs to be cordial are included. The cordiality of compositions involving trees, unicyclic graphs and some other graphs are also investigated.

2. Cartesian product

The Cartesian product of the two graphs G and H is the graph $G \times H$ with vertex set $V(G) \times V(H)$, in which (u, v) is adjacent to (u', v') if and only if either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$. The following example illustrates that the Cartesian product of two cordial graphs is not always cordial.

Example 1: Let G be the star in Figure 1(a) and H be a P_2. Then the Cartesian product $G \times H$, shown in Figure 1(c) is not cordial, because it is eulerian and $|E(G \times H)| = 10 \equiv 2 \pmod 4$.

Theorem 1. *Let G and H be two cordial graphs. Let f and g be cordial labellings of G and H respectively. If any one of the following conditions holds, then the Cartesian product $G \times H$ is cordial:*

(i) $e_f(0) = e_f(1)$ *and* $e_g(0) = e_g(1)$;

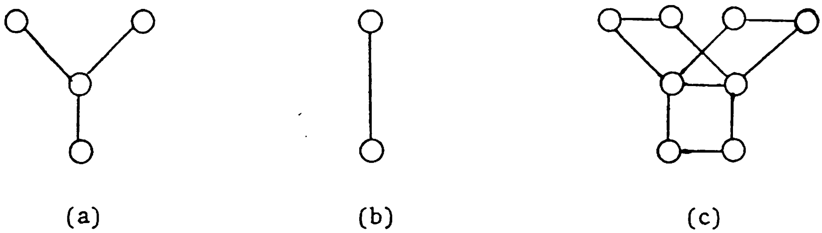


Figure 1

(ii) $e_f(0) < e_f(1), e_g(0) > e_g(1)$ *and* $||V(G)| - |V(H)|| \leq 1$;
(iii) $e_f(0) > e_f(1), e_g(0) < e_g(1)$ *and* $||V(G)| - |V(H)|| \leq 1$.

Proof: Define a binary labelling f^* of $G^* = G \times H$ as follows: for each $u = (u_1, u_2) \in V(G^*)$,

$$f^*(u) = \begin{cases} 0, & \text{if } f(u_1) = g(u_2), \\ 1, & \text{if } f(u_1) \neq g(u_2). \end{cases}$$

We find

$$\begin{aligned} v_{f^*}(0) &= v_f(0) \times v_g(0) + v_f(1) \times v_g(1), \\ v_{f^*}(1) &= v_f(0) \times v_g(1) + v_f(1) \times v_g(0), \\ v_{f^*}(0) + v_{f^*}(1) &= (v_f(0) + v_f(1))(v_g(0) + v_g(1)) = |V(G^*)|, \end{aligned}$$

and

$$|v_{f^*}(0) - v_{f^*}(1)| = |(v_f(0) - v_f(1))(v_g(0) - v_g(1))| \leq 1.$$

Also we find

$$\begin{aligned} e_{f^*}(0) &= e_f(0) \times |V(H)| + e_g(0) \times |V(G)|, \\ e_{f^*}(1) &= e_f(1) \times |V(H)| + e_g(1) \times |V(G)|, \\ e_{f^*}(0) + e_{f^*}(1) &= (e_f(0) + e_f(1))|V(H)| + (e_g(0) + e_g(1))|V(G)| \\ &= |E(G)| \times |V(H)| + |E(H)| \times |V(G)| \\ &= |E(G \times H)|, \end{aligned}$$

and

$$|e_{f^*}(0) - e_{f^*}(1)| = |(e_f(0) - e_f(1))|V(H)| + (e_g(0) - e_g(1))|V(G)||.$$

Case (i): $e_f(0) = e_f(1)$ and $e_g(0) = e_g(1)$. In this case $e_{f^*}(0) - e_{f^*}(1) = 0$, and hence $G^* = G \times H$ is cordial.

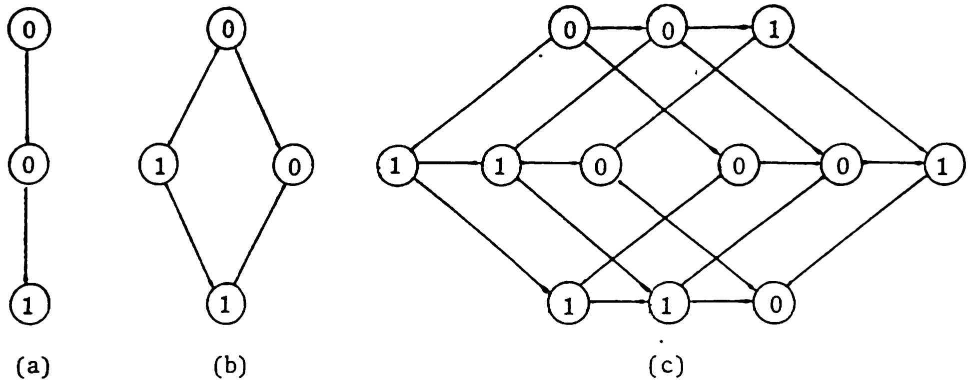


Figure 2

Case (ii): $e_f(0) < e_f(1)$, $e_g(0) > e_g(1)$ and $||V(H)| - |V(G)|| \leq 1$. We have $|e_{f^*}(0) - e_{f^*}(1)| = ||V(H)| - |V(G)|| \leq 1$. It follows that $G^* = G \times H$ is cordial.

The result for Case (iii) follows from the fact that $G \times H$ is isomorphic to $H \times G$. ∎

Example 2: A cordial labelling of the Cartesian product $P_3 \times C_4$, suggested in the proof of Theorem 1 is shown in Figure 2(c).

Corollary 1. *The Cartesian product of two cordial graphs of even sizes is cordial.*

Corollary 2. *The Cartesian product $P_n \times C_{4m}$ is cordial for all m and for all odd n.*

Two cordial labellings f and g of a graph G are said to be complementary if $e_f(0) < e_f(1)$ and $e_g(0) > e_g(1)$. If a graph G has 2 complementary cordial labellings, then G is necessarily of odd order.

Corollary 3. *The Cartesian product $P_n \times P_n$ is cordial for all $n \geq 2$.*

Proof: If n is odd, then P_n is of even size and hence, by Corollary 1, $P_n \times P_n$ is cordial.

Let n be even. To prove that $P_n \times P_n$ is cordial, it is sufficient to show that there exist two complementary cordial labellings of P_n. The cases $n = 4m$ and $n = 4m + 2$ will be considered seperately.

Case (i): $n = 4m$. Define binary labellings f and g of $P_n = (v_1, v_2, \cdots, v_n)$ as follows:

$$f(v) = \begin{cases} 0, & v = v_{2p+1}, \\ 0, & v = v_{2p+2}, \\ 1, & v = v_{2(p+1)+1}, \\ 1, & v = v_{2(p+1)+2}, \quad p = 2\ell, \ell = 0, 1, 2, \cdots, m-1; \end{cases}$$

$$g(v) = \begin{cases} f(v), & v = v_i, \quad i = 1, 2, \cdots, 4m-4, \\ 0, & v = v_{4m-3}, \\ 1, & v = v_{4m-2}, \\ 1, & v = v_{4m-1}, \\ 0, & v = v_{4m}. \end{cases}$$

It is easily checked that f and g are cordial labellings of P_n with

$$e_f(0) = 2m, \quad e_f(1) = 2m-1, \quad e_g(0) = 2m-1, \quad \text{and } e_g(1) = 2m,$$

and hence f and g are complementary cordial labellings of P_n.

Case (ii): $n = 4m+2$. Define binary labellings f_1 and g_1 of $P_n = (v_1, v_2, \cdots, v_{4m+2})$ as follows: Let $P_{4m} = P_n - \{v_{4m+1}, v_{4m+2}\}$,

$$f_1(v) = \begin{cases} f(v), & v \in V(P_{4m}), \\ 1, & v = v_{4m+1}, \\ 0, & v = v_{4m+2}; \end{cases}$$

$$g_1(v) = \begin{cases} g(v), & v \in V(P_{4m}), \\ 0, & v = v_{4m+1}, \\ 1, & v = v_{4m+2}, \end{cases}$$

where f and g are the cordial labellings of P_{4m} in Case (i). Then f_1 and g_1 are complementary cordial labellings of P_n. ∎

A tree is said to be odd or even according to its order is odd or even. Two adjacent vertices u and w of a graph G is said to form a 2-tail of G if $\deg(u) = 2$ and $\deg(w) = 1$. This 2-tail will be denoted by $\langle u, w \rangle$. Any vertex of degree one is called a terminal vertex.

As all trees are cordial and odd trees are of even sizes, we have by Corollary 1 the following:

Theorem 2. *The Cartesian product $T_1 \times T_2$ of two odd trees T_1 and T_2 are cordial.*

Remark 1: The Cartesian product of even trees is not always cordial as illustrated by Example 1.

We shall give a sufficient condition for the Cartesian product $T_1 \times T_2$ of two even trees T_1 and T_2 to be cordial.

Lemma 1. *For any odd tree T and any terminal vertex u of T, there exist two cordial labellings f and g of T such that $f(u) = g(u)$, $v_f(0) < v_f(1)$ and $v_g(0) > v_g(1)$.*

Proof: By induction of $|V(T)|$. Suppose first $|V(T)| = 3$. Then T must be a $P_3 = (v_1, v_2, v_3)$. The two cordial labellings f and g of T with the stated property are as follows:

$$f(v) = \begin{cases} 0, & v = v_1, \\ 1, & v = v_2, \\ 1, & v = v_3; \end{cases}$$

$$g(v) = \begin{cases} 0, & v = v_1, \\ 0, & v = v_2, \\ 1, & v = v_3. \end{cases}$$

Assume that the Lemma is true for all odd trees T of order $2k + 1$, $k \geq 1$, and for any terminal vertex u of T. Let T^* be a tree of order $2k+3$ and u^* be a terminal vertex of T^*. Since $G = T^* - \{u^*\}$ is a tree, there must exist either two terminal vertices, say u_1 and u_2, with a common adjacent vertex, or a 2-tail $\langle u_1, u_2 \rangle$. The tree $H = T^* - \{u_1, u_2\}$ is of order $2k + 1$. By induction hypothesis, there exist cordial labellings f and g of H such that $f(u^*) = g(u^*)$, $v_f(0) < v_f(1)$ and $v_g(0) > v_g(1)$. We now define binary labellings f^* and g^* of T^* by putting

$$f^*(v) = f(v) \quad \text{and} \quad g^*(v) = g(v) \quad \text{for every } v \in V(H);$$

and the images of u_1 and u_2 under f^* and g^* will be determined as follows:

Case 1: u_1 and u_2 are terminal vertices with a common adjacent vertex. In this case we put

$$f^*(u_1) = g^*(u_1) = 0 \quad \text{and} \quad f^*(u_2) = g^*(u_2) = 1.$$

Case 2: $\langle u_1, u_2 \rangle$ is a 2-tail. Let u_3 be the other vertex adjacent to u_1. We put

$$f^*(u_1) = f(u_3), \quad g^*(u_1) = g(u_3)$$
$$f^*(u_2) \neq f^*(u_1) \quad \text{and} \quad g^*(u_2) \neq g(u_1).$$

Then it is easily verified that f^* and g^* are cordial labellings of T^* such that

$$f^*(u^*) = g^*(u^*), \quad v_{f^*}(0) < v_{f^*}(1) \quad \text{and} \quad v_{g^*}(0) > v_{g^*}(1).$$

Remark 2: In Lemma 1 if $f(u) = g(u) = 0$, then by interchanging 0 and 1 in the labels of the vertices under f and g respectively, we have

$$f(u) = g(u) = 1, \quad v_f(0) > v_f(1) \quad \text{and} \quad v_g(0) < v_g(1).$$

Lemma 2. *Every even tree T with a 2-tail has 2 complementary cordial labellings.*

Proof: Let $\langle u, w\rangle$ be a 2-tail of T. Since the tree $T_1 = T - \{w\}$ is of odd order and u is a terminal vertex of T_1, by Lemma 1 and Remark 2, there exist cordial labellings f and g of T_1 such that

$$f(u) = g(u) = 1, \quad v_f(0) < v_f(1) \quad \text{and} \quad v_g(0) > v_g(1).$$

Define binary labellings f^* and g^* of T as follows:

$$f^*(v) = f(v) \quad \text{and} \quad g^*(v) = g(v), \quad \text{for every } v \in V(T_1),$$

and $f^*(w) = 0$ and $g^*(w) = 1$. Then f^* and g^* are complementary cordial labellings of T^*.

Theorem 3. *Let T_1 and T_2 be even trees of equal order such that one of them has a 2-tail. Then the Cartesian product $T_1 \times T_2$ is cordial.*

Proof: It follows from Lemma 2 and Theorem 1.

Corollary 4. *If T is an even tree with a 2-tail, then the Cartesian product $T \times T$ is cordial.*

A connected graph with exactly one cycle is called a unicyclic graph.

Theorem 4. *Let G and H be two unicyclic graphs which are not C_{4k+2} for all $k \geq 1$. If either*

(i) *both G and H are of even orders, or*

(ii) *both G and H are of odd but equal orders such that one of them has a 2-tail,*

then the Cartesian product $G \times H$ is cordial.

Proof: It is proved in [3] that a unicyclic graph G is cordial if and only if $G \not\cong C_{4k+2}$ for all $k \geq 1$. If condition (i) holds, then by Corollary 1, the Cartesian product $G \times H$ is cordial.

Assume that condition (ii) holds and G has a 2-tail. In view of Theorem 1(ii) or 1(iii), we need only to show that G has 2 complementary cordial labellings. Let $\langle u, w\rangle$ be a 2-tail of G. Then the unicyclic graph $G_1 = G - \{w\}$ is of even order. Let h be a cordial labelling of G_1. Assume $h(u) = 0$. Define two binary labellings f and g of G as follows:

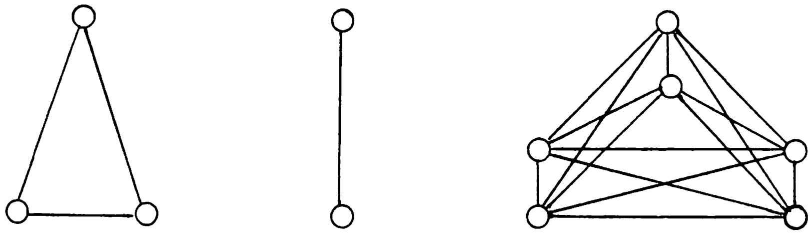
$$f(v) = \begin{cases} h(v), & v \in V(G_1), \\ 1, & v = w; \end{cases}$$

$$g(v) = \begin{cases} h(v), & v \in V(G_1), \\ 0, & v = w. \end{cases}$$

Then f and g are 2 complementary cordial labellings of G. ∎

3. Composition of graphs

Let G and H be two graphs. The composition of G with H, denoted by $G[H]$, is the graph with vertex set $V(G) \times V(H)$ in which (u_1, v_1) is adjacent to (u_2, v_2) if and only if $u_1 u_2 \in E(G)$ or $u_1 = u_2$ and $v_1 v_2 \in E(H)$. For example, $C_3[K_2]$ is shown in Figure 3(c).



(a) (b) (c)

Figure 3

As $C_3[K_2]$ is a K_6, by Theorem A(ii) it is not cordial. But we have

Theorem 5. *For $n \geq 4$, $C_n[K_2]$ is cordial if and only if $n \not\equiv 2 \pmod 4$.*

Proof: Let $C_n = [v_1, v_2, \cdots, v_n]$ and $V(K_2) = \{u_1, u_2\}$. The cases (i) $n = 4m$, (ii) $n = 4m+1$, (iii) $n = 4m+3$, and (iv) $n = 4m+2$ will be handled seperately.

Case (i): $n = 4m$. Define a binary labelling f_1 of $G_1 = C_{4m}[K_2]$ as follows:

$$f(v,u) = \begin{cases} 0, & v = v_{4p+1}, \quad u = u_1, \\ 0, & v = v_{4p+1}, \quad u = u_2, \\ 1, & v = v_{4p+2}, \quad u = u_1, \\ 0, & v = v_{4p+2}, \quad u = u_2, \\ 1, & v = v_{4p+3}, \quad u = u_1, \\ 1, & v = v_{4p+3}, \quad u = u_2, \\ 1, & v = v_{4(p+1)}, \quad u = u_1, \\ 0, & v = v_{4(p+1)}, \quad u = u_2, \quad p = 0, 1, \cdots, m-1. \end{cases}$$

It is not difficult to verify that f_1 is a cordial labelling of G_1.

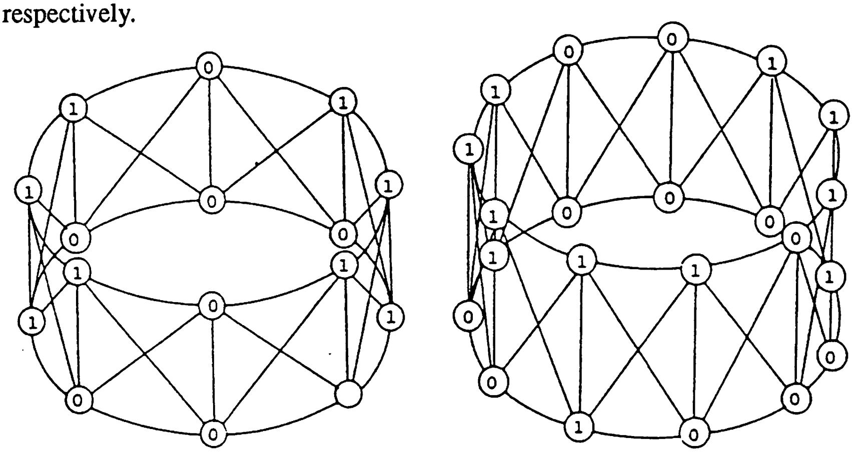
Case (ii): The binary labelling f_2 of $G_2 = C_{4m+1}[K_2]$ defined below is a cordial labelling of G_2:

$$f_2(v,u) = \begin{cases} f_1(v,u), & (v,u) \in V(C_{4m}) \times V(K_2), \\ 1, & v = v_{4m+1}, \quad u = u_1, \\ 0, & v = v_{4m+1}, \quad u = u_2, \end{cases}$$

Case (iii): $n = 4m + 3$. The following binary labelling f_3 of $G_3 = C_{4m+3}[K_2]$ is cordial:

$$f_3(v,u) = \begin{cases} f_2(v,u), & (v,u) \in V(C_{4m+1}) \times V(K_2), \\ 1, & v = v_{4m+2}, \quad u = u_i, \\ 0, & v = v_{4m+3}, \quad u = u_i, \quad i = 1,2. \end{cases}$$

For example, Figure 4(a) and (b) show cordial labellings of $C_8(K_2)$ and $C_{11}[K_2]$ respectively.



(a) $C_8(K_2)$ (b) $C_{11}(K_2)$

Figure 4

Case (iv): $n = 4m + 2$. We observe that $G_4 = C_{4m+2}[K_2]$ is regular of degree 5 and has order $8m + 4$ and size $20m + 10$. If G_4 is cordial, then the joint $G^* = G_4 + \{v^*\}$ will be cordial; but since every vertex of G^* is even and $|E(G^*)| = (20m + 10) + (8m + 4) \equiv 2 \pmod 4$, by Theorem A(iii), G^* cannot be cordial. Hence G_4 cannot be cordial. This completes the proof. ∎

Remark 3: The compositions $K_2[K_2]$ and $C_3[K_2]$ are not cordial, because they are K_4 and K_6 respectively.

Theorem 6. *Let H be a cordial graph of even order such that*

(i) *H has even size, or*
(ii) *H has 2 complementary cordial labellings.*

Then the composition $G[H]$ is cordial for any graph G.

Proof: Let $|V(G)| = n$ and $m = \left[\frac{n}{2}\right]$, the greatest integer less than or equal to $\frac{n}{2}$. Let h be a cordial labelling of H.

Suppose condition (i) holds. Define a binary labelling f of $G[H]$ by setting

$$f(v,u) = h(u) \quad \text{for every } (v,u) \in V(G) \times V(H).$$

Then f can be shown to be a cordial labelling of $G[H]$.

Suppose condition (ii) holds. Let f and g be 2 complementary cordial labellings of H. Then the binary labelling f^* of $G[H]$ defined as follows:

$$f^*(v,u) = \begin{cases} f(u), & \text{for any } m \text{ of the } n \text{ vertices } v \text{ of } G, \\ g(u), & \text{for the remaining } n-m \text{ vertices } v \text{ of } G, \end{cases}$$

is found to be a cordial labelling of $G[H]$. ∎

Corollary 5. *The composition $G[T]$ is cordial for every graph G and every even tree T with a 2-tail.*

Corollary 6. *The composition $G[H]$ is cordial for every graph G and every unicyclic graph H of even order which is not a C_{4k+2}.*

A ladder L_n is the Cartesian product $P_2 \times P_n$. All ladders are cordial [2].

Corollary 7. *The composition $G[L_n]$ is cordial for all even n and for any graph G.*

Theorem 7. *If H is a cordial graph of odd order and even size, then the composition $G[H]$ is cordial for any cordial graph G.*

Proof: Let h be a cordial labelling of H and h' the dual labelling of h. Let g be a cordial labelling of G. Define a binary labelling f of $G[H]$ as follows:

$$f(v,u) = \begin{cases} h(u), & \text{if } g(v) = 0, \\ h'(u), & \text{if } g(v) = 1. \end{cases}$$

Then we find

$$v_f(0) = v_g(0) \times v_h(0) + v_g(1) \times v_{h'}(0),$$
$$v_f(1) = v_g(0) \times v_h(1) + v_g(1) \times v_{h'}(1),$$

and

$$|v_f(0) - v_f(1)| = |v_g(0)(v_h(0) - v_h(1)) + v_g(1)(v_{h'}(0) - v_{h'}(1))|$$
$$= |v_g(0) - v_g(1)| \leq 1.$$

Similarly we can show that

$$|e_f(0) - e_f(1)| = |e_g(0) - e_g(1)| \leq 1.$$

Hence f is a cordial labelling of $G[H]$. ∎

Corollary 8. *The composition $T[T^*]$ is cordial for any tree T and for any odd tree T^*.*

A wheel W_n is obtained by joining all vertices of a C_n to a new vertex. The wheel W_n is cordial if and only if $n \not\equiv 3 \pmod 4$ [2].

Corollary 9. *The composition $G[W_n]$ is cordial for all even n and for any cordial graph G.*

A friendship graph F_n consists of n triangles with a common vertex. F_n is cordial if and only if $n \not\equiv 2 \pmod 4$ [2].

Corollary 10. *The composition $G[F_n]$ is cordial for all even $n \not\equiv 2 \pmod 4$ and for any cordial graph G.*

The pinwheel $P_w(n)$ is obtained from the friendship graph F_n by identifying the outer edge of each triangle in F_n with an edge of a new triangle. All pinwheels are cordial [2].

Corollary 11. *The composition $G[P_w(n)]$ is cordial for every pinwheel of even order and for any cordial graph G.*

References

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