A New Construction of k-Folkman Graphs

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Abstract. Given a graph G and a positive integer k, a graph H is a *k-Folkman graph* for G if for any map $\pi: V(H) \to \{1, \dots, k\}$, there is an induced subgraph of H isomorphic to G on which π is constant. J. Folkman (SIAM. J. Appl. Math. 18 (1970), pp. 19-24) first showed the existence of such graphs. We provide here a new construction of k-Folkman graphs for bipartite graphs G via random hypergraphs. In particular, we show that for any fixed positive integer k, any fixed positive real number ε and any bipartite graph G, there is a k-Folkman graph for G of order $O(|V(G)|^{3+\varepsilon})$ without triangles.

Folkman [5] proved the following vertex partition result. For any graph G and positive integer k, there is a graph H such that

> for any map $\pi: V(H) \to \{1, \dots, k\}$, there is a monochromatic induced copy of G, that is, there is an induced subgraph G' of H that is isomorphic to G all of whose vertices are assigned the same colour under π.

We write $H \to^v_k G$ for such a graph H and call H a *k-Folkman graph* for G (for various results on k-Folkman graphs, see [9], [3], [6], and for the poset analogue [7]).

Several constructions are known for k-Folkman graphs [**5, 9, 10, 8, 3, 4**]. We shall describe a new construction of k-Folkman graphs, via random hypergraphs, for bipartite graphs G; in particular, we construct for each bipartite graph G a k-Folkman graph without triangles that also has chromatic number $k + 1$; these graphs have much smaller order than previous known constructions.

The Construction.

Let k be a fixed positive integer and $G = (X, Y, E)$ be a bipartite graph (our notation for a bipartite graph is a triple, with the first two components being a partition of the vertex set and the final component being a set whose restriction to the cartesian product of the first two sets is the edge set). We construct a graph F (of small order) such that $F \to^v_k G$ in two steps. The key to both of these is the existence of certain hypergraphs. In the proof of the following proposition we will often make use of standard bounds for binomial coefficients (c.f. [2, p. 255]).

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Proposition. *Let H be a hypergraph on vertex set $V_1 \cup \ldots \cup V_t$ ($|V_i| = n$ for all $i = 1, \ldots, t = \lfloor n^\delta \rfloor$ for some fixed $\delta > 1$) with $M = \lfloor n^\varepsilon \rfloor$ edges (for some fixed $\varepsilon > 1$) whose edges are chosen randomly from the n^t edges e such that $|e \cap V_i| = 1$ for all i. Let C be the following property (for fixed real numbers $p, r \geq 2$):*

For any set $S \subseteq E(H)$ of size $\lfloor \frac{M}{p} \rfloor$, there are at least $(1 - \frac{1}{r})t$ of the V_i's that have every vertex on an edge of S.

Then almost every hypergraph H has property C.

Proof: If C fails, then there is a set of $L = \lfloor \frac{M}{p} \rfloor$ edges and $Q = \lfloor \frac{t}{r} \rfloor$ vertices in distinct V_i's such that none of these vertices and edges are adjacent. These vertices can be chosen in $\binom{t}{Q} n^Q$ ways, and the edges can be subsequently chosen in at most $\binom{(n-1)^Q n^{t-Q}}{L}\binom{n^t}{M-L}$ ways. Since $\binom{t}{Q} n^Q \leq 2^t n^{t/2}$ and $\binom{(n-1)^Q n^{t-Q}}{L}\binom{n^t}{M-L} \leq e^M n^{tM} (\frac{n-1}{n})^{QL} \frac{1}{L^L (M-L)^{M-L}}$, we have (for n sufficiently large)

$$\begin{aligned}
\mathrm{Prob}(\overline{C}) &\leq 2^t n^{t/2} e^M n^{tM} \left(\frac{n-1}{n}\right)^{QL} \frac{1}{L^L (M-L)^{M-L}} \binom{n^t}{M}^{-1} \\
&\leq 2^t n^{t/2 + tM} e^M \left(\frac{n-1}{n}\right)^{QL} \frac{1}{L^L (M-L)^{M-L}} 2 n^{t/2} \left(\frac{M}{n^t}\right)^M \\
&\leq 2^{t+1} n^t e^M \left(\frac{n-1}{n}\right)^{QL} \frac{M^M}{(\lfloor \frac{M}{p} \rfloor)^{\lfloor \frac{M}{p} \rfloor} (\frac{p-1}{p} M)^{\frac{p-1}{p} M}} \\
&\leq \exp\left\{ C_1 n^\delta \log n + C_2 n^\varepsilon + C_3 n^{\delta + \varepsilon - 1} \log \left(\frac{n-1}{n}\right)^n \right\} \\
&< \exp\{ C_1 n^\delta \log n + C_2 n^\varepsilon - C_4 n^{\delta+\varepsilon-1} \} \\
&= o(1)
\end{aligned}$$

(here all C_i's are positive constants). Thus almost every hypergraph H has property C. ∎

Returning to our problem, let $|X| = n$ and $|Y| = m$; we may assume $n \geq m$ and that n is sufficiently large. Let $X = \{x^1, \ldots, x^n\}$. We take a hypergraph H as determined in the proposition above with $p = 4k$ and $r = 2k$, and form a new bipartite graph $G^1 = (X^1, Y^1, E^1)$ such that

(i) Y^1 is the disjoint union of $t = \lfloor n^\delta \rfloor$ copies of Y, say $Y_1 \cup \ldots \cup Y_t$;
(ii) $X^1 = E(H)$;
(iii) For $h \in X^1 = E(H)$ and $i \in \{1, \ldots, t\}$, if $h \cap V_i$ is the jth vertex of V_i, we join h to precisely the image of the neighbourhood of x^j in the ith copy of Y.

Then G^1 has the following property:

(*) For any $\pi: X^1 \to \{1, \ldots, k\}$ and for any colour class C with $|C| \geq \lfloor \frac{1}{4k} |X^1| \rfloor$, there is a subset $Z \subseteq C$ such that for some $1 \leq i \leq t$, the bipartite subgraph (Z, Y_i, E^1) induced by Z and Y_i is isomorphic to G.

This follows from the proposition, since if $Z' \subseteq C$ and $|Z'| = \lfloor \frac{1}{4k}|X^1| \rfloor = \lfloor \frac{M}{4k} \rfloor$, there is an i such that Z' meets every vertex of V_i (the proposition in fact states much more). Let $Z \subseteq Z'$ be a set of n edges such that every vertex of V_i lies on an edge of Z. By construction, $(Z, Y_i, E^1) \cong G$. Note that $|Y^1| = m\lfloor n^\delta \rfloor$ and $|X^1| = \lfloor n^\varepsilon \rfloor$.

We now iterate the process with X^1 in the place of Y to form $G^2 = (X^2, Y^2 = X_1^1 \cup \ldots \cup X_s^1, E^2)$, where $s = \lfloor |Y^1|^{\delta'} \rfloor = \lfloor (m\lfloor n^\delta \rfloor)^{\delta'} \rfloor$ and $|X^2| = \lfloor |Y^1|^{\varepsilon'} \rfloor = \lfloor (m\lfloor n^\delta \rfloor)^{\varepsilon'} \rfloor$ (here δ, δ', ε and ε' are fixed real numbers greater than 1). Then G^2 has order $\lfloor (m\lfloor n^\delta \rfloor)^{\delta'} \rfloor \lfloor n^\varepsilon \rfloor + \lfloor (m\lfloor n^\delta \rfloor)^{\varepsilon'} \rfloor$. Moreover, we now show G^2 satisfies the following:

(**) For any map $\pi: V(G^2) \to \{1, \ldots, k\}$, if C_1 and C_2 are, respectively, colour classes of X^2 and Y^2 of orders at least $\lfloor \frac{1}{4k}|X^2| \rfloor$ and $\lfloor \frac{4}{5} \cdot \frac{1}{k}|Y^2| \rfloor$, then there is an induced copy of G in the induced subgraph $(X^2 \cap C_1, Y^2 \cap C_2, E^2)$ of G^2.

Note that as $|C_1| \geq \lfloor \frac{1}{4k}|X^2| \rfloor$, by the proposition there are at least $(1 - \frac{1}{2k})s$ of the copies of X^1, say $X_1^1, \ldots, X_q^1$ ($q = \lceil (1 - \frac{1}{2k})s \rceil$) such that $(X^2 \cap C_1, X_i^1, E^2)$ contains an induced copy of G^1. Now $|C_2 \cap X_i^1| \geq \lfloor \frac{1}{4k}|X_i^1| \rfloor$ for some $i \in \{1, \ldots, q\}$, since otherwise

$$|C_2 \cap (X_1^1 \cup \ldots \cup X_q^1)| < \frac{((1 - \frac{1}{2k})s + 1)|X^1|}{4k}$$

and

$$|C_2 - (X_1^1 \cup \ldots \cup X_q^1)| \leq \frac{s}{2k}|X^1|,$$

so $|C_2| < \frac{s}{k}|X^1|(\frac{1}{4} - \frac{1}{8k} + \frac{1}{4s} + \frac{1}{2}) < \frac{3}{4} \cdot \frac{s}{k}|X^1|$, a contradiction as $|C_2| \geq \lfloor \frac{4}{5} \cdot \frac{s}{k}|X^1| \rfloor$. Now $(X^2 \cap C_1, X_i^1, E^2)$ contains an induced copy of G^1, and from (*) there is an induced copy of G contained in $(X^2 \cap C_1, Y^2 \cap C_2, E^2)$.

Now we choose δ, δ', ε and $\varepsilon' > 1$ such $|X^2| = |Y^2|$. We have assumed $n \geq m$, so by increasing the order of G (by embedding G in a larger bipartite graph) we also suppose $n = m$. We take $\delta = \delta' = \varepsilon > 1$ and $\varepsilon' = \frac{\varepsilon^2 + 2\varepsilon}{\varepsilon + 1}$ to get $|X^2| = \lfloor (n\lfloor n^\varepsilon \rfloor)^{\frac{\varepsilon^2+2\varepsilon}{\varepsilon+1}} \rfloor$ and $|Y^2| = \lfloor (n\lfloor n^\varepsilon \rfloor)^\varepsilon \rfloor \cdot \lfloor n^\varepsilon \rfloor$. Then $|V(G^2)| \leq 2n^{\varepsilon^2+2\varepsilon}$. X^2 and Y^2 are very close in order, so we add in points to get sets $\tilde{X}^2 \supseteq X^2$ and $\tilde{Y}^2 \supseteq Y^2$ such that $|\tilde{X}^2| = |\tilde{Y}^2| = \lfloor n^{\varepsilon^2+2\varepsilon} \rfloor$ (both $|\tilde{X}^2| - |X^2|$ and $|\tilde{Y}^2| - |Y^2|$ are at most $O(n^{\varepsilon^2+\varepsilon})$). We extend G^2 to $\tilde{G}_2$ on $\tilde{X}^2$ and $\tilde{Y}^2$. It is not hard to see that (**) holds for $\tilde{G}_2$ and $\tilde{C}_1$ and $\tilde{C}_2$ being largest colour classes of $\tilde{X}^2$ and $\tilde{Y}^2$ respectively as well (since $|\tilde{C}_1 \cap X^2| \geq \frac{1}{2} \cdot \frac{1}{k}|X^2| > \lfloor \frac{1}{4k}|X^2| \rfloor$ and $|\tilde{C}_2 \cap Y^2| > \frac{9}{10} \cdot \frac{1}{k}|Y^2|$).

To complete the construction of the k-Folkman graph for G, we take a fixed $(k+1)$-chromatic graph H_{k+1} that has no triangles (such graphs are known to exist by various means — see Sachs [**11**]), replace each vertex v by an independent set A_v of size $\lfloor n^{\varepsilon^2+2\varepsilon} \rfloor$, and form a graph F_k on vertex set $\cup\{A_v : v \in V(H_{k+1})\}$ such that for every edge uv of H_{k+1}, the subgraph induced by $A_u \cup A_v$ is isomorphic to $\widetilde{G}_2$. If $\pi: V(F_k) \rightarrow \{1, \ldots, k\}$ is any map and C_v is a largest colour class in A_v, then for some edge uv of H_{k+1}, $\pi^{-1}(C_u) = \pi^{-1}(C_v)$. From (**) we conclude there is a monochromatic induced copy of G in colour class $\pi^{-1}(C_u)$. Thus $F_k \rightarrow_k^v G$ and F_k has order $O(n^{\varepsilon^2+2\varepsilon})$ for all sufficiently large bipartite graphs G of order $n + m \leq 2n$. Clearly, F_k has clique number 2 and is $(k+1)$-colourable, so we have shown the following.

Theorem. *For any fixed number k of colours, any fixed $\varepsilon > 0$ and any bipartite graph G, there is a k-Folkman graph F_k for G of order $O(|V(G)|^{3+\varepsilon})$ that has clique number 2 (and moreover is $(k+1)$-colourable).*

We remark that a k-Folkman graph with small clique number have been constructed for any graph G by a number of methods [**5, 8, 9, 10**]; however, even for bipartite graphs G, these graphs have order at least exponential in $|V(G)|$ for fixed $k \geq 3$, while our graphs are of order polynomial in $|V(G)|$. For $k = 2$, there is a construction in [**8**] that produces for bipartite graphs G a 2-Folkman graph with clique number 2 of order $|V(G)|^4$, but our result here is still an improvement. It is still of interest to determine whether one can construct for any graph G a k-Folkman graph of reasonably small order that has the same clique number as G. In [**4**] we construct, for each fixed $k \geq 2$, k-Folkman graphs of order $O(n^2 \log^2 n)$ for any graph G of order n, but the constructed graphs do not necessarily have small clique number.

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