Improved bounds for the union-closed sets conjecture

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Abstract. It has been conjectured that for any union-closed set $\mathcal{A}$ there exists some element which is contained in at least half the sets in $\mathcal{A}$. This has recently been shown to hold if the smallest set in $\mathcal{A}$ has size one or two, and also to hold if the number of sets in $\mathcal{A}$ is less than eleven. It is shown that the smallest set size approach is unproductive for size three. It is also shown that the conjecture holds for other conditions on the sets in $\mathcal{A}$, and an improved bound is derived: the conjecture holds if the number of sets in $\mathcal{A}$ is less than 19.

1. Introduction

A union-closed set is defined as a non-empty finite collection of distinct non-empty finite sets, closed under union. The following conjecture is rephrased from [1]:

Conjecture. *Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be a union-closed set. Then there exists an element which belongs to at least $\lceil n/2 \rceil$ of the sets in $\mathcal{A}$, where*

$$\lceil n/2 \rceil = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (n+1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

2. Minimal size 3

The authors in [2] showed that if there exists a set of size 1 or 2 in $\mathcal{A}$, one of its elements occurs in at least half the sets of $\mathcal{A}$. However, for minimal size 3 it is possible to construct a union-closed set $\mathcal{A}$ such that no element of the set of minimal size occurs in half the sets in $\mathcal{A}$. This does not of course disprove the conjecture, but it does show that this approach is not immediately useful. Such a case is the following:

$$\mathcal{A} = \{A_1, \ldots, A_{27}\}, \text{ with}$$

$A_1 = \{1,2,3\}$
$A_2 = \{1,2,3,6,7,8,9\}$
$A_3 = \{1,2,3,4,6,7,8,9\}$
$A_4 = \{1,2,3,4,5,6,7,8,9\}$
$A_5 = \{1,2,3,4,5,8,9\}$
$A_6 = \{1,2,3,4,5,6,8,9\}$
$A_7 = \{1,2,3,4,5,6,7\}$
$A_8 = \{1,2,3,4,5,6,7,8\}$
$A_9 = \{6,7,8,9\}$
$A_{10} = \{4,6,7,8,9\}$
$A_{11} = \{4,5,6,7,8,9\}$
$A_{12} = \{4,5,8,9\}$
$A_{13} = \{4,5,6,8,9\}$
$A_{14} = \{4,5,6,7\}$
$A_{15} = \{4,5,6,7,8\}$
$A_{16} = \{1,6,7,8,9\}$
$A_{17} = \{1,4,6,7,8,9\}$
$A_{18} = \{1,4,5,6,7,8,9\}$
$A_{19} = \{2,4,5,8,9\}$
$A_{20} = \{2,4,5,6,8,9\}$
$A_{21} = \{2,4,5,6,7,8,9\}$
$A_{22} = \{3,4,5,6,7\}$
$A_{23} = \{3,4,5,6,7,8\}$
$A_{24} = \{3,4,5,6,7,8,9\}$
$A_{25} = \{1,2,4,5,6,7,8,9\}$
$A_{26} = \{1,3,4,5,6,7,8,9\}$
$A_{27} = \{2,3,4,5,6,7,8,9\}$

Here each of the elements of the minimal set $\{1,2,3\}$ occurs exactly 13 times in 27 distinct sets. (The conjecture still holds - for example, the element 4 occurs in 23 of the 27 sets.)

A more restricted result is still possible here:

Theorem 1. *The conjecture holds provided a minimal set of size three has non-null intersection with all other sets.*

Proof: Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be union-closed with $|A_1| = 3$ minimal such that $A_1 \cap A_i \neq \emptyset$, $i = 1, 2, \ldots, n$. Set $A_1 = \{1,2,3\}$. For $i = 1,2,3$ let x_i be the number of sets whose intersection with A_1 is $\{i\}$; let y_1, y_2, y_3 be the number of sets whose intersection with A_1 is $\{1,2\}, \{1,3\}, \{2,3\}$ respectively and let t be the number of sets which contain A_1.

Now $t + x_1 + x_2 + x_3 + y_1 + y_2 + y_3 = n$. If $t + x_1 + y_1 + y_2 \geq n/2$ then the conjecture holds: assume otherwise. Thus $x_2 + x_3 + y_3 > n/2$: we wish to show that $t \geq x_2$.

Since $t \geq 1$, we may assume $x_2 > 1$. Let A_2, A_3 be any sets whose intersection with A_1 is $\{2\}$, ordered such that $|A_2| \geq |A_3|$. Let $x \neq 2$ be an element of A_2 but not of A_3. This implies $A_1 \cup A_2 \neq A_1 \cup A_3$: hence $t \geq x_2$.

Thus $t + x_3 + y_3 > n/2$ and hence element 3 is in more than half the sets.

3. Further restrictions on set sizes

For $\mathcal{A} = \{A_1, \ldots, A_n\}$, set $w_i = |A_i|$, $i = 1, \ldots, n$. Assume $\mathcal{A}$ ordered such that $w_1 \leq w_2 \leq \ldots < w_n$.

Theorem 3 in [2] shows that the conjecture holds whenever $w_1 \geq w_n/2$. This can be improved by the following result:

Theorem 2. *The conjecture holds whenever* $w_1 + w_2 \geq w_n$.

Proof: Without loss of generality, assume $A_1 = \{1, \ldots, w_1\}, A_n = \{1, \ldots, w_n\}$. By Theorem 3 in [2] it is only necessary to consider the case where $w_1 < w_n/2$: create $\mathcal{A}' = \{A'_1, \ldots, A'_n\}$ by adjoining $w_n + 1, \ldots, 2(w_n - w_1)$ to the sets containing the element 1. Now $w'_1 = w_n - w_1$, $w'_n = 2(w_n - w_1)$, and hence the conjecture holds for $\mathcal{A}'$ (again by Theorem 3 in [2]) provided w'_1 is still minimal. Assuming this proviso, the conjecture must also hold for $\mathcal{A}$ since the additional elements occur exactly as many times as does the element 1 in $\mathcal{A}$.

The proviso will hold if $w_2 \geq w_n - w_1$: that is, if $w_1 + w_2 \geq w_n$.

Corollary. *If* $\cap_{i=1}^{k} A_i \neq \emptyset$ *the conjecture holds whenever* $w_1 + w_{k+1} \geq w_n$.

Proof: Without loss of generality, assume $1 \in \cap_{i=1}^{k} A_i$. Adjoin as above the elements $w_n + 1, \ldots, 2(w_n - w_1)$ to all sets containing 1. Now $w'_1 = w'_n/2$ is still minimal provided $w_n - w_1 \leq w_{k+1}$.

4. Improved bounds on n

Theorem 1 in [2] shows that the validity of the conjecture for odd n leads to its validity for $n+1$. In this section, assume $\mathcal{A} = \{A_1, \ldots, A_n\}$ for odd n, with $|A_1|$ minimal, $|A_n|$ maximum.

Let $\mathcal{B} = \{A_1 \cup A_i, i = 1, 2, \ldots, n\}$. Let b equal the number of distinct sets in $\mathcal{B}$: since A_n always occurs, $b \geq 1$. Let these sets be $A_{k_1}, \ldots, A_{k_b}$. Assume A_{k_j} arises r_j times, $j = 1, \ldots, b$.

If $b \geq (n-1)/2$ then the conjecture holds for $\mathcal{A}$, for then A_i is a subset of b distinct sets in $\mathcal{B}$ (and hence in $\mathcal{A}$) as well as of A_i itself, a total of at least $(n+1)/2$ sets in $\mathcal{A}$. It is thus only necessary to consider the cases $b = 1, \ldots, (n-3)/2$.

This technique allows us to improve the bound of 11 derived in [2] to 19, via the following theorems.

Theorem 3. *The conjecture holds for* $b \leq 5$.

Proof. Case $b = 1$: Here $A_1 \cup A_i = A_n$ for $i = 1, 2, \ldots, n$. Thus there exists at least one element common to each of the A_i (not in A_1), a total of $n-1$ sets.

Case $b = 2$: Now $A_1 \cup A_i \in \{A_{k_i}, A_n\}$, $i = 2, \ldots, n$ for $k_1 \neq 1$ or n. Since $A_{k_1} \subset A_n$, there exists at least one element (not in A_1) common to each of these A_i.

Case $b = 3$: Assume A_n occurs r_3 times. If either of r_1, r_2 is greater than or equal to $(n-1)/2$ then we are done, for this leads to $(n-1)/2$ sets with an element in common (not in A_1), this element also appearing in A_n.

Assume otherwise. Now $r_2 \leq (n-3)/2$, and hence

$$r_1 + r_3 \geq (n-1) - (n-3)/2 = (n+1)/2.$$

Again, each A_i giving A_{k_1} or A_n has an element in common: this element thus occurs in $(n+1)/2$ sets in $\mathcal{A}$.

Case $b = 4$: Let A_n arise r_4 times. If $r_j \geq (n-1)/2$ for $j = 1, 2, 3$ then the conjecture holds, since $r_4 \geq 1$. Assume not: then each of $r_4 + r_1 + r_2$, $r_4 + r_1 + r_3$, $r_4 + r_2 + r_3$ is greater than or equal to $(n+1)/2$.

For the conjecture to be not satisfied, it is necessary that each pairwise intersection of $A_{k_1}, A_{k_2}, A_{k_3}$ be exactly A_1 since otherwise there would exist an element common to the A_i leading to each of the pair and also common to those leading to A_n: at least $(n+1)/2$ such.

Consider the three pairwise unions of $A_{k_1}, A_{k_2}, A_{k_3}$. These cannot be any of $A_{k_1}, A_{k_2}, A_{k_3}$, nor can they be one of the A_i leading to these via $A_i \cup A_i$ (by the intersection property). Thus each must be an A_n or an A_i leading to A_n. But if two are A_n or two are A_i leading to A_n their pairwise intersection contains more than A_1, a contradiction.

Case $b = 5$: Let A_n arise r_5 times. Order $A_{k_1}, \ldots, A_{k_4}$ such that $r_1 \geq r_2 \geq r_3 \geq r_4$. Since $r_1 + r_2 + r_3 + r_4 + r_5 = n - 1$, $r_1 + r_2 + (r_5/2) \geq$

$(n-1)/2$ and hence since $r_5 \geq 1$, $r_1 + r_2 + r_5 \geq (n+1)/2$. Similarly, $r_1 + r_3 + r_5 \geq (n+1)/2$. By the same argument as in the previous case, for the conjecture to not hold this necessitates $A_{k_1} \cap A_{k_2} = A_1$, $A_{k_1} \cap A_{k_3} = A_1$.

Consider $A_{k_1} \cup A_{k_2}$, $A_{k_1} \cup A_{k_3}$. These cannot be equal, since this implies $A_{k_2} = A_{k_3}$. They cannot be any of $A_{k_1}, A_{k_2}, A_{k_3}$ or an A_i leading to these, by the intersection property. Thus each must be one of A_k, A_n, or one of the A_i leading to A_n. The last is impossible, since for the first union this implies

$$A_i \cup A_{k_1} \cup A_{k_2} = A_{k_1} \cup A_{k_2} = A_n$$

And similarly for the second.

Thus one must be A_{k_4}, and one A_n.

Consider the case

$$\left.\begin{array}{l} A_{k_1} \cup A_{k_2} = A_{k_4} \\ A_{k_1} \cup A_{k_3} = A_n \end{array}\right\} \text{ with } \left\{\begin{array}{l} A_{k_1} \cap A_{k_1} = A_1 \\ A_{k_1} \cap A_{k_3} = A_1 \end{array}\right.$$

This implies $A_{k_2} \subset A_{k_3}$, $A_{k_2} \subset A_{k_4}$. But then $r_2 + r_3 + r_4 + r_5$ set have an element in common (the A_i leading to $A_{k_2}, A_{k_3}, A_{k_4}$ and A_n). Now

$$r_1 \leq (n-3)/4 \text{ and } r_1 + r_2 + r_3 + r_4 + r_5 = n-1$$

implies $r_2 + r_3 + r_4 + r_4 \geq (n+1)/2$, and thus the conjecture holds.

A similar argument covers the second case, where $A_{k_1} \cup A_{k_2} = A_n$ and $A_{k_1} \cup A_{k_3} = A_{k_4}$.

Notice that this proves the validity of the conjecture to $n = 13$ and hence to $n = 14$ by Theorem 1 of [2].

Theorem 4. *The conjecture holds for $b \geq (n-5)/2$.*

Proof: In the preamble for this section, it was shown that the conjecture holds for $b \geq (n-1)/2$. It is thus only necessary to examine the following two cases:

Case $b = (n-3)/2$. Assume $r_1 \geq r_2 \geq \ldots \geq r_b$. A_1 is a subset of itself and of A_{k_1} to A_{k_b} : $(n-1)/2$ sets. For the conjecture to be invalid here it is then necessary that $A_1 \cap A_i = \emptyset$ for A_i not one of these. Suppose $A_1 \cup A_{i_1} = A_1 \cup A_{i_2} = A_{k_j}$ for some k_j, $A_1 \cap A_{i_1} = A_1 \cap A_{i_2} = \emptyset$. Then $A_{i_1} = A_{i_2}$: this implies $r_j \leq 2$. But $r_1 + \ldots + r_b = n-1$ then implies $2b \geq n-1$, a contradiction.

Case $b = (n-5)/2$. We seek an element occurring in $(n+1)/2$ sets. Already, A_i is a subset of itself and of A_{k_1} to A_{k_b}, a total of $(n-3)/2$ sets. If two more can be found with an element in common, also in A_1, then we are done.

$r_1 + \ldots + r_b = n-1$. Set $r = \max(r_j, j = 1, \ldots, b)$. Then $br \geq n-1$ implies that $r \geq 3$.

Suppose $r \geq 5$. Then the ordering in $\mathcal{A}$ can be rearranged such that $A_1 \cup A_2 = A_1 \cup A_3 = A_1 \cup A_4 = A_1 \cup A_5 = A_1 \cup A_{k_1} = A_{k_1}$, all of A_1, A_2, A_3, A_4, A_5,

A_{k_1} distinct. No two of A_2 to A_5 can have null intersection with A_1, since this would imply that they were equal: assume A_2 to A_4 have non-null intersection with A_1. Let x be an element common to A_2 and A_1: if we can find one more set containing x then we are done. Now $A_2 \cup A_3$, $A_2 \cup A_4$ contain x and hence must coincide with some set already containing x, and this can only be A_{k_1}. But now A_1 must be a subset of both these unions: A_1 cannot be a subset of A_2 since this would imply $A_2 = A_{k_1}$ and hence A_1, A_3 and A_4 have an element in common.

We may thus assume $r \leq 4$. Considering first the case $r = 4$, as above rearrange $\mathcal{A}$ such that

$$A_1 \cup A_2 = A_1 \cup A_3 = A_1 \cup A_4 = A_1 \cup A_{k_1} = A_{k_1}.$$

Again, no two of $A_1 \cap A_2$, $A_1 \cap A_3$, $A_1 \cap A_4$ can be null, and no two of these intersections can have elements in common without validating the hypothesis. Assume the first two intersections non-null: as above this implies $A_1 \subset A_2 \cup A_3$.

Suppose there exists another r_j-value of at least 3. Then for say A_5, A_6 we again have $A_1 \cap A_5$, $A_1 \cap A_6$ not both null: we thus have $(n+1)/2$ distinct sets with a common element. Hence a maximum r-value of 4 implies all others two or less: a simple counting argument on $r_1 + \ldots + r_b = n - 1$ shows this leads to a contradiction.

We may now assume these ordered such that $A_1 \cap A_2$, $A_1 \cap A_4$ are not null. But then $A_2 \cap A_4 \cap A_1$ must be null to invalidate the hypothesis, and also $A_2 \cup A_4$ must be A_{k_j} for some j: this implies A_1 must be a subset of $A_2 \cup A_4$. But this then implies no other r_j-values are of size 3, and again a counting argument shows this is impossible, leading as it does to a contradiction via $6 + 2(b-2) \geq n - 1$.

The conjecture is now valid for $b \leq 5$ and for $b \geq (n-5)/2$: thus up to $n = 18$ by Theorem 1 in [**2**].

References

1. *Report on the 1987 Australian Applied Mathematics Conference*, Australian Mathematical Society Gazette **14 no. 3** (1987), p. 63.
2. D.G. Sarvate and J-C. Renaud, *On the union-closed sets conjecture*, Ars Combinatoria. (in press).

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