No maximal partial spread of size 10 $PG(3,5)$

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A maximal partial spread in $PG(3,q)$ is a set of mutually skew lines such that no further lines of $PG(3,q)$ are skew to all lines of S.

We shall prove

Theorem. *There is no maximal partial spread of size 10 in $PG(3,5)$.*

This improves, in the special case $q = 5$, the following result of Glynn, see [2] or [5] p. 85.

Theorem(Glynn). *If S is a maximal partial spread in $PG(3,q)$ then $|S| \geq 2q$.*

By Example 8.1 of [4] and the result above, we know almost all possible sizes for a maximal partial spread in $PG(3,5)$. There is no maximal partial spread S with $|S| \leq 10$ or $23 \leq |S| \leq 25$. There are maximal partial spreads of size n for all integers n with $13 \leq n \leq 22$ and for $n = 26$. However we do not know whether or not there is any maximal partial spread of size 11 or 12.

For $q = 2$ and $q = 3$ all possible sizes are known, see [5] p. 79 and p. 82.

For an upper bound on the number of lines of a maximal partial spread see [3].

Proof of the theorem: Assume that S is a maximal partial spread of size 10 in $PG(3,5)$. Let n_i for $i \in \{1,2,\cdots,6\}$ denote the number of lines of $PG(3,5)$ that meet exactly i lines of S. These numbers satisfy the following four equations:

$$\begin{aligned} n_1 + n_2 + n_3 + n_4 + n_5 + n_6 &= 796 \\ n_1 + 2n_2 + 3n_3 + 4n_4 + 5n_5 + 6n_6 &= 1800 \\ n_2 + 3n_3 + 6n_4 + 10n_5 + 15n_6 &= 1620 \\ n_3 + 4n_4 + 10n_5 + 20n_6 &= 720 \end{aligned}$$

(These equations are special cases of the equations that Glynn used to prove the bound mentioned above; see [2] or [5] p 78)

Multiplying the previous four equations by $4, -3, 2$ and -1 respectively we deduce that $n_1 = 304 + n_5 + n_6$. Hence $n_1 \geq 304$ and thus, for some line l of S, there are at least 31 lines of $PG(3,5)$ that intersect l but no other line of S. Now, l is contained in six planes. Hence one of these planes contains at least six of these 31 lines. Any line of $S \backslash \{l\}$ intersects this plane in one and only one point. The theorem will thus be proved when we show that there is no subset B of nine points of a projective plane $\pi = PG(2,5)$ such that seven of the lines of π do not meet any of the points of B. We shall consider the dual problem.

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Let F be a set of seven points of π. As well known, see e.g. Theorem 3.24 of [1] p. 149, for any k-arc of π $k \leq 6$. Further if $k = 6$ then the k-arc is an oval. As $|F| = 7$, F cannot contain two different ovals, see [1] p 149. It follows that either there are at least two lines of π meeting at least three points each or at least one line meeting at least four of the points of F. In both cases, by using elementary counting arguments, we conclude that F meets at least 23 of the 31 lines of π.

The theorem is now proved.

Remark 1: The author of this paper has, without any success, done some attempt to construct a maximal partial spread of size 12 in $PG(3,5)$ and to prove the nonexistence of a maximal partial spread of size 11.

Remark 2: It is not possible to use the same method to improve Glynn's result in general. For example, when $q = 7$ and $|S| = 14$, where S is a maximal partial spread, we would have to prove the nonexistence of a subset B of 13 points of a projective plane $\pi = PG(2,7)$ such that there are 13 lines of π that do not meet any point of B. However, such a subset B of π is not so very difficult to construct.

Let $GF(7)$ denote the finite field with seven elements. The 49 elements of $GF(7) \times GF(7)$ as points and the subsets $[a,k] = \{(q, a + kq) \mid q \in GF(7)\}$ $k, a \in GF(7)$ and $[a,\infty] = \{(a,q) \mid q \in GF(7)\}$ $a \in GF(7)$ as lines will constitute an affine plane. (Below we denote the elements of $GF(7)$ by the elements $0,1,2,\cdots,6$ of $Z/7Z$.)

If B consists of the points $(0,0)$, $(0,1)$, $(0,2)$, $(1,3)$, $(1,4)$, $(1,5)$, $(2,3)$, $(2,4)$, $(3,0)$, $(3,6)$, $(5,2)$, $(5,5)$, and $(5,6)$ then the following 12 lines $[4,\infty]$, $[6,\infty]$, $[6,1]$, $[5,4]$, $[5,1]$, $[4,3]$, $[5,2]$, $[6,5]$, $[4,4]$, $[3,3]$, $[3,5]$ and $[4,2]$ together with the line at infinity will give us 13 lines of π that do not meet any point of B.

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References

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