Decidability of Configuration Theorems in Projective Planes and Other Incidence Structures

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Abstract. The problem of recognizing if a configuration theorem is valid in a given class C of incidence structures is equivalent to the problem of deciding, for an arbitrary finite incidence structure I, if I is embeddable in some incidence structure in C.

Evans proved that the word problem is solvable for a given variety of universal algebras if and only if there exists an algorithm to decide, for any finite partial algebra A, if A can be embedded in some algebra of the variety [1]. It is shown below that the problem of deciding the validity of configuration theorems in a given class of incidence structures can be approached in the same spirit.

An *incidence structure* I is a triple (V, B, R), where $V = V(I)$ and $B = B(I)$ are sets (the sets of *points* and *lines*), and $R = R(I)$ is any subset of $V \times B$, called the *incidence relation*. For $(P, L) \in R$ we shall also write $P \in L$. No restriction will be made to finite sets, unless explicitly stated. The members of V and B are called the *elements* of I. If no two distinct points are incident with more than one line, then I is a *partial projective plane* (called *configuration* in Hughes and Piper [2]).

Isomorphism of incidence structures is defined in the obvious way. An incidence structure (V, B, R) is a *substructure* of (V', B', R') if $V \subseteq V'$, $B \subseteq B'$ and $R = R' \cap (V \times B)$. An incidence structure I *is embeddable* in I' if it is isomorphic to some substructure of I'. Given a class C of incidence structures, we say that the *embeddability problem is solvable for* C if there is an algorithm to decide, for any finite incidence structure I, if there exists a member I' of C such that I is embeddable in I'.

We construct a formal language for incidence structures. Let $\{p_1, p_2, \dots\}$ and $\{\ell_1, \ell_2, \dots\}$ be two disjoint countably infinite sets of symbols, called *p-variables* and *ℓ-variables*. An *atomic formula* is an expression of the form $p_i = p_j$, $\ell_i = \ell_j$, or $p_i \in \ell_j$. An *open sentence* is a formal expression obtained by combining atomic formulas by means of the propositional connectives & (and, conjunction), v (or), $\neg$ (not, negation), $\Rightarrow$ (implies). For negations of atomic formulas we shall write $p_i \neq p_j$, $\ell_i \neq \ell_j$ and $p_i \notin \ell_j$. A particular class of open sentences are the

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configuration theorems of Marshall Hall, introduced in [4] in a less formal way. Formally, a configuration theorem is an open sentence of the form

$$(\alpha_1 \ \& \ \ldots \& \ \alpha_n) \Rightarrow p_i \in \ell_j,$$

where $\alpha_1, \ldots, \alpha_n$ are arbitrary atomic formulas or negations of atomic formulas. As noted by Marshall Hall, the theorems of Pappus and Desargues can be written as configuration theorems. An *interpretation* of the variables in an incidence structure I is a function assigning to each p-variable a point of I and to each ℓ-variable a line of I. Given an interpretation in I, each atomic formula becomes true or false, and hence each open sentence becomes true or false. An open sentence τ is *valid* in I if it is true for every interpretation of the variables in I. For example, the theorem of Desargues is valid in every projective plane coordinatizable over a skewfield. Generally, given a class C of incidence structures, τ is said to be *valid* in C if it is valid in every structure I belonging to C. We say that the *validity problem for open sentences in C is solvable* if there is an algorithm to decide, for any open sentence τ, if τ is valid in C. If such a decision algorithm exists for configuration theorems, then we say that the *validity problem for configuration theorems in C is solvable*. This is a priori a weaker condition.

Let a finite incidence structure I be given, with distinct points $P_1, \ldots P_n$ and distinct lines $L_1, \ldots L_m$. For each i, j, $1 \leq i < j \leq n$ consider the formula $p_i \neq p_j$. Also for each i, j, $1 \leq i < j \leq m$, consider $\ell_i \neq \ell_j$. Let δ be the conjunction of all the $p_i \neq p_j$ and $\ell_i \neq \ell_j$. Further, for each $(P_i, L_j) \in R(I)$, consider the atomic formula $p_i \in \ell_j$ and for each $(P_i, L_j) \notin R(I)$, consider the formula $p_i \notin \ell_j$. Let ρ be the conjunction of all the $p_i \in \ell_j$ and $p_i \notin \ell_j$. Then ρ is the conjunction of nm atomic formulas or negations of atomic formulas. Also δ is a conjunction of negations of atomic formulas. Let us call the open sentence $(\delta \& \rho)$ the *descriptor* of I. The negation of the descriptor is also an open sentence. We can now observe that if I' is another incidence structure, then I is embeddable in I' if and only if the negation of the descriptor of I is not valid in I'.

The following is fairly obvious, but shall be useful:

Proposition 1. *It is decidable, for any open sentence τ and any finite incidence structure I, if τ is valid in I.*

Proposition 2. *Let C be any class of incidence structures and let τ be an open sentence containing n distinct variables. Then τ is valid in C if and only if it is valid in every finite incidence structure I having at most n elements and embeddable in C.*

Proof: Without loss of generality we can assume that the variables occurring in τ are $x_1, \ldots, x_n$. Assume τ is not valid in C. Then there is an interpretation of the variables in some incidence structure J in C, assigning elements $\overline{x}_i$ of J to x_i for each $1 \leq i \leq n$, under which τ is not valid. Let I be the restriction of J to the elements $\overline{x}_1, \ldots, \overline{x}_n$. Clearly τ is not valid in I.

The converse is obvious. ∎

Proposition 3. *Let C be any class of incidence structures. If the embeddability problem is solvable for C, then the validity problem for open sentences in C is solvable.*

Proof: Assume that the embeddability problem is solvable. Let τ be an open sentence containing n distinct variables. Construct all finite incidence structures with at most n elements. For each of them decide if τ is valid in it, and let $I_1, \ldots, I_k$ be those for which the answer is no. By assumption we can determine which I_i are embeddable in C. Then τ is valid in C if and only if none of $I_1, \ldots, I_k$ is embeddable in C. ∎

Let N be the set of non-negative integers. A subset S of N^2 is *recursive* if there is an algorithm to decide, for any $(x, y) \in N^2$, whether $(x, y) \in S$. For an arbitrary set $S \subseteq N^2$, let us define $\overline{S} = \{(x,y) \in N^2\colon \exists\,(a,b) \in S,\, x \leq a,\, y \leq b\}$. S is called *hereditary* if $S = \overline{S}$. We shall need the following lemma. The proof uses some elementary facts about recursive sets that can either be discovered by the reader or found in Rogers [**8**].

Lemma. *Every hereditary subset of N^2 is recursive.*

Proof: The claim is obvious if $S = N^2$ or if S is finite. Assume that $S \neq N^2$ and that S is infinite. Let

$$S_1 = \{x \in N\colon \quad \forall\, b \in N \quad \exists\, y \geq b\, (x,y) \in S\},$$
$$S_2 = \{y \in N\colon \quad \forall\, a \in N \quad \exists\, x \geq a\, (x,y) \in S\}.$$

Since $S \neq N^2$, both S_1 and S_2 must be finite. Also $T = S \cap (N \backslash S_1) \times (N \backslash S_2)$ must be finite. We have $S = (S_1 \times N) \cup (N \times S_2) \cup \overline{T}$. From the finiteness of S_1, S_2 and T it follows that $S_1 \times N$, $N \times S_2$ and $\overline{T}$ are all recursive. Consequently their union S must also be recursive. ∎

Borrowing from the notation of graph theory, let $K_{n,m}$ denote an incidence structure with n points, m lines and such that all points are incident with all lines. For any class C of incidence structures, the set of pairs (n, m) such that $K_{n,m}$ is embeddable in C is a hereditary subset of N^2, and therefore it is recursive. This means that it is decidable, for any n, m, whether $K_{n,m}$ is embeddable in C.

Proposition 4. *Let C be any class of incident structures. If the validity problem for configuration theorems in C is solvable, then the embeddability problem is solvable for C.*

Proof: Assume that the validity problem for configuration theorems in C is solvable. Let a finite incidence structure I be given, with distinct points $P_1, \ldots, P_n$

and distinct lines $L_1, \ldots, L_m$. If all points are incident with all lines, then I is isomorphic to some $K_{n,m}$ and by the previous remark we can decide whether I is embeddable in C. Otherwise we can assume, without loss of generality, that P_1 is not incident with L_1. Let α be the descriptor of I. The structure I is embeddable in C if and only if for some member I' of C, $\neg\, \alpha$ is not valid in I', that is, if and only if $\neg\, \alpha$ is not valid in C. But the descriptor α being a conjunction of atomic formulas or negations of atomic formulas one of which is precisely $p_1 \notin \ell_1$, the open sentence $\neg\, \alpha$ is logically equivalent to the configuration theorem $\alpha \Rightarrow p_1 \in \ell_1$ (that is, true for exactly the same interpretation of variables as $\alpha \Rightarrow p_1 \in \ell_1$). Therefore I is embeddable in C if and only if this configuration theorem is not valid in C. The Proposition follows. ∎

Since configuration theorems are a subset of open sentences, Propositions 3 and 4 combined yield the following:

Proposition 5. *For any class C of incidence structures the following statements are equivalent:*

(i) *the embeddability problem is solvable for C,*
(ii) *the validity problem for open sentences in C is solvable,*
(iii) *the validity problem for configuration theorems in C is solvable.*

Let now C be the class of all projective planes. By a construction of Marshall Hall [**4**], a finite incidence structure I can be embedded in C if, and obviously only if, I is a partial projective plane. Thus the embeddability problem is solvable for projective planes and therefore:

Proposition 6. *The validity problem for open sentences in projective planes is solvable.*

It is not known whether every finite partial projective plane can be embedded in a finite projective plane. (The problem was raised by Evans [**2**]). If it were so, then the validity problem for open sentences in finite projective planes would be solvable. However, a result of Quackenbush [**7**] implies that every finite partial projective plane can be embedded in a (finite) balanced incomplete block design with $\lambda = 1$, also called $(2, k)$ Steiner system. Since every substructure of such a design is a partial projective plane, we have the following:

Proposition 7. *The validity problem for open sentences in balanced incomplete block designs with $\lambda = 1$ is solvable.*

It was shown by Treash that every partial Steiner triple system can be embedded in a Steiner triple system [**9**]. (See also Lindner [**6, 3**].) This implies the following result.

Proposition 8. *The validity problem for open sentences in Steiner triple systems is solvable.*

Remark: The above proposition is true regardless whether Steiner triple systems are understood to be finite, or if they can be also infinite. It can also be viewed as a generalization of the solvability of the word problem for Steiner quasigroups. This solvability is a consequence of Evans' theorem mentioned in the introduction, via the embedding theorem of Treash for Steiner triple systems.

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