Mutually Orthogonal Frequency Squares with Non-Constant Frequency Vectors

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1. Introduction.

It is well known that Latin squares and sets of mutually orthogonal Latin squares (MOLS) are useful in the design of statistical experiments, see Denes and Keedwell [3]. In [5] Hedayat and Seiden studied a number of properties of frequency squares and orthogonal frequency squares and showed that these generalizations of Latin squares, in which repetitions are allowed, are also useful in statistical design theory.

A frequency square $F(n; \lambda_1, \dots, \lambda_m)$ of order n is an $n \times n$ array consisting of the numbers $1, 2, \dots, m$ with the property that for each $i = 1, \dots, m$, the number i occurs exactly λ_i times in each row and in each column. Clearly $n = \lambda_1 + \dots + \lambda_m$ and an $F(n; 1, \dots, 1)$ frequency square is a Latin square of order n. Two frequency squares $F_1(n; \lambda_1, \dots, \lambda_{m_1})$ and $F_2(n; \mu_1, \dots, \mu_{m_2})$ are said to be orthogonal if upon superposition, each ordered pair (i, j) occurs exactly $\lambda_i \mu_j$ times for $i = 1, \dots, m_1$; $j = 1, \dots, m_2$. A set $\{F_1, \dots, F_t\}$ of $t \geq 2$ frequency squares is said to be orthogonal if F_i is orthogonal to F_j whenever $i \neq j$.

In this paper we present a simple method for the construction of sets of orthogonal frequency squares in which the frequency vectors may be non-constant. Our method involves making substitutions on the symbols within a set of mutually orthogonal frequency squares (MOFS) to obtain a set of MOFS with a different frequency vector.

Because of the statistical properties of sets of MOFS, considerable attention has been focused by a number of authors on the problem of constructing these sets. In [11] Pellegrino and Malara showed that if for $i = 1, \dots, t$ the square F_i contains

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m_i distinct symbols, then the maximum number t of MOFS of order n is bounded by

$$\sum_{i=1}^{t} m_i - t \leq (n-1)^2. \qquad (1)$$

If t satisfies the bound (1), then the set of MOFS is said to be complete.

To give a flavor of some of the constructions that are available in the literature, the reader should consult the references in Mullen [10]. Many of these constructions, which deal with squares which have constant frequency vectors, that is, $\lambda_1 = \ldots = \lambda_m$, are based upon properties of statistical designs, and in particular on symmetric factorial experimental design theory.

Several papers have been written concerning the problem of constructing sets of MOFS with varying numbers of symbols. Using symmetric factorial designs, Mandeli and Federer [8] discussed such complete sets of prime power order while Lancellotti and Pellegrino [6] discussed such sets of composite order by using a method of substitution. While the frequency vectors in both cases can vary from square to square within an orthogonal set, each is constant for any given square.

2. Derived frequency squares.

Given two partitions $A = \{\mu_1, \mu_2, \ldots, \mu_k\}$ and $B = \{\lambda_1, \lambda_2, \ldots, \lambda_\ell\}$ of n with $\ell \leq k$, we say that A is a refinement of B if the set A can be partitioned into ℓ subsets $A_1, A_2, \ldots, A_\ell$ such that if $A_i = \{\mu_{i_1}, \mu_{i_2}, \ldots, \mu_{i_m}\}$ then $\lambda_i = \mu_{i_1} + \mu_{i_2} + \ldots + \mu_{i_m}$ for $i = 1, 2, \ldots, \ell$. Alternatively we will describe B as being coarser than A. For example if $n = 6$, A is the partition $2 + 2 + 1 + 1$ and B is the partition $3 + 3$, then A is a refinement of B.

Hedayat and Seiden [5] introduced the concept of a derived frequency square as follows. A frequency square F_2 of type $F_2(n; \lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is said to be derived from the frequency square F_1 of type $F_1(n; \mu_1, \mu_2, \ldots, \mu_k)$ if F_2 may be obtained from F_1 by some mapping of the symbols $\{1, 2, \ldots, k\}$ of F_1 onto the symbols $\{1, 2, \ldots, \ell\}$. This definition implies that the partition $\{\mu_1, \mu_2, \ldots, \mu_k\}$ is a refinement of $\{\lambda_1, \lambda_2, \ldots, \lambda_\ell\}$. It is given in [5] that if two frequency squares F_1 and F_2 are orthogonal then any two squares $F_1^\star$ and $F_2^\star$ derived respectively from them must also be orthogonal. From this it follows that for all $n > 2$ and $n \neq 6$ a pair of orthogonal frequency squares with any combination of frequency vectors will exist. Using the pair of orthogonal frequency squares of types $F(6; 1, 1, 1, 1, 1, 1)$ and $F(6; 2, 1, 1, 1, 1)$ displayed in [5] to derive other cases the above observation can be extended to $n = 6$ provided both squares are not Latin. Furthermore, if $N(n)$ denotes the largest number of MOLS of order n currently known, then there exist $N(n)$ MOFS of type $F(n; \lambda_1, \ldots, \lambda_\ell)$ for any partition $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ of n. A table of $N(n)$ for $n \leq 100$ as of 1985 can be found in Beth, Jungnickel, and Lenz [1, pp. 643-644] and for $n \leq 10,000$ in Brouwer [2].

Hedayat, Raghavarao, and Seiden [4] by the theory of statistical designs and Mullen [10] by finite field theory, have shown the existence of complete sets of $(p^s - 1)^2/(p^{s/i} - 1)$ MOFS of type $F(p^s; p^{s(i-1)/i}, \ldots, p^{s(i-1)/i})$ where p is a prime and i divides s with $s \geq 1$. These complete sets can be used in several ways to derive sets of MOFS with coarser frequency vectors.

Theorem 1. *For p prime and i a divisor of s, there exist $(p^s - 1)^2/(p^{s/i} - 1)$ MOFS of type $F(p^s; c_1 p^{s(i-1)/i}, c_2 p^{s(i-1)/i}, \ldots, c_m p^{s(i-1)/i})$ where $c_1 + c_2 + \ldots + c_m = p^{s/i}$ and $m \neq 1$.*

Proof: Starting with the $(p^s - 1)^2/(p^{s/i} - 1)$ MOFS of type $F(p^s; p^{s(i-1)/i}, \ldots, p^{s(i-1)/i})$ shown to exist in [4] and [10], we can derive a set with the same number of MOFS with an arbitrarily selected coarser frequency vector by partitioning the $p^{s/i}$ symbols into m subsets of sizes $c_1, c_2, \ldots, c_m$.

As an illustration of Theorem 1, let $p = 5$ and $s = 2$ so that we are considering squares with 25 rows and columns. Since there exist 24 MOLS of order 25, when $i = 1$, we can construct 24 MOFS of type $F(25; c_1, c_2, \ldots, c_m)$ where $c_1 + c_2 + \ldots + c_m$ is any one of the 1957 non-trivial partitions of 25. For example there are 24 MOFS of type $F(25; 8, 8, 4, 3, 2)$. If $i = 2$ we can construct 144 MOFS of each of the following types by considering the 6 non-trivial partitions of 5.

Partition	**Frequency vector**
$4+1$	(25; 20, 5)
$3+2$	(25; 15, 10)
$3+1+1$	(25; 15, 5, 5)
$2+2+1$	(25; 10, 10, 5)
$2+1+1+1$	(25; 10, 5, 5, 5)
$1+1+1+1+1$	(25; 5, 5, 5, 5, 5)

From (1) we note that in Theorem 1 the only complete set of MOFS arises from the finest partition, that is, when $c_i = 1$ for $i = 1, \ldots, m$.

In the construction from Theorem 1 each square of the initial set produces a single square in the derived set. Therefore, one would wish to start with as large a set as possible. This implies starting with as coarse a frequency vector as possible. For example, in deriving squares of type $F(64; 48, 16)$ we can obtain a mutually orthogonal set of 1323 squares if we begin with a complete set of $F(64; 16, 16, 16, 16)$ squares; 567 from a complete set of $F(64; 8, 8, \ldots, 8)$ squares and 63 from a complete set of Latin squares of order 64.

In Theorem 1, we restricted n to the prime power case in order to begin with a complete set of MOFS for some partition of n. The derivation of MOFS will still apply in the non prime power case if we can find some initial orthogonal set. There are several ways to obtain sets of MOFS in the non prime power case. We may prove

Corollary 2. *For* $n = p_1^{s_1} p_2^{s_2} \ldots p_r^{s_r}$ *where* $p_1^{s_1} < p_2^{s_2} < \ldots < p_r^{s_r}$ *and* $1 \leq j \leq r$, *there are* $(p_j^{s_j} - 1)\ (p_1^{s_1} - 1)/\ (p_j^{s_j/i} - 1)$ *MOFS of type*

$$F\left(Ap_j^{s_j}; Ac_1 p_j^{s_j(i-1)/i}, Ac_2 p_j^{s_j(i-1)/i}, \ldots, Ac_m p_j^{s_j(i-1)/i}\right)$$

where i *is a divisor of* s_j, $A = \Pi_{\substack{k=1 \\ k \neq j}}^{r} p_k^{s_k}$, *and* $c_1 + c_2 + \ldots c_m = p_j^{s_j}$ *and* $m \neq 1$.

Proof: Laywine [7] has given a construction in the non prime power case which gives $(p_j^{s_j} - 1)\ (p_1^{s_1} - 1)/\ (p_j^{s_j/i} - 1)$ MOFS of type

$$F\left(Ap_j^{s_j}; Ap_j^{s_j(i-1)/i}, \ldots, Ap_j^{s_j(i-1)/i}\right)$$

where $A = \Pi_{\substack{k=1 \\ k \neq j}}^{r} p_k^{s_k}$. The result follows by starting with this set of MOFS.

In [9] Mandeli and Federer constructed sets of MOFS with non prime power order by extending MacNeish's theorem for orthogonal Latin squares. In particular their construction provides squares in which the frequency vector could vary from square to square but was constant within a given square. Similarly in [6] Lancellotti and Pellegrino provided a generalization of the Mandeli and Federer construction.

If we wish to construct MOFS which have varying frequency vectors, the best strategy is to use the more numerous coarser squares whenever possible. In particular we may prove

Theorem 3. *Suppose* s *has divisors* $i_0 = 1 < i_1 < i_2 < \ldots < i_k = s$. *Then there exists a set of MOFS of cardinality* $\sum_{\alpha=0}^{k} A_\alpha (p^s - 1)/(p^{s/i_\alpha} - 1)$ *consisting of* $A_\alpha (p^s - 1)/(p^{s/i_\alpha} - 1)$ *squares of type*

$$F\left(p^s; c_{\alpha 1} p^{s(i_\alpha - 1)/i_\alpha}, c_{\alpha 2} p^{s(i_\alpha - 1)/i_\alpha}, \ldots, c_{\alpha j_\alpha} p^{s(i_\alpha - 1)/i_\alpha}\right)$$

where $A_0 + A_1 \ldots + A_k = p^s - 1$ *and* $c_{\alpha 1} + c_{\alpha 2} + \ldots + c_{\alpha j_\alpha} = p^{s/i_\alpha}$ *for* $\alpha = 0, 1, \ldots, k$.

Proof: Partition a complete set of Latin squares of order p^s into $k + 1$ subsets of cardinalities $A_0, A_1, \ldots, A_k$. For $\alpha = 0, 1, \ldots, k$ construct $A_\alpha (p^s - 1)/(p^{s/i_\alpha} - 1)$ MOFS of type

$$F(p^s; p^{s(i_\alpha - 1)/i_\alpha}, \ldots, p^{s(i_\alpha - 1)/i_\alpha})$$

from the subset with A_α elements. Then use Theorem 1 to obtain the result.

Hedayat, Raghavarao, and Seiden [4] have shown that if $F_1 = F_1(n_1; \lambda_1, \ldots, \lambda_p)$ and $F_2 = F_2(n_2; \mu_1, \ldots, \mu_q)$ are frequency squares then $F_1 \otimes F_2$ is an

$F(n_1 n_2; \lambda_1 \mu_1, \ldots, \lambda_p \mu_q)$ frequency square where $\otimes$ is the Kronecker product of matrices. Furthermore if $F_1 \perp F_3$ and $F_2 \perp F_4$, then $F_1 \otimes F_2 \perp F_3 \otimes F_4$. We can combine this operation together with the earlier results to construct various sets of orthogonal frequency squares in the non prime power case.

As an illustration of the effectiveness of Theorem 1 and Corollary 2 when used in conjunction with the $\otimes$ operation, we may build the following table for frequency squares of order 225. In the table we will use the notation b^c to represent $b, \ldots, b$ a total of c times.

Table. Number of mutually orthogonal frequency squares.

Frequency vector	Number	Construction
$(225; e_1, \ldots, e_m),\ e_1 + \ldots + e_m = 225,\ m > 1$	8	Reference [2]
$(225; 3^{75})$	24	Thm. 1 & $\otimes$
$(225; 6^{25}, 3^{25})$	24	Thm. 1 & $\otimes$
$(225; 15^{15})$	32	Thm. 1 & $\otimes$
$(225; 30^5, 15^5)$	32	Thm. 1 & $\otimes$
$(225; 60^3, 15^3)$	32	Thm. 1 & $\otimes$
$(225; 120, 60, 30, 15)$	32	Thm. 1 & $\otimes$
$(225; 45^3, 30^3)$	32	Thm. 1 & $\otimes$
$(225; 90, 60, 45, 30)$	32	Thm. 1 & $\otimes$
$(225; 45^3, 15^6)$	32	Thm. 1 & $\otimes$
$(225; 90, 45, 30^2, 15^2)$	32	Thm. 1 & $\otimes$
$(225; 30^6, 15^3)$	32	Thm. 1 & $\otimes$
$(225; 60^2, 30^3, 15)$	32	Thm. 1 & $\otimes$
$(225; 30^3, 15^9)$	32	Thm. 1 & $\otimes$
$(225; 60, 30^4, 15^3)$	32	Thm. 1 & $\otimes$
$(225; 150, 75)$	32	Corollary 2
$(225; 75^3)$	32	Corollary 2
$(225; 180, 45)$	48	Corollary 2
$(225; 135, 90)$	48	Corollary 2
$(225; 135, 45^2)$	48	Corollary 2
$(225; 90^2, 45)$	48	Corollary 2
$(225; 90, 45^3)$	48	Corollary 2
$(225; 45^5)$	48	Corollary 2

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