Edge Contractions in 3-Connected Graphs

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Abstract. For $v \geq 4$ we determine the largest number $f(v)$, such that every simple 3-connected graph on v vertices has $f(v)$ edge contractions which result in a smaller 3-connected graph. We also characterize those simple 3-connected graphs on v vertices which have exactly $f(v)$ such edge contractions.

Preliminaries.

We use the notation and terminology of Bondy and Murty [2]. We restrict ourselves to simple graphs.

Let $e = x_1x_2$ be an edge in a graph G. The graph, $G \cdot e$, obtained by *contracting* e is defined to be

$$(G - \{x_1, x_2\}) + \{x\} + \{xy \mid x_1y \in E(G) \text{ or } x_2y \in E(G),\ x_1 \neq y \neq x_2\}.$$

If G is simple, then so is $G \cdot e$. We say G is obtained from $G \cdot e$ by *vertex splitting at* x. Note that more than one graph can be obtained from a given graph by vertex splitting at a particular vertex. If the minimum degree of G is at least 3, then we define the graph $(G-e)^{\sim}$ obtained by *reducing at* $e = x_1x_2$ as follows: In $G - e$ we contract one edge incident with x_i if $d_{G-e}(x_i) = 2$, $i = 1, 2$. We note that $(G - e)^{\sim}$ may have double edges even if G is simple. If $N_{G-e}(x_i) = \{y_i, z_i\}$, then the edge y_iz_i of $(G-e)^{\sim}$ is referred to as e_{x_i}, $i = 1, 2$. We say G is obtained from $(G - e)^{\sim}$ by *edge addition*.

If G is 3-connected, then we say that e is *contractible* (respectively, *reducible*) if $G \cdot e$ (respectively, $(G - e)^{\sim}$) is 3-connected. Let $E_c = E_c(G)$ be the set of contractible edges of a 3-connected graph G, and let $|E_c| = \epsilon_c$.

A 3-connected graph G is *critically 3-connected* if no proper spanning subgraph of G is 3-connected.

The following three classical theorems are due to Tutte [3, Ch. 4].

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Theorem 1. *If G is 3-connected and $G \neq K_4$, then $G \cdot e$ is 3-connected for some e in $E(G)$.*

Theorem 2. *Let G' be 3-connected. If G is obtained from G' by edge addition, then G is 3-connected. If G is obtained from G' by vertex splitting and G has minimum degree at least 3, then G is 3-connected.*

Theorem 3. *If G is 3-connected, $G \neq K_4$, and $e \in E(G)$, then $(G - e)^{\sim}$ or $G \cdot e$ is 3-connected.*

Theorem 3 has the following corollary:

Corollary 1. *If G is a critically 3-connected graph and e in $E(G)$ joins vertices of degree at least four, then $G \cdot e$ is 3-connected.*

Some lemmas.

Let $e_i = u_i v_i$, $i = 1, 2$ be edges of G such that u_1 and u_2 are distinct. We say that

$$(G - \{e_1, e_2\}) + \{x_1, x_2\} + \{x_1 u_1, x_1 v_1, x_2 u_2, x_2 v_2, x_1 x_2\}$$

is obtained from G by *adding an edge across* e_1 and e_2. We say that

$$(G - \{e_1\}) + \{x\} + \{x u_1, x v_1, x u_2\}$$

is obtained from G by *adding an edge obliquely across* e_1 and e_2.

An *independent* 3-*edge cut* S is an edge cut consisting of three pairwise non-adjacent edges. We say that an edge is *(obliquely) added across* S if an edge is (obliquely) added across two edges in S.

A triangle of G is a K_3 subgraph of G.

Lemma 1. *Let xu be a non-contractible edge of a 3-connected graph G. Suppose $d(z) = 3$ for every z in $V(G) - \{x\}$, and xu is not on a triangle of G. If $d(x) = 3$ (respectively, $d(x) = 4$), then $G' = (G - xu)^{\sim}$ is a cubic 3-connected graph and G is obtained from G' by adding an edge (respectively, adding an edge obliquely) across an independent 3-edge cut S.*

Proof: Since xu is not contractible, Theorem 3 implies that G' is 3-connected. By the assumptions on the vertex degrees of G, G' is cubic.

There exists some vertex v such that $\{x, u, v\}$ is a 3-vertex cut because xu is not contractible. Then $d_G(v) = 3$ implies that there exists some edge e incident with v which is a cut edge of $G - \{x, u\}$.

If $d(x) = 3$, then $S = \{e_x, e_u, e\}$ is a 3-edge cut of G'. Since xu is not on a triangle in G, e_x and e_u are not adjacent. The 3-connectivity of G' then implies that S is independent. G is obtained from G' by adding an edge across S.

If $d(x) = 4$, then x is a cut vertex of $G' - \{e_u, e\}$. Then $d_{G'}(x) = 3$ implies there exists an edge f incident with x such that $S = \{e_u, e, f\}$ is a 3-edge cut of

G'. Since xu is not on a triangle of G and G' is 3-connected, S is independent. G is obtained from G' by adding an edge obliquely across S. ∎

Let $V_t(G)$ (respectively, $E_t(G)$) be the set of vertices (respectively, edges) of G which are in $V(K_3)$ (respectively, $E(K_3)$), for some triangle K_3 of G such that $d_G(v) = 3$, for each v in $V(K_3)$. If $G \neq K_4$ and G is 3-connected, then every vertex in V_t is incident with a unique edge in $E - E_t$.

Suppose $e \in E_t(G)$ and $e \in E(K_3)$. We say $(G - e)^{\sim}$ is obtained from G by *contracting* K_3 *to vertex.* Suppose $d_G(v) = 3$. The graph obtained from G by adding an edge across two edges incident with v is said to be obtained by *replacing* v *by a triangle.*

Lemma 2. *Let* G *be a 3-connected graph such that* $G \neq K_4$. *If* $v \in V_t(G)$, *then the only contractible edge incident with* v *is the edge in* $E - E_t$. *If* $d_G(v) = 3$ *and* $v \notin V_t(G)$, *then* v *is incident with at least two contractible edges.*

Proof: Let $e_i = vx_i, i = 1,2,3$ be the edges incident with v.

Suppose $v \in V_t(G)$, $e_1 \in E - E_t$, and $e_2, e_3 \in E_t$. It is easy to see that e_1 is not reducible and e_2 and e_3 are not contractible. By Theorem 3, e_1 is contractible.

Suppose $v \notin V_t(G)$ and $G \cdot e_3$ is not 3-connected. Then $\{v, x_3, y\}$ is a 3-cut for some y in $V(G)$. Let H_1 and H_2 be the components of $G - \{v, v_3, y\}$, where $x_i \in V(H_i), i = 1,2$.

Suppose $v(H_1) = 1$. Then x_1 is adjacent to v, x_3, and y. Since $v \notin V_t$, $d(x_3) \geq 4$. Therefore, $G \cdot e_1 = (G - e_3)^{\sim}$. By Theorem 3, $(G - e_3)^{\sim}$ is 3-connected, and so e_1 is contractible.

Suppose $v(H_2) \geq 2$. Then $\{x_3, y\}$ is a 2-cut of $(G - e)^{\sim}$, and so e_1 is not reducible. Therefore, e_1 is contractible by Theorem 3.

Similarly, e_2 is contractible. ∎

Lemma 3. *Let* G *be a critically 3-connected graph such that* $G \neq K_4$. *Suppose* $d_G(v) = 4$. *If* v *is not on a triangle, then* v *is incident with at least two contractible edges. If* v *is not incident with a contractible edge, then* v *is only adjacent with vertices of degree three and* v *is on two edge-disjoint triangles.*

Proof: Suppose v is not on a triangle. Suppose some edge e incident with v is not contractible. By Theorem 3, $(G-e)^{\sim}$ is 3-connected. If v is on a triangle in $(G-e)^{\sim}$, then it is on a triangle in G, a contradiction. Hence, $v \notin V_t((G - e)^{\sim})$ and v has degree 3 in $(G - e)^{\sim}$. Therefore, v is incident with two contractible edges, e_1 and e_2, in $E((G - e)^{\sim})$ by Lemma 2. Hence, $(G - e)^{\sim} \cdot e_i$ is 3-connected, $i = 1,2$. But since v is not on a triangle in G, $(G - e)^{\sim} \cdot e_i = ((G \cdot e_i) - e)^{\sim}$, $i = 1,2$. Then $G \cdot e_i$ is obtained from the 3-connected graph $((G \cdot e_i) - e)^{\sim}$ by edge addition, and so $G \cdot e_i$ is 3-connected by Theorem 2, $i = 1,2$.

Suppose v is not incident with a contractible edge. Since G is critically 3-connected, Corollary 1 implies that all neighbours of v have degree 3. By the first part of Lemma 3, v is on some triangle vu_1u_2. Let u_3 and u_4 be the other two neighbours of v. Since $vu_3 \notin E_c$, there exists y in $V(G)$ such that $\{v, u_3, y\}$ is a 3-cut of G. Since $d(u_3) = 3$, $G - \{v, u_3, y\}$ has two components, G_1 and G_2.. Vertices u_1 and u_2 can not be in different components of $G - \{v, u_3, y\}$ because $u_1u_2 \in E$, and so we may assume that $u_1, u_2 \in V(G_1) \cup \{y\}$. Then $u_4 \in V(G_2)$ since G is 3-connected. If $v(G_2) \geq 2$, then $\{u_3, y\}$ is a 2-cut of $(G-e_4)^{\sim}$. But then Theorem 3 implies that $G \cdot e_3$ is 3-connected, a contradiction. Thus, $V(G_2) = \{u_4\}$. Then $u_3u_4, u_4y \in E$ since $d(u_4) = 3$. ∎

Main theorems.

Let G be a 3-connected graph. Let $V_3 = V_3(G) = \{v \in V(G) \mid d(v) = 3\}$ and $V_f = V_f(G) = \{v \in V(G) \mid d(v) \geq 4\}$. Let $v_3 = v_3(G) = |V_3|$ and $v_f = v_f(G) = |V_f|$. Let $V_3^i = V_3^i(G)$ be the set of vertices in V_3 that are adjacent to exactly i vertices in V_f and let $v_3^i = v_3^i(G) = |V_3^i|$, $i = 0, 1, 2, 3$. Let $V_f^i = V_f^i(G)$ be the set of vertices in V_f that are adjacent to exactly i vertices in V_f and let $v_f^i = v_f^i(G) = |V_f^i|$, $i = 0, 1, \ldots, v-1$.

Theorem 4. *If G is a critically 3-connected graph and $G \neq K_4$, then $|E_c(G)| \geq \frac{1}{2}(v + 3v_f)$.*

Proof: Let $E_{3,3}$, $E_{3,f}$, and $E_{f,f}$ be the sets of contractible edges joining two vertices in V_3, a vertex in V_3 to a vertex in V_f, and two vertices in V_f, respectively. Let A_i be the set of vertices in V_3^i that are only joined to vertices in V_f by contractible edges, let $a_i = |A_i|$, and let $B_i = V_3^i - A_i$, $i = 1, 2$.

By Lemma 2, every vertex in V_3^0 is incident with an edge in $E_{3,3}$. Suppose $v \in V_3^1$ and v is incident with at most one contractible edge. Then Lemma 2 implies $v \in V_t$ and the edge joining v to the vertex in V_f is contractible, and so $v \in A_1$. Thus, every vertex in B_1 is incident with two contractible edges, and by the definition of B_1 these two edges are in $E_{3,3}$. By Lemma 2, every vertex in B_2 is incident with two contractible edges, and by the definition of B_2, one of these two edges is in $E_{3,3}$. Thus,

$$|E_{3,3}| \geq \frac{1}{2}\left[v_3^0 + 2(v_3^1 - a_1) + (v_3^2 - a_2)\right].$$

By definition every vertex in A_i is incident with i edges in $E_{3,f}$. By Lemma 2 every vertex in B_2 or V_3^3 is incident with two contractible edges. Therefore, every vertex in B_2 is incident with at least one edge in $E_{3,f}$ and every vertex in V_3^3 is incident with at least two edges in $E_{3,f}$. Thus,

$$|E_{3,f}| \geq a_1 + 2a_2 + (v_3^2 - a_2) + 2v_3^3.$$

By Corollary 1, every edge joining two vertices in V_f is in E_c, so $|E_{f,f}| = \frac{1}{2}\sum_{i=1}^{v-1} i\, v_f^i$.

Thus,

$$
\begin{aligned}
|E_c| &= |E_{3,3}| + |E_{3,f}| + |E_{f,f}| \\
&\geq \frac{1}{2}[\, v_3^0 + 2(v_3^1 - a_1) + (v_3^2 - a_2)] \\
&\quad + [\, a_1 + 2a_2 + (v_3^2 - a_2) + 2v_3^3] + \frac{1}{2}\sum_{i=1}^{v-1} i\, v_f^i \\
&= \frac{1}{2}\sum_{i=0}^{3} v_3^i + \frac{1}{2}\left(\sum_{i=1}^{3} i\, v_3^i + \sum_{i=1}^{v-1} i\, v_f^i\right) + \frac{1}{2}a_2 \\
&\geq \frac{1}{2}v_3 + \frac{1}{2}\sum_{v\in V_f} d(v) \geq \frac{1}{2}(v - v_f) + \frac{1}{2}\cdot 4v_f = \frac{v + 3v_f}{2}.
\end{aligned}
\qquad \text{(A)}
$$

We will construct sets of graphs Γ_i, $i = 0, 1, \ldots, 5$, such that for every $G_i \in \Gamma_i$, $v(G_i) \equiv i \pmod 6$ and

$$
\epsilon_c(G_i) = \begin{cases} \frac{v}{2}, & \text{if } i = 0 \\ \frac{v}{2} + 1, & \text{if } i = 2, 4 \\ \frac{v+3}{2}, & \text{if } i = 1, 3, 5 \end{cases} .
$$

In Figure 1 we give an example of each of these constructions.

Let K be the triangular prism. Let Γ_0 be the union of $\{K\}$ and the set of all graphs obtained by replacing all vertices of a 3-connected cubic graph by triangles.

Let Γ_1 be the set of all graphs obtained in the following two ways:

i) In a graph in Γ_0, add an edge obliquely across an independent 3-edge cut.
ii) Let H be a 3-connected graph in which one vertex has degree 4 and all other vertices have degree 3. Replace all vertices of H of degree 3 by a triangle.

Let Γ_2 be the set of all graphs obtained by adding an edge across an independent 3-edge cut of any graph in Γ_0.

Let Γ_4 be the set of all graphs obtained by replacing $v(G) - 1$ vertices in a 3-connected graph G by triangles.

Let Γ_3 be the set of all graphs obtained in the following three ways:

i) Let $\{e_1, e_2, e_3\}$ be an independent 3-edge cut in some G in Γ_2 such that e_1 is incident with a vertex v not in $V_t(G)$, and add an edge from v across e_2 to obtain a graph H.
ii) Let H in Γ_3 be obtained as in (i) such that $V - V_t$ is a 3-vertex cut. Add an edge to H joining the two vertices in $V - V_t$ of degree three.
iii) In a graph in Γ_4, contract an edge incident with the vertex which is not in $V_t(G)$.

Let Γ_5 be the set of all graphs obtained in the following two ways:

i) Let $\{e_1, e_2, e_3\}$ be an independent 3-edge cut in some G in Γ_4 such that e_1 is incident with the vertex v not in $V_t(G)$, and add an edge from v across e_2.

ii) In a graph H in Γ_0, contract an edge in $E(H) - E_t(H)$.

Let $\Gamma = \cup_{i=0}^{5} \Gamma_i$.

Theorem 2 has the following corollary:

Corollary 2. *Let G be a simple 3-connected graph. Any graph obtained from G by adding an edge across two edges, adding an edge from a vertex across an edge, or replacing a vertex of G by a triangle is 3-connected.*

Theorem 5. *If $G \in \Gamma_i, 0 \leq i \leq 5$, then G is 3-connected, $v(G) \equiv i \pmod 6$, and*

$$\epsilon_c(G) = \begin{cases} \frac{v}{2}, & \text{if } i = 0 \\ \frac{v}{2} + 1, & \text{if } i = 2, 4 \\ \frac{v+3}{2}, & \text{if } i = 1, 3, 5 \end{cases} .$$

Proof: Corollary 2 implies that the graphs in Γ are 3-connected. For graphs in Γ_3 and Γ_5 obtained from graphs in Γ_4 and Γ_0, respectively, by contracting an edge e, we must also use Lemma 2 to show that e is contractible. It is easy to see that $v(G) \equiv i \pmod 6$ by using the fact that any graph has an even number of vertices of odd degree.

Using Lemma 2, we see that $\epsilon_c(G_0) = \frac{v}{2}, \epsilon_c(G_1) = \frac{v+3}{2}$, and $\epsilon_c(G_4) = \frac{v}{2} + 1$ for all graphs G_i in Γ_i, $i = 0, 1, 4$, except those in Γ_1 obtained from graphs in Γ_0 by edge addition.

Consider graphs G_2 in Γ_2, G_3 in Γ_3, and G_5 in Γ_5 obtained from graphs H_0 in Γ_0, H_2 in Γ_2, and H_4 in Γ_4, respectively, by adding an edge $e = uv$. Since e is added in such a way that $\{u, v\}$ is contained in a 3-vertex cut of G_i, $e \notin E_c(G_i)$, $i = 2, 3, 5$. Let $f = ux$ be the edge in $E(H_2)$ that was added to a graph in Γ_0 to obtain H_2. Then $\{u, x, y\}$ is a 3-cut for some y in $V(H_2)$. Then $\{u, x, y\}$ is a 3-cut in G_3, since e is incident with u. Hence, $f \notin E_c(G_3)$. Lemma 2 implies that the edges joining vertices in $V_t(G_i)$ are not in $E_c(G_i), i = 2, 3, 5$. Thus, we have shown that $\epsilon(G_i) - \epsilon_c(G_i) \geq v - 1$, $i = 2, 3, 5$. By their construction, $\epsilon(G_2) = \frac{3v}{2}$ and $\epsilon(G_3) - \epsilon(G_5) = \frac{v+1}{2}$, and by Theorem 1, $\epsilon_c(G_2) \geq \frac{v}{2} + 1$ and $\epsilon_c(G_i) \geq \frac{v+3}{2}, i = 3, 5$. Thus, $\epsilon_c(G) = \frac{v}{2} + 1$ and $\epsilon_c(G_3) = \epsilon_c(G_5) = \frac{v+3}{2}$. If G'_3 in Γ_3 is obtained by adding an edge uv to a graph H'_3 in Γ_3, then $\{u, v\}$ is contained in a 3-vertex cut of H_3 by definition. Now it is easy to see that $\epsilon_c(G'_3) = \epsilon_c(H_3) = \frac{v+3}{2}$.

Consider a graph G'_1 in Γ_1 obtained from a graph in Γ_0 by adding an edge $e = uv$, and consider graphs G'_3 in Γ_3 and G'_5 in Γ_5 obtained from graphs in Γ_4 and Γ_0, respectively, by contracting an edge e. Since e is added in such a way that $\{u, v\}$ is contained in a 3-cut of G'_1, $e \notin E_c(G'_1)$. For G'_1, G'_3, and G'_5, the edges

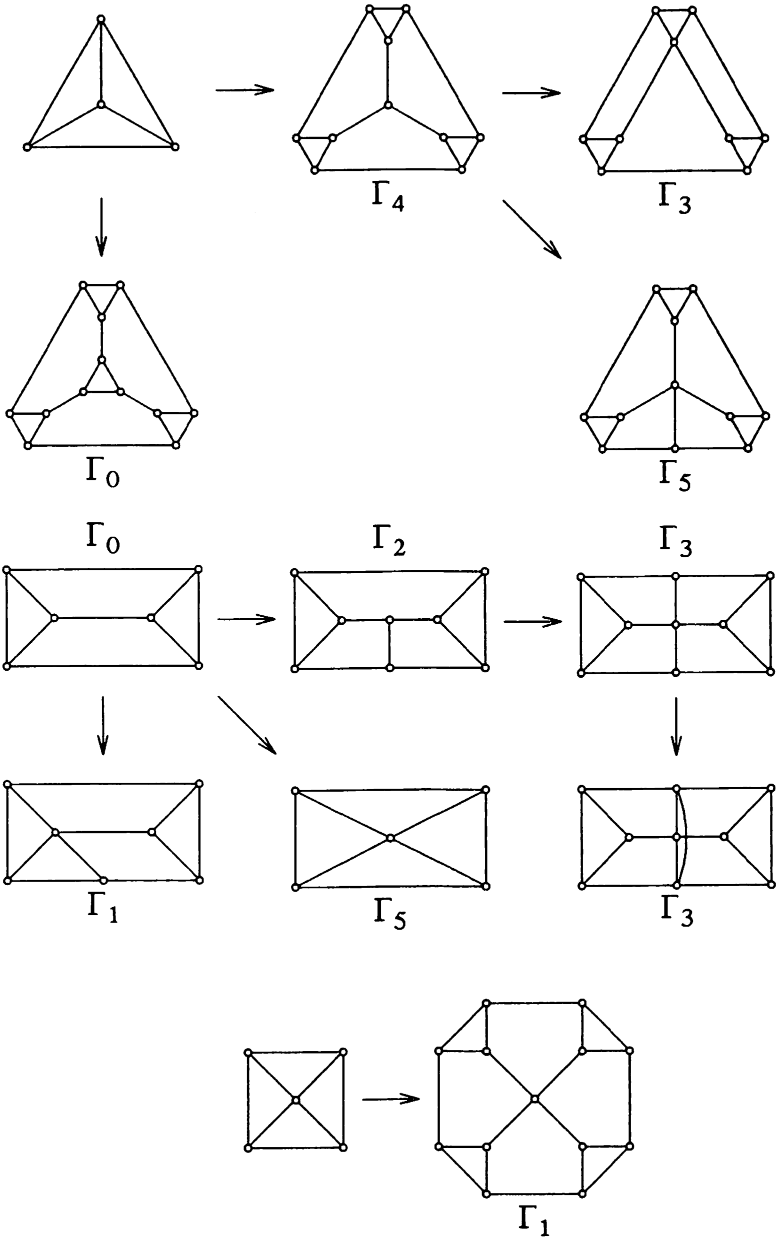
Γ_4
Γ_3
Γ_0
Γ_5
Γ_0
Γ_2
Γ_3
Γ_1
Γ_5
Γ_3
Γ_1

Figure 1

incident with the vertex of degree 4 and on a triangle are not contractible. Lemma 2 implies that the edges joining vertices in $V_t(G'_i)$ are not in $E_c(G'_i)$, $i = 1, 3, 5$. Thus, we have $\epsilon(G'_i) - \epsilon_c(G'_i) \geq v - 1$, $i = 3, 5$. As before this implies that $\epsilon_c(G'_3) = \epsilon_c(G'_5) = \frac{v+3}{2}$. ∎

Theorem 6. *Let $f(v)$ be the largest integer such that every 3-connected graph on v vertices has $f(v)$ contractible edges, $v \geq 5$. Then*

$$f(v) = \begin{cases} \frac{v}{2}, & \text{if } v \equiv 0 \pmod 6 \\ \frac{v}{2} + 1, & \text{if } v \equiv 2, 4 \pmod 6 \\ \frac{v+3}{2}, & \text{if } v \equiv 1, 3, 5 \pmod 6 \end{cases} .$$

If $\epsilon_c(G) = f(v(G))$, then $G \in \Gamma$.

Proof: Theorem 5 shows that the given values are upper bounds for $f(v)$. By Theorem 4, $f(v) \geq \frac{v+3v_f}{2}$. Thus, $f(v) = \frac{v}{2}$ when $v \equiv 0 \pmod 6$. If $v(G)$ is odd, then the vertices of G cannot all have odd degree 3, and so $v_f(G) \geq 1$. Thus, $f(v) = \frac{v+3}{2}$ when $v \equiv 1, 3, 5 \pmod 6$.

Suppose $\varepsilon_c(G) = \frac{v}{2}$. By Theorem 4, $V(G) = V_3$, so Lemma 2 implies that every vertex is incident with a contractible edge. Since $\epsilon_c(G) = \frac{v}{2}$, every vertex is incident with exactly one contractible edge. Now Lemma 2 implies $V = V_t$. If $G \not\cong K$, Theorem 3 implies that the graph obtained by contracting all the triangles to vertices is 3-connected. Therefore, $G \in \Gamma_0$.

Since Theorem 4 implies $f(v) \geq \frac{v}{2}$, and $\epsilon_c = \frac{v}{2}$ implies $v \equiv 0 \pmod 6$, $f(v) \geq \frac{v}{2} + 1$ when $v \equiv 2, 4 \pmod 6$. Hence, $f(v) = \frac{v}{2} + 1$.

Suppose G_i is a critically 3-connected graph such that $v(G_i) \geq 5$, $v(G_i) \equiv i \pmod 6$, and $\epsilon_c(G_i) = f(v(G_i))$, $i = 1, \ldots, 5$.

Consider G_2 and G_4. Since $\epsilon_c(G_i) = \frac{v}{2} + 1$, Theorem 4 implies that $v_f = 0$, $i = 2, 4$. Hence, for $i = 2, 4$, Lemma 2 implies

$$\frac{v}{2} + 1 = \epsilon_c(G_i) \geq \frac{1}{2}[v_t + 2(v - v_t)]. \qquad \text{(B)}$$

Therefore, $2 \geq v - v_t$. Since $v_t(G_i) \equiv 0 \pmod 3$, $i = 2, 4$, we have $v(G_2) - v_t(G_2) = 2$ and $v(G_4) - v_t(G_4) = 1$. Let $V(G_2) - V_t(G_2) = \{u_2, v_2\}$ and $V(G_4) - V_t(G_4) = \{u_4\}$.

We have equality in (B) for G_2, and so u_2 and v_2 are both incident with exactly two contractible edges. Since a vertex in $V - V_t$ can only be joined to a vertex in V_t by a contractible edge, $u_2 v_2 \in E(G_2)$ and $u_2 v_2 \notin E_c(G_2)$. By Lemma 1, G_2 is obtained from the cubic 3-connected graph $G'_2 = (G_2 - u_2 v_2)^{\sim}$ by adding an edge across an independent 3-edge cut. Since $V(G'_2) = V_t(G'_2)$, $G'_2 \in \Gamma_0$. Hence, $G_2 \in \Gamma_2$.

Theorem 3 implies $G_4 \in \Gamma_4$.

Consider G_i, where $i \in \{1,3,5\}$. Since $v_f \geq 1$, $\frac{v+3v_f}{2} \geq \frac{v+3}{2} = \epsilon_c(G_i)$, and so we have equality in Equation (A) from Theorem 4. Hence, $v_f = 1$ and $d_{G_i}(x_i) = 4$, where $V_f(G_i) = \{x_i\}$. Let x_i be incident with r_i contractible edges and adjacent to t_i vertices in $V_t(G_i)$. By Lemma 2,

$$\frac{v+3}{2} = \epsilon_c(G_i) \geq \frac{1}{2}[v_t + 2(v_3 - v_t) + r_i] = \frac{1}{2}[(v-1) + (v_3 - v_t) + r_i],$$

and so $4 \geq (v_3 - v_t) + r_i$. Since $d_{G_i}(x_i) = 4$, $v_3 - v_t \geq 4 - t_i$. A vertex in V_t can only be joined to x_i with a contractible edge, and so $r_i \geq t_i$. Thus, $4 \geq (v_3 - v_t) + r_i \geq (4 - t_i) + r_i \geq 4$. Therefore, $r_i = t_i$ and $4 - t_i = v_3 - v_t$, and so x_i is incident with all vertices in $V_3 - V_t$ and none of the edges joining x_i and a vertex in $V_3 - V_t$ is contractible. Since $t_i + (v_3 - v_t) = 4$ and $v_3 - v_t \equiv i - 1 \pmod 3$, $(i, t_i, v_3 - v_t) \in \{(1,1,3),(1,4,0),(3,2,2),(5,0,4),(5,3,1)\}$.

Consider G_1. Suppose $t_1 = 1$ and $v_3 - v_t = 3$. Since x_1 is incident with only one contractible edge, there is a triangle $x_1 v_1 w_1$ by Lemma 3. No vertex in V_t can be on a triangle with a vertex of degree 4, and so $v_1, w_1 \in V_3 - V_t$. Let $V_3 - V_t = \{u_1, v_1, w_1\}$. By Theorem 3, $G_1' = \widetilde{(G_1 - x_1 u_1)}$ is 3-connected. If $x_1 u_1$ is on a triangle, then we may assume that it is $x_1 u_1 v_1$. Let S be the set of edges in $E(G_1')$ which are incident with x_1, v_1, or w_1 but not on the triangle $x_1 v_1 w_1$. Then G_1 is obtained from G_1' by adding an edge obliquely across the independent 3-edge cut S. If $x_1 u_1$ is not on a triangle, then Lemma 1 implies that G_1 is obtained from G_1' by adding an edge obliquely across an independent 3-edge cut. Regardless of whether $x_1 u_1$ is on a triangle, $V(G_1') = V_t(G_1')$, so $G_1' \in \Gamma_0$. Hence, $G_1 \in \Gamma_1$.

Suppose $t_1 = 4$ and $v_3 - v_t = 0$ for G_1. Then Theorem 3 implies $G_1 \in \Gamma_1$.

Consider G_3. Then $t_3 = 2$ and $v_3 - v_t = 2$. Let $N_{G_3}(x_3) = \{u_3, v_3, w_3, z_3\}$, where $\{u_3, v_3\} = V_3 - V_t$. Suppose x_3 is not on a triangle. Then $G_3' = \widetilde{(G_3 - x_3 u_3)}$ is 3-connected and cubic by Theorem 3 since $x_3 u_3 \notin E_c$. Since $x_3 v_3 \notin E_c$, there exists some vertex w such that $T = \{x_3, v_3, w\}$ is a 3-vertex cut. Since x_3 is not on a triangle, the components of $G - T$ both have at least two vertices. Hence, if $w \neq u_3$ then T is a 3-vertex cut of G_3', and so $x_3 v_3 \notin E_c(G_3')$. If $w = u_3$, then e_{u_3} is a cut edge of $G_3' - \{x_3, v_3\}$ and so $x_3 v_3 \notin E_c(G_3')$. Since $V_t(G_3') = V(G_3') - \{x_3, v_3\}$, and $x_3 v_3 \notin E_c(G_3')$, $\epsilon_c(G_3') = \frac{v}{2} + 1$. Therefore, $G_3' \in \Gamma_2$. Since $x_3 u_3 \notin E_c(G)$, and x_3 is not on a triangle, Lemma 1 implies that G_3 is obtained from G_3' by adding an edge obliquely across an independent 3-edge cut. Thus, $G_3 \in \Gamma_3$. Suppose x_3 is on a triangle. Then the triangle is $x_3 u_3 v_3$. Let

$$G_3'' = (G_3 - x_3) + \{x_3', x_3''\} + \{u_3 x_3', v_3 x_3', x_3' x_3'', w_3 x_3'', z_3 x_3''\}$$

be obtained from G_3 by splitting x_3. By Theorem 2, G_3'' is 3-connected. Since $V_t(G_3'') = V(G_3'') - \{x_3''\}$, $G_3'' \in \Gamma_4$. Since $G_3 = G_3'' \cdot (x_3' x_3'')$, $G_3 \in \Gamma_3$.

Consider G_5. Suppose $t_5 = 0$ and $v_3 - v_t = 4$. Since x_5 is not incident with a contractible edge, Lemma 3 implies x_5 is on two edge disjoint triangles $x_5 u_5 v_5$ and $x_5 w_5 z_5$. No vertex in V_t can be on a triangle with x_5, and so $V_3 - V_t = \{u_5, v_5, w_5, z_5\}$. Let

$$G_5'' = (G_5 - x_5) + \{x_5', x_5''\} + \{x_5' u_5, x_5' v_5, x_5' x_5'', x_5'' y_5, x_5'' z_5\}$$

be obtained from G_5 by splitting x_5. By Theorem 2, G_5'' is 3-connected. Since we also have $V(G_5'') = V_t(G_5'')$, $G_5'' \in \Gamma_0$. Since $G_5 = G_5'' \cdot (x_5' x_5'')$, $G_5 \in \Gamma_5$.

Suppose $t_5 = 3$ and $v_3 - v_t = 1$. Let $V_3 - V_t = \{u_5\}$. Since $x_5 u_5$ is not on a triangle and $x_5 u_5 \notin E_c$, Lemma 1 implies that G_5 is obtained from $G_5' = (G_5 - x_5 u_5)^{\sim}$ by adding an edge obliquely across an independent 3-edge cut. Since $V_t(G_5') = V(G_5') - \{x_5\}$, $G_5' \in \Gamma_4$, and so $G_5 \in \Gamma_5$.

Finally, suppose G is a 3-connected graph such that $v(G) \geq 5$, $\epsilon_c(G) = f(v(G))$, and G is not critical. Let G' be a critically 3-connected spanning subgraph of G. Then $\epsilon_c(G') = f(v(G'))$, and so $G' \in \Gamma$ and $\epsilon_c(G') = \epsilon_c(G)$.

Consider e in $E(G) - E(G')$. Suppose e is incident with a vertex v in $V_t(G')$. Let v be on the triangle vwx and let y be the other vertex adjacent to v in G'. It is easy to see that $\{w, x, y\}$ is the only 3-vertex cut in G' containing w and x. Since $e \in G$, $\{w, x, y\}$ is not a 3-vertex cut in G. Hence, $wx \in E_c(G)$. But then $E_c(G') \cup \{wx\} \subseteq E_c(G)$, and we have a contradiction with $\epsilon_c(G') = \epsilon_c(G)$. Therefore, e joins two nonadjacent vertices in $V(G') - V_t(G')$. This is only possible if $G' \in \Gamma_3$, $V(G') - V_t(G')$ is a 3-vertex cut, and e joins the two vertices of degree two in $V(G') - V_t(G')$. Hence, $\epsilon(G) - \epsilon(G') = 1$ and $G \in \Gamma_3$.

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Since the completion of this work it has come to the author's attention that Ando, Enomoto, and Saito [1] have independently obtained similar results. Using slightly different methods they have determined f and proven that $E_c(G) = \frac{v}{2}$ implies $G \in \Gamma_0$.

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