A Note on the Characterizations of Type-1 λ-Designs in Terms of r_{ij}

Xiang-dong Hou

University of Illinois at Chicago
Chicago, Illinois 60680
U.S.A.

Abstract. In a λ-design D, the points $1, 2, \dots, n$ are divided into two classes with replications r_1 and r_2, respectively. For any $1 \leq i, j \leq n$, let r_{ij} be the number of the blocks containing i and j. It is proven that D is type-1 iff for any i, j $(i \neq j)$ in the same class, r_{ij} depends only on the class.

1. Introduction.

A λ-design is a family of subsets $S_1, S_2, \dots, S_n$ of $\{1, 2, \dots, n\}$ such that $|S_i| = k_i > \lambda > 0$, $(1 \leq i \leq n)$, $|S_i \cap S_j| = \lambda$, $(1 \leq i \neq j \leq n)$, and not all k_i's are equal. In terms of the point-block incidence matrix, it is equivalent to an $n \times n$ $(0, 1)$-matrix A such that

$$A^t A = \lambda J + \text{ diag } [k_1 - \lambda, \dots, k_n - \lambda].$$

Ryser [8] and Woodall [10] proved that such an A has precisely two row sums r_1 and r_2 $(r_1 > r_2)$ with $r_1 + r_2 = n + 1$. Let $AA^t = (r_{ij})$. i is said of class 1 (class 2) if $r_{ii} = r_1$ $(r_{ii} = r_2)$.

The only known examples of λ-designs are of type-1, namely, obtained from (v, k, λ')-designs by complementing with respect to a fixed block. It was conjectured that all λ-designs are of type-1. The conjecture has been verified for $1 \leq \lambda \leq 9$ ([2],[4],[5],[6],[8]) and for all prime values of λ ([9]). On the other hand, work has also been done to seek characterizations of type-1 λ-designs ([1],[3],[7]). The following two characterizations are in terms of r_{ij}:

Theorem 1.1. *(Kramer [7]). A λ-design is type-1 iff r_{ij} $(i \neq j)$ depends only on the classes of i and j.*

Theorem 1.2. *(Bridges [3]). A λ-design is type-1 iff $r_{ij} = \lambda$ for all pairs i, j of different classes.*

Here we prove an improvement of Theorem 1.1: "A λ-design is type-1 iff for any i, j $(i \neq j)$ in the same class, r_{ij} depends only on the class."

The notations and most of the basic results used in this paper are from [2] and [8].

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

2. The characterization.

For any λ-design A, we always assume that its first e_1 rows have row sum r_1, the remaining e_2 rows have row sum r_2.

Lemma 2.1. *Let A be a λ-design, and*

$$AA^t = \begin{bmatrix} E_1 & E_2 \\ F_1 & F_2 \end{bmatrix}$$

where E_1 is of size $e_1 \times e_1$, F_2 is of size $e_2 \times e_2$. If one of E_1 and E_2 has constant row sums, so does $[E_1, E_2]$. Same is true for $[F_1, F_2]$.

Proof: Let $A = (a_{ij})$. Without loss of generality, assume E_1 has constant row sums. Then for any i, j of class 1,

$$a_{i1} k'_1 + \ldots + a_{in} k'_n = a_{j1} k'_1 + \ldots + a_{jn} k'_n.$$

Since $a_{i1} + \ldots + a_{in} = r_1 = a_{j1} + \ldots + a_{jn}$, and $k'_m = \frac{\lambda(\rho+1)-k_m}{\rho-1}$, we have

$$a_{i1} k_1 + \ldots + a_{in} k_n = a_{j1} k_1 + \ldots + a_{jn} k_n$$

which means $[E_1, E_2]$ has constant row sums. ∎

Theorem 2.2. *A λ-design A is type-1 iff for any i, j $(i \neq j)$ in the same class, r_{ij} depends only on the class.*

Proof: The necessity is obvious. We only prove the sufficiency. Ignoring the trivial case $n = 2$, we can assume $n > 2$. Since

$$AA^T = \begin{bmatrix} E_1 & E_2 \\ F_1 & F_2 \end{bmatrix}$$

where E_1 and F_2 are of sizes $e_1 \times e_1$ and $e_2 \times e_2$, and have constant row sums, by Lemma 2.1 and Theorem 8 of [7], A has two column sums. Write A as

$$A = \begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix}$$

where A_1 and B_2 are of sizes $e_1 \times f_1$ and $e_2 \times f_2$, $[A_1, B_1]$, $[A_2, B_2]$ have row sums r_1 and r_2, $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ have column sums k_1 and k_2 $(k_1 > k_2)$, A_1, A_2, B_1, B_2 have column sums k'_1, k^*_1, k'_2, k^*_2. Clearly, A_1, B_1, A_2, B_2 have constant row inner products, say $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

If $e_1 \neq e_2$ or $f_1 \neq f_2$, one of A_1, A_2, B_1, B_2, say A_1, has more rows than columns. Hence, $A_1 = 0$ or J. Then r_{ij} is constant for all i, j of different classes. By Theorem 1.1, A is type-1.

Therefore, it is sufficient to eliminate the case "$e_1 = e_2 = f_1 = f_2 = \frac{n}{2}$". In this case, one can find that

$$n = \frac{2\lambda(1+\rho)^2 - 2\rho}{\rho^2 + 1}. \tag{3.1}$$

Since A_1, A_2, B_1, B_2 are all symmetric designs, and $\lambda_1 + \lambda_3 = \lambda = \lambda_2 + \lambda_4$, we have

$$k_i'(k_i' - 1) = \left(\frac{n}{2} - 1\right)\lambda_i, \quad i = 1,2 \tag{3.2}$$

$$k_i^\star(k_i^\star - 1) = \left(\frac{n}{2} - 1\right)(\lambda - \lambda_i), \quad i = 1,2. \tag{3.3}$$

Adding (3.2) and (3.3), we have

$$k_i'(k_i' - 1) + k_i^\star(k_i^\star - 1) = \left(\frac{n}{2} - 1\right)\lambda, \quad i = 1,2. \tag{3.4}$$

Using $k_i' = \frac{\lambda(\rho+1)-k_i}{\rho-1}$, $k_i^\star = \frac{\rho k_i - \lambda(\rho+1)}{\rho-1}$, and (3.1) in (3.4), we have

$$k_i^2 - (n+1)k_i + \frac{\lambda}{(\rho^2+1)^2}[\lambda(1+\rho)^4 + (\rho-1)^2(1+\rho+\rho^2)] = 0, \quad i = 1,2.$$

Solve to get

$$k_1 = \frac{1}{2}\left[n + 1 + \frac{\rho - 1}{\rho^2 + 1}\sqrt{(\rho-1)^2 + 4\lambda\rho}\right] \tag{3.5}$$

$$k_2 = \frac{1}{2}\left[n + 1 - \frac{\rho - 1}{\rho^2 + 1}\sqrt{(\rho-1)^2 + 4\lambda\rho}\right]. \tag{3.6}$$

Let $\rho = \frac{b}{a}$, where $a, b \in \mathbf{Z}$ (the set of integers), $(a,b) = 1$. Then

$$\frac{\rho - 1}{\rho^2 + 1}\sqrt{(\rho-1)^2 + 4\lambda\rho} = \frac{b - a}{a^2 + b^2}\sqrt{(b-a)^2 + 4\lambda ab} \tag{3.7}$$

is an odd integer. We claim $b - a$ is odd. Otherwise, both a and b are odd. Since $a^2 + b^2 = (b-a)^2 + 2ab$, we have $4 \nmid (a^2 + b^2)$. Then the right side of (3.7) is even. Contradiction. Now by (3.7), $\exists\, 0 < z \in \mathbf{Z}$, such that

$$(b-a)^2 + 4\lambda ab = z^2 \tag{3.8}$$

and

$$(a^2 + b^2) \mid z \tag{3.9}$$

(since $(b-a, a^2+b^2)=1$). Multiplying (3.8) by z^2, we get

$$z^2(b-a)^2 + 4\lambda^2 a^2 b^2 = (z^2 - 2\lambda ab)^2.$$

Hence $\exists\, x, y \in \mathbf{Z}$, $x > y > 0$, such that

$$\begin{aligned} z(b-a) &= x^2 - y^2 \\ 2\lambda ab &= 2xy \\ z^2 - 2\lambda ab &= x^2 + y^2 \end{aligned}$$

or

$$z = x + y \tag{3.10}$$

$$\lambda = \frac{xy}{ab} \tag{3.11}$$

$$b - a = x - y. \tag{3.12}$$

Now we can express n, $k_1 - \lambda$, $k_2 - \lambda$ in terms of a, b, x, y:

$$n = \frac{2xy(a+b)^2}{ab(a^2+b^2)} - \frac{2ab}{a^2+b^2} \tag{3.1'}$$

$$k_1 - \lambda = \frac{x(x+y)}{a^2+b^2} \tag{3.13}$$

$$k_2 - \lambda = \frac{y(x+y)}{a^2+b^2}. \tag{3.14}$$

Hence,

$$k_1' = \frac{1}{\rho - 1}[\lambda\rho - (k_1 - \lambda)] = \frac{x(b^2 y - a^2 x)}{a(b-a)(a^2+b^2)} \tag{3.15}$$

$$k_2' = \frac{1}{\rho - 1}[\lambda\rho - (k_2 - \lambda)] = \frac{y(b^2 x - a^2 y)}{a(b-a)(a^2+b^2)}. \tag{3.16}$$

By (3.2),

$$\begin{aligned} \lambda_2 - \lambda_1 &= \frac{1}{(\frac{n}{2} - 1)}[k_2'(k_2' - 1) - k_1'(k_1' - 1)] \\ &= \frac{ab(a+b)(x+y)(2xy - ab)}{(a^2+b^2)[xy(a+b)^2 - ab(a^2+b^2+ab)]} \in \mathbf{Z}. \end{aligned}$$

Multiplying by $(a+b)$, we have

$$\begin{aligned}&\frac{ab(a+b)^2(x+y)(2xy-ab)}{(a^2+b^2)[xy(a+b)^2-ab(a^2+b^2+ab)]}\\ &\quad=\frac{ab(x+y)[2xy(a+b)^2-ab(a+b)^2]}{(a^2+b^2)[xy(a+b)^2-ab(a^2+b^2+ab)]}\\ &\quad=\frac{2ab(x+y)}{a^2+b^2}+\frac{ab(x+y)[2ab(a^2+b^2+ab)-ab(a+b)^2]}{(a^2+b^2)[xy(a+b)^2-ab(a^2+b^2+ab)]}\\ &\quad=\frac{2ab(x+y)}{a^2+b^2}+\frac{a^2b^2(x+y)}{xy(a+b)^2-ab(a^2+b^2+ab)}\in\mathbf{Z}.\end{aligned}$$

But $\frac{2ab(x+y)}{a^2+b^2}\in\mathbf{Z}$ by (3.9), hence,

$$\frac{a^2b^2(x+y)}{xy(a+b)^2-ab(a^2+b^2+ab)}\in\mathbf{Z}.$$

Therefore,

$$xy(a+b)^2-ab(a^2+b^2+ab)\le a^2b^2(x+y)$$

which implies

$$(x+y)\left[\frac{x+y}{4}(a+b)^2-a^2b^2\right]\le ab(a^2+b^2+ab)+\frac{(b-a)^2(a+b)^2}{4}<\frac{(a+b)^4}{4}.$$

By (3.9), $x+y\ge a^2+b^2$. Then the above inequality implies

$$(a^2+b^2)\frac{(a+b)^4}{16}<\frac{(a+b)^4}{4}.$$

Contradiction. ∎

References

1. Ákos Seress, *A numerical characterization of type-1 λ-designs.* (to appear).
2. W.G. Bridges, *Some results on λ-designs*, J. Combin. Theory **8** (1970), 350-360.
3. W.G. Bridges, *A characterization of type-1 λ-designs*, J. Combin. Theory Ser. A **22** (1977), 361-367.
4. W. G. Bridges and E.S. Kramer, *The determination of all λ-designs with* $\lambda = 3$, J. Combin. Theory **8** (1970), 343-349.
5. N.G. deBruijn and P. Erdös, *On a combinatorial problem*, Indagationes Math. **10** (1948), 421-423.
6. E.S. Kramer, *On λ-designs*, Ph.D. Dissertation (June, 1969), University of Michigan.
7. E.S. Kramer, *On λ-designs*, J. Combin. Theory Ser. A **16** (1974), 57-75.
8. H.J. Ryser, *An extension of a theorem of deBruijn and Erdös on combinatorial designs*, J. Algebra **10** (1968), 246-261.
9. N.M. Singhi and S.S. Shrikhande, *On the λ-design conjecture*, Utilitas Mathematica **9** (1976), 301-318.
10. D.R. Woodall, *Square λ-linked designs*, Proc. London Math. Soc. (3)**20** (1970), 669-687.