Bipartite Analogues of Split Graphs and Related Topics

Howard B. Frost
Department of Mathematics
University of Arizona
Tucson, AZ 85721

Michael S. Jacobson
Department of Mathematics
University of Louisville
Louisville, KY 40292

Jerald A. Kabell
Department of Computer Science
Central Michigan University
Mt. Pleasant, MI 48859

F.R. McMorris
Department of Mathematics
University of Louisville
Louisville, KY 40292

Abstract. Following up on the bipartite analogue of an interval graph developed in a previous work, we investigate several possibilities for a bipartite analogue of the concept of a split graph. We also give bipartite analogues of threshold graphs and of perfect graphs.
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1. Introduction.

In [6], Harary, Kabell and McMorris defined a bipartite intersection graph as a bipartite graph in which each partite set of vertices is represented by some family of distinct subsets of a universal set, and two vertices are adjacent if and only if their corresponding sets are in different families and have a nonempty intersection. This led naturally to the notion of a bi-interval graph, and a natural analogue of the results of Lekkerkerker and Boland[8]. Attempts to extend a bipartite intersection graph analogy to bi-chordal graphs (bipartite graphs with no induced cycles C_n for $n > 4$) following the result of Buneman[1], Gavril[4], and Walter[12] failed when it was shown, by applying a result of McMorris and Shier[11] on the intersection graph of subtrees of a star $K_{1,n}$. Upon reflection, this is not really surprising, as it corresponds to the obvious observation that *every* graph is an intersection graph of complete subgraphs of a complete graph. (A star is the bipartite version of a complete graph when we require all *edges* to be adjacent instead of all vertices

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adjacent. Of course, the other bipartite version of a complete graph is the complete bipartite graph which corresponds to the case of all edges *present*.)

We now investigate bipartite analogues for split, threshold and perfect graphs.

2. Bi-split graphs.

The natural question left unanswered by [6] was, of course, just how far can such an analogy extend? As noted above, a result about split graphs caused our search for a bipartite intersection graph characterization of bi-chordal graphs to break down. This suggests that class of graphs as a possible starting point, and so we begin by examining various definitions for bi-split graphs. The class of split graphs was first characterized by Foldes and Hammer[3] as those having the property that the vertices may be partitioned into two sets, one of which induces a complete subgraph while the other induces a set of independent vertices. They derived from this two equivalent conditions. First, split graphs are precisely the chordal graphs (no induced C_n for $n > 3$) that have chordal complements. Second, split graphs are characterized by the set of forbidden induced subgraphs $\{C_4, C_5, 2K_2\}$. In [11], it was shown that they are also precisely the intersection graphs of subtrees of $K_{1,n}$ for some n.

Upon which of these properties should the definition of bi-split be based? We would like to retain as many of these properties as possible: We want our bi-split graph to have a bi-split bipartite complement , to be bi-chordal and have a bi-chordal bipartite complement (*i.e.* no induced C_6, C_8, or $3K_2$), and we want our bi-split definition to be hereditary (i.e., every induced subgraph of a bi-split graph should be bi-split). We have already seen [6] that we cannot use the characterization in [11] since it yields all bipartite graphs.

McKee[10] gives us logical reasons as to why the various "intuitive" bipartite analogies often break down, so let us start by looking at a definition for bi-split very similar to one proposed by him. First recall that the bipartite complement of a bipartite graph is the bipartite graph with the same bipartition, but with the complement edge set. An independent bipartite graph is a bipartite graph with no edges. Now suppose we require a bipartite graph to have its vertex set partitioned into a complete bipartite graph and an independent bipartite graph. Keeping in mind that the bipartition for such a graph is fixed, this property is preserved under bipartite complementation and is hereditary, but is not necessarily bi-chordal so it does not accomplish what we would like.

Since edge conditions play an important role in bi-chordal graphs[5], it is natural to try various edge partitions. Unfortunately, there does not seem to be a choice for which the property thus defined is hereditary, or preserved under bipartite complementation. Thus the only recourse we have is to try to find some vertex partition that "forces" our bi-split graph to be bi-chordal. The following definition does the trick. Call a bipartite graph "bi-split" if and only if the vertex set can be partitioned into a complete bipartite graph and an independent bipartite graph, and does not contain $2K_2$ as an induced subgraph. Because C_6, C_8, and $3K_2$ all contain $2K_2$, this guarantees that our bi-split graphs are bi-chordal and have bichordal bipartite complements. Thus we have a definition that satisfies all our requirements. We have the following result.

Proposition. *A bipartite graph is bi-split if and only if it does not contain* $2K_2$ *as an induced subgraph.*

Proof: One way comes from the definition of bi-split while the other follows from the observation that if G is a bipartite graph that contains no $2K_2$, then G has at most one nontrivial connected component G_1 and in G_1, there exists vertices in both partite sets that are adjacent to all the vertices in the other part. To see this, simply take a vertex x of maximum degree in one of the partite sets X. Suppose that there exists a vertex y in the other set Y that is not adjacent to x. Since G_1 is connected, there exists a vertex z in X adjacent to y. Since x has maximum degree, there is a w in Y adjacent to x but not z. But now the edges xw and zy induce $2K_2$.

Now if G contains no nontrivial component, then the "split" is obvious. Otherwise, consider any vertex in one of the parts in the nontrivial connected component which is adjacent to all the vertices in the other part. This defines the complete part of the "split", while the remaining vertices are independent and hence G is bi-split.

From this result, it can be easily seen that bi-split graphs are precisely bipartite graphs with one non-trivial component that is a complete bipartite graph with possibly the center of a star identified with one vertex in each partite set.

3. Bi-threshold graphs.

The class of graphs most closely related to the split graphs are the threshold graphs introduced by Chvatal and Hammer[2]. These are the graphs for which there exists a weighting of the vertices such that a set of vertices is independent if and only if the sum of its weights does not exceed some fixed threshhold value. It is also known that a graph is a threshold graph if and only if it does not contain any of the graphs from $\{C_4, P_4, 2K_2\}$ as induced subgraphs.

Recall that two edges are *adjacent* if they share a common vertex and *nonadjacent* otherwise. We say that a set of edges is *independent* if the edges in the set are pairwise nonadjacent. Call a bipartite graph *bi-threshold* if and only if there exists a weighting of the edges such that a set of edges is independent precisely when the sum of its weights does not exceed some fixed threshold value. Call a $K_{1,n}$ with exactly one edge subdivided a *star with one subdivided edge.* Such a graph is shown in Figure 1 with labels on the edges added.

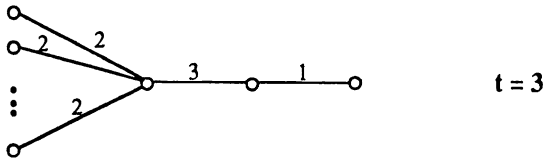

Figure 1

We can now show that very few graphs are bi-threshold.

Theorem. *A connected bipartite graph is bi-threshold if and only if it is a star with possibly one subdivided edge.*

Proof: Let G be a bi-threshold graph. We first show that G does not contain C_4 as an induced subgraph. Suppose it does, and let the weights of the edges of C_4 be a, b, c, d in cyclic order. Assume the threshold value is t. Then it must be the case that $a + b + c + d \leq 2t$. On the other hand, $a + b > t$, and $c + d > t$, so we also have $a + b + c + d > 2t$, a contradiction. A similar argument will show that G cannot contain an induced path of length 4. Thus G must be a tree of diameter 3 or less. (*i.e.*, the "largest" that G can be is two stars that share a common edge.) It remains to show that G cannot contain two adjacent vertices of degee three or more. Suppose uv is an edge and $deg(u), deg(v) \geq 3$. Label the weights of two edges incident with u (other than uv) by a and b, and two edges incident with v (other than uv) by c and d. Then we get exactly the same contradiction as above.

The converse follows from the weighting of the star with one subdivided edge given in Figure 1, and the observation that every induced subgraph of a bi-threshold graph is bi-threshold.

This theorem together with the structural characterization of bi-split graphs gives us that all bi-threshold graphs are bi-split graphs.

4. Bi-perfect graphs.

Since chordal graphs, and hence split and threshold graphs, are perfect, we will now analyse this concept in the bipartite setting. Recall that a graph G is *perfect* if and only if for every induced subgraph H of G, the chromatic number of H is equal to the clique number of H (the number of vertices in a maximum clique of H.) Lovasz[9] showed that this is equivalent to the complement of G being perfect. That is, for every induced subgraph H of G, the minimum number of complete subgraphs needed to partition the vertices of H is equal to the order of the largest independent set of vertices of H. Two bipartite versions of chromatic number and clique number are edge-chromatic number and maximum degree respectively and Konig[7] showed that these parameters are equal in all bipartite graphs. The minimum number of complete subgraphs needed to partition the vertices of a graph translates to the minimum number of stars($K_{1,n}$'s) required to partition the vertices of a bipartite graph, while the order of the largest independent set of vertices of a graph translates to the size of a maximum matching of a bipartite graph. In order to have the standard relationship between these parameters, we make the following definition. The *star cover number* of a bipartite graph G, $s(G)$, is the minimum number of stars which partition the non-isolated vertices of G. If we let $m(G)$ denote the size of a maximum matching of G, then clearly we have $s(G) \leq m(G)$ for all bipartite graphs G. Also note that if we let I_G denote the set of isolated vertices of G, then $s(G)$ is just the classical domination number of $G - I_G$.

Call a bipartite graph G *weakly bi-perfect* if $s(G) = m(G)$. As in the case of weakly perfect graphs, weakly bi-perfect is not an hereditary property. For example, if G is an arbitrary bipartite graph with n vertices, we can form the graph G^* by adding n new vertices, each adjoined to exactly one vertex of G. It is easy to see that G^* is bipartite and $s(G^*) = m(G^*)$. Thus no forbidden subgraph charactization of weakly bi-perfect graphs is possible. A bipartite graph is *bi-perfect* if every induced subgraph of G is weakly bi-perfect. We first note that an analogue of Lovasz's theorem does not exist, for P_6 (the path with 6 vertices) is not bi-perfect whereas the bipartite complement of P_6 is $P_4 \cup K_2$ which is bi-perfect. We now characterize bipartite graphs that are bi-perfect.

Theorem. *A bipartite graph is bi-perfect if and only if it does not contain P_6 as a subgraph.*

Proof: Let G be a bipartite graph that contains P_6 as a subgraph, and let H be an induced subgraph containing a P_6 in G. Clearly, $s(H) \leq 2$, while $m(H) = 3$ so that G is not bi-perfect.

For the converse, suppose that G is a bipartite graph that does not contain a P_6. Since the only possible cycles in G are C_4's, we break the remainder of the proof into two cases: G a forest and G containing a C_4.

Assume G is a forest, so each component of G is a tree. Recall that the center of a tree consists of either one central vertex or two adjacent central vertices. Since G does not contain P_6, the vertices in the center of each component tree are a distance at most two from all the other vertices in the component. It follows that each component is of the form of a star with an arbitrary number of vertices of degree one adjoined. Such graphs are easily seen to be bi-perfect so that every component of G is bi-perfect and hence G itself is bi-perfect.

Now suppose that G' is a componet of G that contains a C_4. Let x, y, z, w be vertices in G' that induce C_4, with xy, yz, zw, wx edges. If there were additional vertices adjacent to two consecutive vertices on this C_4, then P_6 world be a subgraph of G. Thus we may assume that each vertex in G' not on this C_4 is adjacent only to x or to z or to both. Using this we have if H is an induced subgraph of G', and C_4 is a subgraph of H, then $s(H) = m(H) = 2$ and so H is weakly bi-perfect. If C_4 is not a subgraph of H then H is a forest and our previous case gives that H is weakly bi-perfect. Therefore we have again shown that every component of G is bi-perfect and the proof is complete.

Corollary. *A bipartite graph G is bi-perfect if and only if every subgraph of G is weakly bi-perfect.*

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