On the Vulnerability of Cycle Permutation Graphs

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Abstract. Several measures of the vulnerability of a graph have been examined previously. These include connectivity, toughness, binding number, and integrity. In this paper the authors examine the toughness and binding number of cycle permutation graphs (sometimes called generalized prisms). In particular, we determine the binding number for any cycle permutation graph and find upper and lower bounds for the toughness of such graphs. A class of cycle permutation graphs where the lower bound is always achieved and a class of cycle permutation graphs (which are also generalized Petersen graphs) where the lower bound is never achieved are also presented.

Preliminaries.

Measures of the vulnerability of graphs are currently of growing interest among graph theorists and network designers. In particular, much has been done recently on the toughness and binding number of different classes of graphs since these parameters are more sensitive to the structure of the graph than is the connectivity of the graph.

The toughness of a graph G, $t(G)$, is defined by $t(G) = \min\{\frac{|S|}{\omega(G-S)}\}$, where the minimum is taken over all disconnecting subsets S of $V(G)$ and $\omega(G-S)$ is the number of components of $G-S$. The binding number of G, $b(G)$, is defined by $b(G) = \min\{\frac{|N(S)|}{|S|}\}$, where the minimum is taken over all non-empty subsets S of $V(G)$ such that $N(S) \neq V(G)$, where $N(S)$ is the open neighborhood of S. The connectivity of G, $\kappa(G)$, is the order of a minimum disconnecting set of vertices.

Given a graph G with vertices labeled $1, 2, \ldots, n$ and a permutation α in S_n, the permutation graph $P_\alpha(G)$ is obtained by taking two copies of G, say G_x with vertices $x_1, x_2, \ldots, x_n$ and G_y with vertices $y_1, y_2, \ldots, y_n$, along with a set of permutation edges joining x_i in G_x and $y_{\alpha(i)}$ in G_y. Note that permutation graphs are in some ways generalizations of the Cartesian product since if $\alpha = (1)$ then $P_\alpha(G) \cong G \times K_2$. If the graph C_n is an n-cycle labeled $1, 2, \ldots, n$ where vertex i is joined to $i+1$ and $i-1 \pmod{n}$ then $P_\alpha(C_n)$ is called a cycle permutation graph (sometimes called a generalized prism since if $\alpha = (1)$ the graph $P_\alpha(C_n)$

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is the n-prism). Cycle permutation graphs have been studied extensively as generalizations of the Petersen graph.

Another class of graphs which are generalizations of the Petersen graph are the generalized Petersen graphs, denoted $G(n, k)$. For integers n and k, with $1 \leq k \leq n-1$ and $2k \neq n$, the graph $G(n, k)$ has vertex set $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and edge set consisting of all pairs of the form (u_i, u_{i+1}), (u_i, v_i), or (v_i, v_{i+k}), where subscripts are read modulo n and the symbol "n" is zero [6]. The graph induced by the u_i is called the outer rim and the graph induced by the v_i is called the inner rim(s). The edges of the form (u_i, v_i) are called the spokes. The Petersen graph in this notation is $G(5,2)$. Of particular interest will be those generalized Petersen graphs which have a single inner rim which is an n-cycle, that is, $gcd(n, k) = 1$. In this case there is a natural correspondence between the outer rim, the inner rim, and the spokes of $G(n, k)$ and the cycle C_x, the cycle C_y, and the permutation edges of $P_\alpha(C_n)$, respectively.

In this paper the authors find the binding number of cycle permutation graphs and determine upper and lower bounds for the toughness of these graphs. A class of cycle permutation graphs where the lower bound is always achieved and a class of cycle permutation graphs (which are also generalized Petersen graphs) where the lower bound is never achieved are also determined.

For the remainder of this paper let C_x and C_y be the two copies of C_n in the cycle permutation graph $P_\alpha(C_n)$. For a set S of vertices in $P_\alpha(C_n)$ let $S_x = S \cap V(C_x)$ and $S_y = S \cap V(C_y)$. Since S_x and S_y form a partition of S, $|S_x| + |S_y| = |S|$. Moreover, let $N(S)$ be the (open) neighborhood of S in $P_\alpha(C_n)$ and let $N(S\colon H)$ be the neighborhood of S when restricted to the subgraph H, that is, $N(S\colon H) = N(S) \cap V(H)$. The subgraph induced by S shall be denoted $\langle S \rangle$.

Toughness.

We begin with some useful elementary propositions.

Proposition 1. *If S is a disconnecting set of a cycle C then $|S| \geq \omega(C - S)$. Moreover, if S contains two vertices which are adjacent in C then $|S| > \omega(C-S)$.*

Proposition 2. *The following are equivalent:*

(i) *$P_\alpha(C_n)$ is bipartite;*

(ii) *n is even and $\alpha(1), \alpha(3), \ldots, \alpha(n-1)$ have the same parity; and*

(iii) *$P_\alpha(C_n)$ contains an independent set S of order n. Furthermore, in this case, the set $V(P_\alpha(C_n)) - S$ is also an independent set of order n.*

Proposition 3 ([3]). $\kappa(P_\alpha(C_n)) = 3$.

Now we determine a lower bound for $t(P_\alpha(C_n))$.

Theorem 4.

$$t(P_\alpha(C_n)) \geq \begin{cases} 1 & \text{if } n \text{ is even} \\ \frac{n}{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

Proof: Let S be a disconnecting set of $P_\alpha(C_n)$ such that $t(P_\alpha(C_n)) = \frac{|S|}{\omega(P_\alpha(C_n)-S)}$. Then S_x and S_y are both non-empty; otherwise S is not a disconnecting set. We examine three cases depending upon the value of $|S|$.

Case (i): Suppose $3 \le |S| < n$. By Proposition 1, $|S_x| \ge |\omega(C_x - S_x)|$ and $|S_y| \ge |\omega(C_y - S_y)|$. Since $|S| < n$, there exist vertices $u \in C_x$ and $v \in C_y$ such that $u \notin S_x$, $v \notin S_y$, and (u, v) is an edge of $P_\alpha(C_n)$. So $\omega(P_\alpha(C_n) - S) < \omega(C_x - S_x) + \omega(C_y - S_y) \le |S|$. Thus $t(P_\alpha(C_n)) = \frac{|S|}{\omega(P_\alpha(C_n)-S)} \ge \frac{|S|}{|S|-1} > \frac{n}{n-1}$ since $|S| < n$.

Case (ii): Suppose $|S| > n$. Then $\omega(P_\alpha(C_n) - S) \le |V(P_\alpha(C_n) - S)| < n$ and $t(P_\alpha(C_n)) = \frac{|S|}{\omega(P_\alpha(C_n)-S)} > \frac{n}{\omega(P_\alpha(C_n)-S)} \ge \frac{n}{n-1}$.

Case (iii): Suppose $|S| = n$. First we show that if S_x or S_y contain adjacent vertices in C_x or C_y, respectively, then $\frac{|S|}{\omega(P_\alpha(C_n)-S)} \ge \frac{n}{n-1}$. Without loss of generality assume that S_x contains adjacent vertices x_i and x_{i+1}. By Proposition 1, $\omega(C_x - S_x) < |S_x|$ and $\omega(C_y - S_y) \le |S_y|$ and so $\omega(P_\alpha(C_n) - S) \le \omega(C_x - S_x) + \omega(C_y - S_y) < |S|$. Thus $\frac{|S|}{\omega(P_\alpha(C_n)-S)} = \frac{n}{\omega(P_\alpha(C_n)-S)} \ge \frac{n}{n-1}$.

If n is odd, either S_x or S_y contains two adjacent vertices and $t(P_\alpha(C_n)) = \frac{|S|}{\omega(P_\alpha(C_n)-S)} \ge \frac{n}{n-1}$. If n is even, $|V(P_\alpha(C_n) - S)| = n$ and so $\omega(P_\alpha(C_n) - S) \le n$. Thus $t(P_\alpha(C_n)) = \frac{|S|}{\omega(P_\alpha(C_n)-S)} \ge 1$. ∎

Furthermore, this bound is sharp in the sense that for any n-cycle C_n there is always at least one permutation $\alpha = (1)$, the identity permutation, for which this lower bound is achieved.

Corollary 5.

$$t(P_{(1)}(C_n)) = \begin{cases} 1 & \text{if } n \text{ is even} \\ \frac{n}{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

Proof: The set $S = \{x_1, x_3, \ldots, x_{n-1}, y_2, y_4, \ldots, y_n\}$ forms a disconnecting set of order n whose removal from $P_\alpha(C_n)$ leaves:

(i) a subgraph consisting of n isolated vertices if n is even; or
(ii) a subgraph consisting of $n-2$ isolated vertices and one copy of K_2 if n is odd.

∎

Moreover, we may characterize exactly those cycle permutation graphs with toughness equal to 1.

Theorem 6. *$t(P_\alpha(C_n)) = 1$ if and only if $P_\alpha(C_n)$ is bipartite.*

Proof: Let $P_\alpha(C_n)$ be bipartite. Then n is even and $\alpha(1), \alpha(3), \ldots, \alpha(n-1)$ have the same parity. So $S = \{x_1, x_3, \ldots, x_{n-1}, y_2, y_4, \ldots, y_n\}$ and $T = \{x_2, x_4, \ldots, x_n, y_1, y_3, \ldots, y_{n-1}\}$ are independent sets in $P_\alpha(C_n)$. But $P_\alpha(C_n) - S = \langle T \rangle$ which implies $\omega(P_\alpha(C_n) - S) = n$ and $t(P_\alpha(C_n)) = 1$.

Now suppose $t(P_\alpha(C_n)) = 1$. In the proof of Theorem 3 it was shown that this can occur only when:

(i) n is even;
(ii) $|S| = n$;
(iii) $\omega(P_\alpha(C_n) - S) = n$; and
(iv) S_x and S_y contain no adjacent vertices in C_x and C_y, respectively.

Thus $|S_x| = |S_y| = \frac{n}{2}$. Let $T = V(P_\alpha(C_n)) - S$. Since $|T| = n$ and $\omega(\langle T \rangle) = \omega(P_\alpha(C_n) - S) = n$, T is an independent set of order n in $P_\alpha(C_n)$ and thus $P_\alpha(C_n)$ is bipartite. ∎

Next, we determine a class of cycle permutation graphs which are also generalized Petersen graphs for which the lower bound of Theorem 4 is never achieved.

Proposition 7. *If* $gcd(n, k) = 1$, *then* $gcd(n, n-k) = 1$.

Proposition 8 ([6]). $G(n, k) \cong G(n, n-k)$.

Proposition 9 ([4]). *If* $gcd(n, k) = 1$, *then* $G(n, k) \cong P_\alpha(C_n)$, *where* α *is given by* $\alpha(i) = k^{-1}(i + k - 1)$ *and* k^{-1} *is the multiplicative inverse of* k $(\mathrm{mod}\ n)$.

Theorem 10. *If* n *is odd,* $gcd(n, k) = 1$, *and* $k \notin \{1, n-1\}$, *then* $t(G(n, k)) > \frac{n}{n-1}$.

Proof: By Proposition 7 and Proposition 8, we need only consider the case where k is even. Also, by Proposition 9, we may assume $G(n, k) \cong P_\alpha(C_n)$ where $\alpha \neq (1)$ is a permutation such that $\alpha(i) = k^{-1}(i + k - 1)$. Since n is odd, by Theorem 4, $t(G(n, k)) \geq \frac{n}{n-1}$. Thus suppose $t(G(n, k)) = \frac{n}{n-1}$. In this case, $|S| = n$, $\omega(C_x - S_x) \leq \lfloor \frac{n}{2} \rfloor$, and $\omega(C_y - S_y) \leq \lfloor \frac{n}{2} \rfloor$. In order to have $n - 1$ components in $P_\alpha(C_n) - S$, both $C_x - S_x$ and $C_y - S_y$ must have $\lfloor \frac{n}{2} \rfloor$ components. Thus $|S_x| \geq \lfloor \frac{n}{2} \rfloor$ and $|S_y| \geq \lfloor \frac{n}{2} \rfloor$. Therefore without loss of generality assume $|S_x| = \lceil \frac{n}{2} \rceil$ and $|S_y| = \lfloor \frac{n}{2} \rfloor$.

If there exist vertices u and v in $C_x - S_x$ such that u and v are adjacent, then $\omega(C_x - S_x) < \lfloor \frac{n}{2} \rfloor$, a contradiction. Thus assume $V(C_x - S_x) = \{x_{i+2j} \mid j = 0, 1, \ldots, \frac{n-3}{2}\}$. In order to have $n - 1$ components in $P_\alpha(C_n) - S$, all of the vertices in $C_x - S_x$ must be isolated in $P_\alpha(C_n) - S$. Therefore $N(V(C_x - S_x) : C_y)$ is S_y, that is $S_y = \{y_{\alpha(i+2j)} \mid j = 0, 1, \ldots, \frac{n-3}{2}\}$. Since n is odd, k is even, and $k \neq n - 1$, it follows that $k \leq n - 3$. Thus there exists a j', $0 \leq j' \leq \frac{n-3}{2}$, such that $j' = \frac{k}{2}$. Now $\alpha(i + 2j') = \alpha(i + k) = k^{-1}(i + k + k - 1) = k^{-1}(i + k - 1) + k^{-1}k = \alpha(i) + 1$. So $y_{\alpha(i)}$ and $y_{\alpha(i+2j')}$ are adjacent elements of S_y. Therefore, $\omega(C_y - S_y) < \lfloor \frac{n}{2} \rfloor$ and we obtain the desired contradiction. ∎

Chvátal [1] showed that an upper bound for the toughness of G is $\frac{\kappa(G)}{2}$. For cycle permutation graphs, this upper bound is $\frac{3}{2}$. This bound may be improved by considering some of the structure of $P_\alpha(C_n)$.

Theorem 11. *If $P_\alpha(C_n) \cong P_\beta(C_n)$ where β has the property that for some $k \leq n-2$, $\beta(i) = i$ for all $1 \leq i \leq k$, then $t(P_\alpha(C_n)) \leq \frac{k+2}{k+1}$.*

Proof: We need only show that $t(P_\beta(C_n)) \leq \frac{k+2}{k+1}$. For $1 \leq i \leq k+1$, let

$$s_i = \begin{cases} x_i & \text{if } i \text{ is odd} \\ y_i & \text{if } i \text{ is even} \end{cases} \quad \text{and let} \quad ti = \begin{cases} y_i & \text{if } i \text{ is odd} \\ x_i & \text{if } i \text{ is even.} \end{cases}$$

Then $S = \{s_1, s_2, \ldots, s_k, s_{k+1}, y_n\}$ is a disconnecting set of $P_\beta(C_n)$ such that $P_\beta(C_n) - S$ will have $k+1$ components, the subgraphs of $P_\beta(C_n)$ induced by the sets $\{t_1\}, \{t_2\}, \ldots, \{t_k\}$, and $\{t_{k+1}, x_{k+2}, x_{k+3}, \ldots, x_n, y_{k+2}, y_{k+3}, \ldots, y_{n-1}\}$. Thus $t(P_\beta(C_n)) \leq \frac{|S|}{\omega(P_\alpha(C_n)-S)} \leq \frac{k+2}{k+1}$. ∎

In [5], Stueckle presents a set of permutations which generate all nonisomorphic cycle permutation graphs of C_n, $n \leq 8$. The toughness of the corresponding cycle permutation graphs was determined. For $n = 3$, all cycle permutation graphs are isomorphic to $P_{(1)}(C_3)$ which, by Corollary 5, has toughness equal to $\frac{3}{2}$. For $4 \leq n \leq 8$, all of the cycle permutation graphs have toughness at most $\frac{4}{3}$. If $n \geq 4$ and $P_\alpha(C_n) \cong P_\beta(C_n)$, where $\beta(1) = 1$ and $\beta(2) = 2$ (that is, $P_\alpha(C_n)$ contains an induced 4-cycle containing two permutation edges), Theorem 11 implies $t(P_\alpha(C_n)) \leq \frac{4}{3}$. For those cycle permutation graphs without such an induced 4-cycle, Theorem 11 implies $t(P_\alpha(C_n)) \leq \frac{3}{2}$. Yet, for $4 \leq n \leq 8$, all of these cycle permutation graphs also have toughness at most $\frac{4}{3}$. Furthermore, for all permutation graphs of C_9 which have no induced 4-cycle, the toughness was shown to be at most $\frac{4}{3}$. Thus, the evidence seems to indicate the following conjecture.

Conjecture. *For $n \geq 4$, $t(P_\alpha(C_n)) \leq \frac{4}{3}$.*

If this upper bound cannot be obtained, perhaps the following looser upper bound can be obtained.

Conjecture. *For $n \geq 4$, $t(P_\alpha(C_n)) < \frac{3}{2}$.*

Furthermore, for $n = 4, 5, 8$, and 9, there exists some permutation α for which $t(P_\alpha(C_n)) = \frac{4}{3}$. Is it possible to construct an infinite class of cycle permutation graphs for which $t(P_\alpha(C_n))$ is equal to $\frac{4}{3}$? Since $G(5,2)$ and $G(9,2)$ have toughness equal to $\frac{4}{3}$, could such a class be the generalized Petersen graphs when $n \equiv 1 \pmod 4$ and $k = 2$?

Conjecture. *If $n \geq 5$ and $n \equiv 1 \pmod 4$, then $t(G(n,2)) = \frac{4}{3}$.*

Binding number.

In this section we determine the binding number of any cycle permutation graph.

Proposition 12 ([7]). *Let S be a set of vertices in C_n. If $|S| \notin \{0, n, \frac{n}{2}\}$, then $|N(S)| \geq |S| + 1$. Furthermore, if $|S| \in \{0, n, \frac{n}{2}\}$, then $|N(S)| = |S|$.*

Proposition 13 ([7]).

$$b(C_n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ \frac{n-1}{n-2} & \text{if } n \text{ is odd.} \end{cases}$$

Proposition 14 ([2]). *If G is a spanning subgraph of H, then $b(G) \leq b(H)$.*

Proposition 15 ([2]). *If G and H are disjoint graphs such that $b(G) = g$, and $b(H) = h$, then $b(G \cup H) = \min\{1, g, h\}$.*

Lemma 16. *Let S be a subset of C_n such that $|S| \leq \frac{n}{2}$ and let P be a component of maximum order in $\langle S \rangle$. If, for some $k \geq 2$, $P \cong P_k$ (a path on k vertices), then $|N(S)| \geq |S| + 2$.*

Proof: Suppoe $|S| \leq \frac{n}{2}$ and that P is a path on $k \geq 2$ vertices. Let C' be the cycle formed by contracting P to a single point in C and let S' be the vertices in C' which correspond to S in C. Then $|S'| = |S| - k + 1 < |S|$, $|N(S')| = |N(S)| - k$, and $|V(C') - S'| = |V(C) - S| \geq |S|$. So, by Proposition 12, $|N(S')| \geq |S'| + 1 = |S| - k + 2$. Thus $|N(S)| = |N(S')| + k \geq |S| + 2$. ∎

Theorem 17.

$$b(P_\alpha(C_n)) = \begin{cases} 1 & \text{if } P_\alpha(C_n) \text{ is bipartite} \\ \frac{2n-1}{2n-3} & \text{otherwise.} \end{cases}$$

Proof: Since $C_x \cup C_y$ is a spanning subgraph of $P_\alpha(C_n)$, it follows from Propositions 13, 14, and 15 that $b(P_\alpha(C_n)) \geq 1$. Furthermore, for $T = V(P_\alpha(C_n)) - N(\{x_1\})$, $\frac{|N(T)|}{|T|} = \frac{2n-1}{2n-3}$. So, $b(P_\alpha(C_n)) \leq \frac{2n-1}{2n-3}$.

Now suppose S is a non-empty subset of $V(P_\alpha(C_n))$ such that $N(S) \neq V(P_\alpha(C_n))$ and $b(P_\alpha(C_n)) = \frac{|N(S)|}{|S|} < \frac{2n-1}{2n-3}$. Then $|N(S)| \geq |N(S_x : C_x)| + |N(S_y : C_y)| \geq |S_x| + |S_y| = |S|$ and $|N(S)| \leq |S| + 1$ since $|N(S)| \geq |S| + 2$ would imply that $\frac{|N(S)|}{|S|} \geq \frac{2n-1}{2n-3}$. Note that $|S_x| \neq n$ and $|S_y| \neq n$; otherwise $N(S) = V(P_\alpha(C_n))$. If S_x is the empty set, $|N(S)| = |N(S_y : C_x)| + |N(S_y : C_y)| \geq 2|S_y| = 2|S|$ and $\frac{|N(S)|}{|S|} \geq 2 > \frac{2n-1}{2n-3}$, a contradiction. Similarly, S_y is non-empty.

Suppose $|S_x| \neq \frac{n}{2}$ and $|S_y| \neq \frac{n}{2}$. Then $|N(S)| \geq |N(S_x : C_x)| + |N(S_y : C_y)| \geq (|S_x| + 1) + (|S_y| + 1) = |S| + 2$, a contradiction. Thus n is even and either $|S_x|$ or $|S_y|$ is equal to $\frac{n}{2}$. Without loss of generality assume $|S_x| = \frac{n}{2}$. If two vertices of S_x are adjacent, Lemma 16 implies that $|N(S_x : C_x)| \geq |S_x| + 2$; thus $|N(S)| \geq |N(S_x : C_x)| + |N(S_y : C_y)| \geq (|S_x| + 2) + |S_y| = |S| + 2$. Therefore, S_x is independent in C_x and $N(S_x : C_x) = V(C_x - S_x)$.

If $|S_y| > \frac{n}{2}$, some vertex u of S_y is adjacent to a vertex v of S_x. Thus $v \in N(S)$ and $|N(S)| \geq |N(S_x: C_x)| + |N(S_y: C_y)| + 1 \geq |S_x| + (|S_y| + 1) + 1 = |S| + 2$.

If $|S_y| \leq \frac{n}{2}$ and S_y contains two adjacent vertices it follows from Lemma 16 that $|N(S_y: C_y)| \geq |S_y| + 2$; thus $|N(S)| \geq |N(S_x: C_x)| + |N(S_y: C_y)| \geq |S_x| + (|S_y| + 2) = |S| + 2$. Therefore, S_y is independent in C_y and $N(S_y: C_y) \subseteq V(C_x - S_x)$. If some vertex u of S_y is adjacent to some vertex v of S_x, $u, v \in N(S)$ and $|N(S)| \geq |N(S_x: C_x)| + |N(S_y: C_y)| + 2 \geq |S_x| + |S_y| + 2 = |S| + 2$. So, S is an independent set in $P_\alpha(C_n)$.

If $|N(S)| = |S| + 1$, then $|S| = n - 1$; otherwise $\frac{|N(S)|}{|S|} > \frac{2n-1}{2n-3}$. Therefore $P_\alpha(C_n)$ has the structure given in Figure 1. In this case, $Z = S \cup \{v\}$ is a set with $N(Z) = N(S)$ and $\frac{|N(Z)|}{|Z|} = \frac{|N(S)|}{|S|+1} < \frac{|N(S)|}{|S|}$, a contradiction. Thus $|N(S)| = |S|$ and $\frac{|N(S)|}{|S|} = 1$. If $|S_y| < \frac{n}{2}$, then $|N(S)| \geq |N(S_x: C_x)| + |N(S_y: C_y)| \geq |S_x| + (|S_y| + 1) = |S| + 1$, a contradiction. So, $|S_y| = \frac{n}{2}$ and $|S| = n$. Since S is an independent set of order n, it follows that $P_\alpha(C_n)$ is bipartite. Hence, if $P_\alpha(C_n)$ is bipartite, $b(P_\alpha(C_n)) = 1$; otherwise $b(P_\alpha(C_n)) = \frac{2n-1}{2n-3}$. ∎

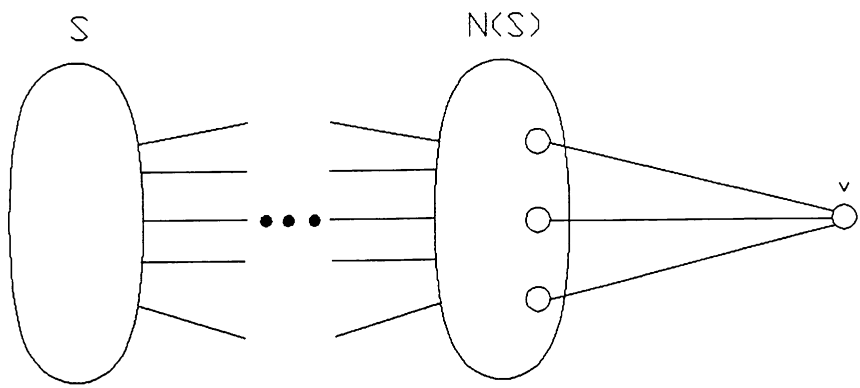

Figure 1

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