New Classes of Orthogonal Designs

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Abstract. An extension of a method of Hammer, Sarvate and Seberry is given. As a result, from an $OD(s_1, s_2, \dots, s_r)$ of order n and a $w(nm, p)$ an $OD(ps_1, ps_2, \dots, ps_r)$ of order $nm(n+k)$ for each integer $k \geq 0$ is constructed.

1. Introduction

A weighing matrix of weight p and order n is a $\{0, 1, -1\}$ matrix A of order n such that $A^tA = AA^t = pI_n$, where A^t is the transpose of A and I_n is the identity matrix of order n. An orthgonal design of order n and type $(s_1, s_2, \dots, s_r)$, s_i positive integers, is a complex $n \times n$ matrix X, with entries from $\{0, \pm x_1, \pm x_2, \dots, \pm x_r\}$ (the x_i indeterminates) satisfying $XX^t = \left(\sum_{i=1}^r s_i x_i^2\right) I_n$. An orthgonal design of order n and type $(s_1, s_2, \dots, s_r)$ will be denoted by $OD(n; s_1, s_2, \dots, s_r)$. Note that if every variable in such a design is replaced by 1, then one gets a $w(n, p)$, where $p = \sum_{i=1}^r s_i$.

Hammer, Sarvate and Seberry [2] extended a result of Kharaghani [3] and constructed new ODs. In this note we extend their result even further and construct new ODs. For the application of ODs and details refer to Geramita and Seberry [1].

2. The main construction

We will start with the following extension of Kharaghani [3, Lemma 1] and Hammer, Sarvate and Seberry [2, Theorem 3.1].

Lemma 2.1. *If there is an $OD(n; s_1, s_2, \dots, s_r)$ on the variables $x_1, x_2, \dots, x_r$ and a $w(nm, p)$ then there exist nm matices $C_{11}, C_{12}, \dots, C_{1n}, \dots, C_{i1}, C_{i2}, \dots, C_{in}, \dots, C_{m1}, C_{m2}, \dots, C_{mn}$ satisfying:*

(i) $C_{il}C_{ij}^t = 0$ *if* $l \neq j$

(ii) $\sum_{i=1}^m \sum_{j=1}^n C_{ij}C_{ij}^t = \left(\sum_{i=1}^r ps_i x_i^2\right) I_{nm}$.

Proof: Let A_j denote the j^{th} row of the OD and B_k the k^{th} row of the weighing matrix. For $1 \leq i \leq m$ and $1 \leq j \leq n$, let $C_{ij} = B_{j+n(i-1)}^t \times A_j$. Then

$$\begin{aligned} C_{il}C_{ij}^t &= \left(B_{l+n(i-1)}^t \times A_l\right)\left(B_{j+n(i-1)} \times A_j^t\right) \\ &= B_{l+n(i-1)}^t B_{j+n(i-1)} \times A_l A_j^t \\ &= \begin{cases} 0 & \text{if } l \neq j \\ \left(\sum_{k=1}^r s_k x_k^2\right) B_{j+n(i-1)}^t B_{j+n(i-1)} & \text{if } l = j, \end{cases} \end{aligned}$$

because A is an $OD(s_1, s_2, \dots, s_r)$ on the variables $x_1, x_2, \dots, x_r$. Since B_ks are the rows of a $w(nm, p)$, it follows from [3,Lemma 1] that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} B_{j+n(i-1)}^t B_{j+n(i-1)} = pI_{nm}.$$

Hence from above $\sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}C_{ij}^t = \left(\sum_{k=1}^{r} ps_k x_k^2\right) I_{nm}$. ∎

Theorem 2.2. *If there is an $OD(n; s_1, s_2, \dots, s_r)$ and a $w(nm, p)$, then there is an $OD(ps_1, ps_2, \dots, ps_r)$ of order $nm(n+k)$ for each integer $k \geq 0$.*

Proof: Let $k \geq 0$ be an integer. Let $L_1, L_2, \dots, L_m$ be Latin squares of order $n+k$. Replace n elements of L_i by

$$C_{i1}, C_{i2}, \dots, C_{in}$$

constructed in Lemma 2.1, and the rest by the zero matrix of order $nm \times n$, for each i, $1 \leq i \leq m$. Call the resulting matrix M_i. Then the block matrix $[M_1 | M_2 | \cdots | M_m]$ is an $D(ps_1, ps_2, \dots, ps_r)$ of order $mn(n+k)$. ∎

Corollary 2.3 [Hammer, Sarvate and Seberry]. *Suppose there exists an $OD(s_1, s_2, \dots, s_r)$, where $\sum_{i=1}^{r} s_i = w$, of order n. Then there exists an $OD(s_1 w, s_2 w, \dots, s_r w)$ of order $n(n+k)$ for $k \geq 0$.*

Proof: Let $m = 1$ and $p = w$ in Theorem 2.2.

Example 2.4: Let $a = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & 1 & - \\ 1 & - & - & 1 \end{bmatrix}$ and $k = 1$. Then $n = 2, m = 2$, so,

$$C_{11} = \begin{bmatrix} a & b \\ a & b \\ a & b \\ a & b \end{bmatrix}, C_{12} = \begin{bmatrix} b & -a \\ b & -a \\ -b & a \\ -b & a \end{bmatrix},$$

$$C_{21} = \begin{bmatrix} a & b \\ -a & -b \\ a & b \\ -a & -b \end{bmatrix} \text{ and } C_{22} = \begin{bmatrix} b & -a \\ -b & a \\ -b & a \\ b & -a \end{bmatrix}.$$

Let $L_1 = L_2 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$. Then for one choice

$$M_1 = \begin{bmatrix} C_{11} & C_{12} & 0 \\ 0 & C_{11} & C_{12} \\ C_{12} & 0 & C_{11} \end{bmatrix}, M_2 = \begin{bmatrix} C_{21} & C_{22} & 0 \\ 0 & C_{21} & C_{22} \\ C_{22} & 0 & C_{21} \end{bmatrix}.$$

Hence we get

$$[M_1|M_2] = \begin{bmatrix} C_{11} & C_{12} & 0 & C_{21} & C_{22} & 0 \\ 0 & C_{11} & C_{12} & 0 & C_{21} & C_{22} \\ C_{12} & 0 & C_{11} & C_{22} & 0 & C_{21} \end{bmatrix}$$

which is an $OD(12;4,4)$. Note that the method of Hammer, Sarvate and Seberry doesn't give this design.

3. Some rectangular designs

In this section we will use a $w(q,p)$ in which q is not necessarily a multiple of n.

Theorem 3.1. *If there is an $OD(n; s_1, s_2, \dots, s_r)$ on the variables $x_1, x_2, \dots, x_r$ and a $w(q,p)$ $q > n$, then there is a matrix B of order $nq \times n^2(m+1)$ such that $BB^t = \left(\sum_{i=1}^r ps_ix_i^2\right) I_{nq}$.*

Proof: Following the line of proof of Theorem 2.3 we get d additional matrices, where d is a positive integer such that $q = mn + d$, $d < n$. Add one more Latin Square and fill d of its elements by the d additional matrices and the rest by the proper zero matrix. ∎

Example 3.2: Let $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, $w = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. So $q = 3, n = 2, m = 1$, $d = 1$. Then $C_{11} = \begin{bmatrix} a & b \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $C_{12} = \begin{bmatrix} 0 & 0 \\ b & -a \\ 0 & 0 \end{bmatrix}$, additional matrix $= C_{13} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ a & b \end{bmatrix}$. Let $M_1 = M_2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, then for one choice we have

$$[M_1|M_2] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 \\ C_{12} & C_{11} & 0 & C_{13} \end{bmatrix}.$$

Note that again as in Theorem 3.2 one can add an equal number of zero blocks to each Latin square.

Theorem 3.3. *If there is an $OD(n; s_1, s_2, \dots, s_r)$ on the variables $x_1, x_2, \dots, x_r$ and a $w(m,p)$, $m \leq n$, then there is a matrix C of order $m^2 \times nm$ such that $CC^t = \left(\sum_{i=1}^r ps_ix_i^2\right) I_{m^2\times m^2}$.*

Proof: Let $C_{1j} = B_j \times A_j$, $1 \leq j \leq m$, be the matrices constructed in Lemma 2.1. Form a Latin square of order m and replace its elements by the above matrices. ∎

Example 3.4: Let

$$A = \begin{bmatrix} a & a & a & a \\ a & -a & -a & a \\ a & a & -a & -a \\ a & -a & a & -a \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix}.$$

Then $C_{11} = \begin{bmatrix} a & a & a & a \\ a & a & a & a \end{bmatrix}$, $C_{12} = \begin{bmatrix} a & -a & -a & a \\ -a & a & a & -a \end{bmatrix}$. Let $M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, then for one choice the design is

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{12} & C_{11} \end{bmatrix} = \begin{bmatrix} a & a & a & a & a & -a & -a & a \\ a & a & a & a & -a & a & a & -a \\ a & -a & -a & a & a & a & a & a \\ -a & a & a & -a & a & a & a & a \end{bmatrix}.$$

Note that for $a = 1$ this is a BIBD with parameters $(4,8,2,1,0)$. This is not a coincidence. In fact we have the following result of Shrikhande [7]. See also [4].

Corollary 3.5. *Let $2s, 2t$ be the order of Hadamard matrices with $t \geq s$. Then there is a BIBD with parameters $(4s^2, 4st, 2st-t, 2s^2-s, st-t)$.*

Proof: Normalize the two Hadamard matrices and apply Theorem 3.3. ∎

4. Designs constructed from two circulant block matrices

In this section we will first modify a recent result of the author in [5] and then use Lemma 2.1 to construct new ODs.

Theorem 4.1. *Let $C_1, C_2 \ldots, C_k, H$ be matrices with entries of C_i from $\{0, \pm x_1, \pm x_2, \ldots, \pm x_r\}$ such that:*

(i) *H is an $OD(n; l_1, l_2, \ldots, l_a)$ on the variables $y_1, y_2, \ldots, y_a$,*
(ii) *$C_iC_j^t = C_jC_i^t = 0$ if $i \neq j$,*
(iii) *$\sum_{i=1}^k C_iC_i^t = \sum_{i=1}^k C_i^tC_i = \left(\sum_{i=1}^r s_ix_i^2\right) I_n$,*
(iv) *$HC_i = C_iH, i = 1, 2, \ldots, k$. Then there is an $OD(2nk+2n; 2s_1, 2s_2, \ldots, 2s_r, 2l_1, 2l_2, \ldots, 2l_a)$ which is constructed from two circulant block matrices.*

Proof: Let A, B be the circulant block matrices with the first row $[H, C_1, C_2, \ldots, C_k]$ and $[-H, C_1, C_2, \ldots, C_k]$ respectively. Then the matrix $\begin{bmatrix} A & B \\ -B^t & A^t \end{bmatrix}$ is the desired design. To see this, note that $B = A - [2H, 0, 0, \ldots, 0]$. Since $HC_i = C_iH$ for each i, it follows that $AB = BA$. It is not hard to see that $AA^t + BB^t = A^tA + B^tB =$ the circulant block matrix with the first row

$$\left[\left(\sum_{i=1}^{a} 2l_iy_i^2\right) I_n + \left(\sum_{i=1}^{r} 2s_ix_i^2\right) I_n, 0, 0, \ldots, 0\right].$$

∎

Theorem 4.2. *If there is an $OD(n; s_1, s_2, \ldots, s_r)$ and a $w(n,p)$, then there is an $OD(2, 2ps_1, 2ps_2, \ldots, 2ps_r)$ of order $2n(n+k+1)$ for each integer $k \geq 0$.*

Proof: Let $C_{11}, C_{12}, \ldots, C_{1n}$ be the matrices constructed in Lemma 2.1 from the $OD(n; s_1, s_2, \ldots, s_r)$ and the $w(n,p)$. Denote C_{1i} by C_i and consider the matrices $C_1, C_2, \ldots, C_n, \ldots, C_k, y_1 I_n$, where $C_{n+1} = C_{n+2} = \cdots = C_k =$ *zero* matrix of order n. Apply the preceding theorem. ∎

Remarks:

(i) Theorem 4.2 provide many new ODs. For example, from an $OD(4; 1, 1, 1, 1)$, Theorem 4.2 provides $OD(40+8k, 2, 2p, 2p, 2p, 2p)$, $p = 1, 2, 3, 4$ and k any integer ≥ 0.

(ii) By applying recent results of the author in [6] and a method similar to the above one can make more new ODs.

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