Modified Group Divisible Designs

Ahmed Assaf
Department of Algebra, Combinatorics and Analysis
Auburn University

ABSTRACT. Let X be a finite set of order mn, and assume that the points of X are arranged in an array of size $m \times n$. The columns of the array will be called groups. In this paper we consider a new type of group divisible designs called modified group divisible designs in which each $\{x, y\} \subset X$ such that x and y are neither in the same group nor in the same row occurs λ times. This problem was motivated by the problem of resolvable group divisible designs with $k = 3$, $\lambda = 2$, [1], and other constructions of designs.

1. Introduction

1.1 Designs

Let X be a finite set of order v and $\beta = \{B_i : i \in I\}$ be a family of subsets B_i of X called blocks. The pair (X, β) is called a design.

1.2 Balanced incomplete block designs (BIBD)

Let $v \geq k \geq 2$ and λ be positive integers. A design (X, β) is called a balanced incomplete block design (BIBD) $B[k, \lambda, v]$ if

(i) $|X| = v$
(ii) the blocks are of size k
(iii) every $\{x, y\} \subset X$ is contained in exactly λ blocks.

We shall use the following

Theorem 1.1. *(Hanani [2]) Let λ and $v \geq 3$ be positive integers. Necessary and sufficient conditions for the existence of a BIBD $B[3, \lambda, v]$ are that $\lambda(v-1) \equiv 0 \pmod 2$ and $\lambda v(v-1) \equiv 0 \pmod 6$.*

Let a design (X, β) be given. A parallel class of blocks is a subfamily $P \subset \beta$ of pairwise disjoint blocks, the union of which equal X. A BIBD $B[k, \lambda, v]$ is called resolvable and denoted by $RB[k, \lambda, v]$ if its blocks can be partioned into parallel classes.

1.3 Modified group divisible designs

We shall consider designs of the form $(X, \mathcal{Y}, \mathcal{P})$ where X is a finite set of points, $\mathcal{Y}$ is a parallel class of subsets of X called groups and $\mathcal{P}$ is a family of subsets of X called blocks.

Let m, k, λ and v be positive integers. A design $(X, \mathcal{Y}, \mathcal{P})$ is called a modified group divisible design and denoted by $MGD[k, \lambda, m, v]$ if

(i) $|X| = v$
(ii) $|G_i| = m$ for every $G_i \in \mathcal{Y}$

(iii) $|B_j| = k$ for every $B_j \in \mathcal{P}$

(iv) $|G_i \cap B_j| \leq 1$ for every $G_i \in \mathcal{Y}$ and every $B_j \in \mathcal{P}$

(v) every $\{x, y\} \subset X$ such that x and y are neither in the same group nor in the same row is contained in exactly λ blocks of $\mathcal{P}$. (We may look at the points of X as the points of an array of size $m \times n$ and then the groups of $(X, \mathcal{Y}, \mathcal{P})$ are precisely the columns of A).

If in the definition of a modified group divisible design the condition (v) is changed as follows (v′) every $\{x, y\} \subset X$ such that x and y belong to distinct groups, is contained in exactly λ blocks then the design $(X, \mathcal{Y}, \mathcal{P})$ will be called a group divisible design.

1.4 Modified transversal designs

A modified group divisible design $MGD[k, \lambda, m, km]$ is called a modified transversal design and denoted by $MT[k, \lambda, m]$, and a group divisible design $GD[k, \lambda, m, km]$ is called a transversal design and denoted by $T[k, \lambda, m]$.

Lemma 1.1. *Let p and q be prime numbers and assume that $q \geq p$. Then there exists a $MGD[q, 1, q, pq]$.*

Proof: Let $X = Z_q \times Z_p$, then the required blocks are

$$< (0,0), (1,\alpha), (2,2\alpha), (3,3\alpha), \ldots, (q-1, \alpha(q-1)) > \quad \text{mod } (-,p) \;\; \alpha = 1, 2, \ldots, p-1.$$

Lemma 1.2. *If there exists a Latin square of size n such that $a_{ii} = i$, then there exists a $MT[3, 1, n]$.*

Proof: Let $X = Z_3 \times Z_n$. Then the required blocks are

$$< (1, i), (2, j), (3, a_{ij}) >, \; i \neq j.$$

Lemma 1.3. *There exists a $MT[3, 1, n]$ for every positive integer $n \neq 2$.*

Proof: It is well known that for every $n \neq 2$ or 6 there exist two orthogonal Latin squares $A = (a_{ij})$ and $B = (b_{ij})$ of order n (see for example Hanani [2]). Permute the elements of the two orthogonal Latin squares such that $a_{1j} = b_{1j} = j$, $1 \leq j \leq n$. It is clear that the two Latin squares are still orthogonal. Let $X = Z_3 \times Z_n$. Then the required blocks of $MT[3, 1, n]$ are $< (1, j), (2, a_{ij}), (3, b_{ij}) >$ $2 \leq i \leq n$, $1 \leq j \leq n$. To complete the proof of Lemma 1.3 we have to prove the existence of $MT[3, 1, 6]$. Apply Lemma 1.2; we only have to prove the existence of a Latin square of order 6 such that $a_{ii} = i$, and this is done below

1	3	4	6	2	5
4	2	5	1	6	3
6	1	3	5	4	2
2	5	6	4	3	1
3	6	1	2	5	4
5	4	2	3	1	6

2. Pairwise balanced designs

Let v and λ be positive integers and K a set of positive integers. A design (X, β) is a pairwise balanced design $B[K, \lambda, v]$ if

(i) $|X| = v$
(ii) $\{|B_i| : B_i \in \beta\} \subset K$
(iii) every $\{x, y\} \subset X$ is contained in exactly λ blocks of β.

The set of all integers n for which a $B[K, 1, n]$ exists will be denoted by $B(K, 1)$.

We shall use the following

Lemma 2.1. *Hanani [2]: (i) If* $n \equiv 0$ *or* $1 \pmod 3$ *then* $n \in B[\{3,4,6\}, 1]$ *(ii) for every* $n \geq 3$ $n \in B(K, 1)$ *where* $K = \{3,4,5,6,8\}$.

Assume there exists a $T[4, 1, n]$ and remove $(n - m)$ points from one of its groups (where $0 \leq m \leq n$) to get a $GD[\{3,4\}, 1, \{n, m\}, 3n + m]$. Call the underlying pointset of this design X, so that $|X| = 3n + m$, and construct $GD[3, 1, \{2n, 2m\}, 2(3n+m)]$ on $X \times I_2$ by taking for each group of the original design a new group $G \times I_2$ and for each block B the blocks of $GD[3, 1, 2, 2|B|]$ constructed on $B \times I_2$ in such a way that it has groups $\{b\} \times I_2$ for $b \in B$. The constructions of $GD[3, 1, 2, 6]$ and $GD[3, 1, 2, 8]$ are very easy, see Hanani [2]. The above discussion enables us to prove the following.

Lemma 2.2. *There exists a* $B[\{3, 5^*\}, 1, v]$ *for every* $v \equiv 5 \pmod 6$*, where* $*$ *means there is exactly one block of size 5.*

Proof: This lemma is a special case of a known result see [3]. The proof given here is completely different. We distinguish two cases.

Case 1: $v \equiv 17 \pmod{18}$, $v \geq 53$. Since $v \equiv 5 \pmod 6$ then $(v - 1)$ is even. On $(v-1)/2$ points construct $GD[\{3,4\}, 1, \{n, 5\}, 3n+5]$ by removing $(n - 5)$ points from one group of a $T[4, 1, n]$. Simple calculations show that $n \equiv 1 \pmod 3$. Construct a $GD[3, 1, \{2n, 10\}, 2(3n+5)]$ on $X \times I_2$ as above. Now to construct a $B[\{3, 5^*\}, 1, v]$ add an extra point ∞ to the groups of the $GD[3, 1, \{2n, 10\}, 2(3n + 5)]$, and on the groups of size $2n$ with the extra point ∞ construct a $B[3, 1, 2n+1]$ (Note that $2n+1 \equiv 3 \pmod 6$), and on 11 points construct a $B[\{3, 5^*\}, 1, 11]$. (We shall see later that a $B[\{3, 5^*\}, 1, 11]$ exists). The blocks of the $GD[3, 1, \{2n, 10\}, 2(3n+5)]$ and the blocks of the designs constructed on the groups of the $GD[3, 1, \{2n, 10\}, 2(3n+5)]$ with ∞ are the blocks of the $B[\{3, 5^*\}, 1, v]$.

Case 2: $v \equiv 5$ or $11 \pmod{18}$, $v \neq 41$. In this cae on $(v-1)/2$ points construct a $GD[\{3,4\}, 1, \{n, 2\}, 3n+2]$ by removing $(n-2)$ points from one group of a $T[4, 1, n]$ $n \neq 6$, and the proof of case 2 now is the same as case 1. To complete the proof of Lemma 2.2 we have to prove the existence of a $B[\{3,5^*\},1,v]$ for $v = 11,17,35,41$.

To construct a $B[\{3,5^*\},1,11]$ let $X = Z_6 \cup \{\infty_i\}$, $i = 1,2,\dots,5$, then the required blocks are

$$
\begin{array}{lll}
<0,1,\infty_1> & <0,2,\infty_2> & <0,3,\infty_3> \\
<2,3,\infty_1> & <1,4,\infty_2> & <1,5,\infty_3> \\
<4,5,\infty_1> & <3,5,\infty_2> & <2,4,\infty_3> \\
<0,4,\infty_4> & <1,3,\infty_4> & <2,5,\infty_4> \\
<0,5,\infty_5> & <1,2,\infty_5> & <3,4,\infty_5> \\
<\infty_1,\infty_2,\infty_3,\infty_4,\infty_5> & &
\end{array}
$$

For $B[\{3,5^*\},1,17]$ see Hanani [2, p. 363].

For $B[\{3,5^*\},1,35]$ let $X = Z_{18} \cup \{\infty_i\}$, $i = 1,2,\dots,17$. On Z_{18} construct a $RB[2,1,18]$. There are precisely 17 parallel classes, to each parallel class add a point ∞_i and then construct a $B[\{3,5^*\},1,17]$ on the 17 points ∞_i, $i = 1,2,\dots,17$.

For $B[\{3,5^*\},1,41]$ take $T[3,1,12]$ and add five points to the three groups and construct a $B[\{3,5^*\},1,17]$ on each group with the 5 points.

3. Constructions

It is clear that the necessary conditions for the existence of modified group divisible designs are different from those of group divisible designs. The following lemma will state the necessary conditions for the existence of modified group divisible designs.

Lemma 3.1. *Let m, λ, v and k be positive integers. The necessary conditions for the existence of modified group divisible designs are that $v \equiv 0 \pmod{m}$, $v \geq km$, $m \geq k$, $\lambda(v+1-m-n) \equiv 0 \pmod{(k-1)}$ and $\lambda v(v+1-m-n) \equiv 0 \pmod{k(k-1)}$, where $n = v/m$.*

Proof: $v \equiv 0 \pmod{m}$, $v \geq km$ and $m \geq k$ follow from the definition of modified group divisible designs. Further, $\lambda(v+1-m-n)/(k-1)$ is the replication number of every point and $\lambda v(v+1-m-n)/k(k-1)$ is the total number of blocks.

The conditions of Lemma 3.1 are not sufficient for the existence of an $MGD[k, \lambda, m, v]$. For example an $MGD[4,1,6,24]$ does not exist because if an $MGD[4, 1,6,24]$ did exist, then the rows and the blocks of the $MGD[4,1,6,24]$ are the blocks of a $GD[4,1,6,24]$. But it is well known that a $GD[4,1,6,24]$ does not exist [3].

The main purpose of this paper is to prove the following.

Theorem 3.1. *Let m, λ and v be positive integers. The necessary and sufficient conditions for the existence of a modified group divisible design $MGD[3, \lambda, m, v]$ are that*

$$v \equiv 0 \pmod m, \; v \geq 3m, \; m \geq 3, \; \lambda(v + 1 - m - n) \equiv 0 \pmod 2$$

and $\lambda v(v + 1 - m - n) \equiv 0 \pmod 6$.

It is clear that to prove Theorem 3.1 we need only to handle the cases $\lambda = 1, 2, 3, 6$.

First we need the following lemmas.

Lemma 3.2. *If there exist a $B[k, \lambda, n]$ and a $MT[k, 1, m]$ then there exists a $MGD[k, \lambda, m, mn]$.*

Proof: On n groups of size m construct $B[k, \lambda, n]$ and then on each block, where the points of the blocks are groups of size m, construct an $MT[k, 1, m]$.

The above lemma can be generalized as follows.

Lemma 3.3. *If there exists a pairwise balanced design $B[K, \lambda, n]$ and if for every $k \in K$ there exists an $MGD[r, 1, m, km]$ then there exists an $MGD[r, \lambda, m, mn]$.*

Proof: The proof of this lemma is similar to the proof of the previous lemma.

Lemma 3.4. *There exists an $MGD[3, 1, m, 4m]$ for every m odd.*

Proof: That m is odd follows from the necessary conditions. To prove this lemma we distinguish two cases.

Case 1: $m \equiv 1$ or $3 \pmod 6$. In this case there exists a $B[3, 1, m]$ by Theorem 1.1 and there exists a $MT[3, 1, 4]$ by lemma 1.3. From lemma 3.2, it follows that there exists an $MGD[3, 1, m, 4m]$ for $m \equiv 1$ or $3 \pmod 6$.

Case 2: $m \equiv 5 \pmod 6$. In this case there exists a $B[\{3, 5^*\}, 1, m]$. From Lemma 3.3, we only have to prove the existence of an $MGD[3, 1, 4, 12]$ and an $MGD[3, 1, 4, 20]$. But an $MGD[3, 1, 4, 12]$ is an $MT[3, 1, 4]$ and this exists by Lemma 1.3.

To construct an $MGD[3, 1, 4, 20]$, let $X = Z_5 \times Z_4$. Then the required blocks are

$$< (0,0),(1,3),(2,2) > \ (mod(-,4))$$
$$< (0,0),(1,2),(3,1) > \ (mod(-,4))$$
$$< (0,0),(1,1),(4,3) > \ (mod(-,4))$$
$$< (1,0),(2,2),(3,1) > \ (mod(-,4))$$
$$< (1,0),(2,3),(4,1) > \ (mod(-,4))$$
$$< (0,0),(2,1),(3,2) > \ (mod(-,4))$$
$$< (0,0),(2,3),(4,2) > \ (mod(-,4))$$
$$< (0,0),(3,3),(4,1) > \ (mod(-,4))$$
$$< (1,0),(3,2),(4,3) > \ (mod(-,4))$$
$$< (2,0),(3,2),(4,1) > \ (mod(-,4))$$

Lemma 3.5. *There exists a* $MGD[3,1,m,5m]$ *for every* $m \equiv 0$ *or* 1 (mod 3).

Proof: That $m \equiv 0$ or 1 (mod 3) follows from the necessary conditions. To prove this lemma we distinguish two cases.

Case 1: $m \equiv 1$ or 3 (mod 6). In this case apply Lemma 3.2 and then the proof of this case is the same as case 1 of Lemma 3.4.

Case 2: $m \equiv 0$ or 4 (mod 6). By Lemma 2.1 there exists a $B[\{3,4,6\},1,m]$. Now apply Lemma 3.3. According to this lemma we only have to prove the existence of an $MGD[3,1,5,15]$, an $MGD[3,1,5,20]$ and an $MGD[3,1,5,30]$. But an $MGD[3,1,5,15]$ is the same as a $MT[3,1,5]$, which exists by Lemma 1.3, and an $MGD[3,1,5,20]$ is the same as an $MGD[3,1,4,20]$ and this design exists by Lemma 3.4. For an $MGD[3,1,5,30]$, let $X = Z_5 \times Z_5 \cup \{\infty_i\}$ $i = 0,1,\ldots,4$. Then the required blocks are

$<(0,0),(1,1),(2,3)>$ (mod $(-,5)$)	$<(0,0),(2,1),(4,2)>$ (mod $(-,5)$)
$<(1,0),(2,3),(3,2)>$ (mod $(-,5)$)	$<(0,0),(1,3),(3,1)>$ (mod $(-,5)$)
$<(2,0),(3,1),(4,4)>$ (mod $(-,5)$)	$<(1,0),(2,1),(4,3)>$ (mod $(-,5)$)
$<(0,0),(3,2),(4,1)>$ (mod $(-,5)$)	$<(0,0),(2,2),(3,4)>$ (mod $(-,5)$)
$<(0,0),(1,2),(4,3)>$ (mod $(-,5)$)	$<(1,0),(3,1),(4,2)>$ (mod $(-,5)$)
$<(0,0),(1,4),\ \infty_4>$ (mod $(-,5)$)	$<(0,0),(2,4),\ \infty_3>$ (mod $(-,5)$)
$<(2,0),(3,3),\ \infty_4>$ (mod $(-,5)$)	$<(1,0),(4,4),\ \infty_3>$ (mod $(-,5)$)
$<(0,0),(3,3),\ \infty_1>$ (mod $(-,5)$)	$<(0,0),(4,4),\ \infty_2>$ (mod $(-,5)$)
$<(2,0),(4,3),\ \infty_1>$ (mod $(-,5)$)	$<(1,0),(3,4),\ \infty_2>$ (mod $(-,5)$)
$<(1,0),(2,4),\ \infty_0>$ (mod $(-,5)$)	
$<(3,0),(4,2),\ \infty_0>$ (mod $(-,5)$)	

Lemma 3.6. *There exists an* $MGD[3,1,m,6m]$ *for every* m *odd.*

Proof: That m is odd follows from the necessary conditions. Again to prove this lemma we distinguish two cases.

Case 1: $m \equiv 1$ or $3 \pmod 6$. The proof of this case is the same as case 1 of Lemma 3.4.

Case 2: $m \equiv 5 \pmod 6$. By Lemma 2.2 there exists a $B[\{3,5^*\},1,m]$. From Lemma 3.3, we only have to prove the existence of $MGD[3,1,6,30]$. But an $MGD[3,1,6,30]$ is the same as $MGD[3,1,5,30]$, and the construction of an $MGD[3,1,5,30]$ was given in Lemma 3.5.

Now we are able to prove the following

Theorem 3.2. *The necessary conditions for the existence of an* $MGD[k,\lambda,m,$ $mn]$ *are sufficient in the case* $\lambda = 1$ *and* $k = 3$.

Proof: We distinguish four cases.

Case 1: $n \equiv 1$ or $3 \pmod 6$. Apply Lemma 3.2. We have to prove the existence of a $B[3,1,n]$ and an $MT[3,1,m]$. But a $B[3,1,n]$ exists by Theorem 1.1 and an $MT[3,1,m]$ exists by Lemma 1.3.

Case 2: $n \equiv 0$ or $4 \pmod 6$. In this case m should be odd. By Lemma 2.1 there exists a $B[\{3,4,6\},1,n]$. From Lemma 3.3, we have to prove the existence of an $MGD[3,1,m,3m]$, an $MGD[3,1,m,4m]$ and an $MGD[3,1,m,6m]$. But an $MGD[3,1,m,3m]$ is an $MT[3,1,m]$ and the other designs were shown to exist in Lemmas 3.4 and 3.6 respectively.

Case 3: $n \equiv 2 \pmod 6$. In this case m should be congruent to 1 or 3 modulo 6 ($m \equiv 1$ or $3 \pmod 6$), so there exists a $B[3,1,m]$ by Theorem 1.1 and a $MT[3,1,n]$ also exists for every n by Lemma 1.3. Now from Lemma 3.2, it follows that there exists an $MGD[3,1,m,mn]$ for $m \equiv 1$ or $3 \pmod 6$. Notice that in this case instead of "breaking" n into a pairwise balanced design, we broke m.

Case 4: $n \equiv 5 \pmod 6$. In this case $m \equiv 0$ or $1 \pmod 3$. By Lemma 2.2 there exists a $B[\{3,5^*\},1,n]$. Now apply Lemma 3.3. We only have to prove the existence of an $MGD[3,1,m,3m]$ and an $MGD[3,1,m,5m]$. But an $MGD[3,1,m,3m]$ is an $MT[3,1,m]$ and an $MGD[3,1,m,5m]$ was shown to exist in Lemma 3.5.

Theorem 3.3. *The necessary conditions for the existence of an* $MGD[k,\lambda,m,$ $mn]$ *are sufficient in the case* $k = 3$ *and* $\lambda = 2$.

Proof: We distinguish two cases.

Case 1: $n \equiv 0$ or $1 \pmod 3$. By Theorem 1.1 there exists a $B[3,2,n]$ and by Lemma 1.3 there exists an $MT[3,1,n]$, so we apply Lemma 3.2 to get our result.

Case 2: $n \equiv 2 \pmod 3$. In this case m should be congruent to 0 or 1 $(\bmod 3)$. Again by Theorem 1.1 there exists a $B[3,2,m]$, and the proof of this case is the same as that of case 1.

In order to prove a similar result in the case $\lambda = 3$ we need the following lemma:

Lemma 3.7. *For every* $v \geq 3$, $v \in B(K,3)$ *where* $K = \{3,4,6\}$.

Proof: By Lemma 2.1 for every $v \geq 3$, $v \in B(M,1)$ where $M = \{3,4,5,6,8\}$. But $5,8 \in B(4,3)$. Hence for every $v \geq 3$, $v \in [\{3,4,6\},3]$.

Theorem 3.4. *The necessary conditions for the existence of an* $MGD[k,\lambda,m,mn]$ *are sufficient in the case* $k = 3$ *and* $\lambda = 3$.

Proof: We distinguish the following cases

Case 1: n is odd. In this case there exists a $B[3,3,n]$. From Lemmas 3.2 and 1.3 it follows that an $MGD[3,3,m,mn]$ exists for every odd n and any m.

Case 2: n is even. In this case m should be odd. By Lemma 3.7 there exists a $B[\{3,4,6\},3,n]$ for every n. Apply Lemma 3.3. We have to prove the existence of an $MGD[3,1,m,rm]$ where $r = 3,4,6$. For $r = 3$ this is precisely $T[3,1,m]$. For $r = 4,6$ this was done in Lemmas 3.4 and 3.6.

Theorem 3.5. *The necessary conditions for the existence of an* $MGD[k,\lambda,m,mn]$ *are sufficient in the case* $k = 3$ *and* $\lambda = 6$.

Proof: There exists a $B[3,6,n]$ for every $n \geq 3$ (Theorem 1.1). From Lemmas 3.2 and 1.3 it follows that a $MGD[3,6,m,mn]$ exists for every $n \geq 3$ and any m.

Now the proof of Theorem 3.1 follows from Theorems 3.2, 3.3, 3.4 and 3.5.

References

1. A. Assaf and A. Hartman, *Resolvable group divisible designs with* $k=3$, $\lambda=2$, in preparation.
2. H. Hanani, *Balanced incomplete block designs and rlated designs*, Discrete Math. **11** (1975).
3. Huang, E. Mendelsohn and A. Rosa, *Partially resolvable t-partitions*, Annals Discrete Math. **12** (1982), 169–183.
4. G. Tarry, *Le probleme des 36 officiers*, Compt. Rend. Assoc. Fr. Av. Sci. **1** (1900), 122–123. **2** (1901) 170–203.