ON PBIBD CONSTRUCTIONS USING t-DESIGNS

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Abstract. Constructions of partially balanced incomplete block designs with three and four associate classes are given. The constructions use t-designs for $t = 6$ and $t = 8$.

1. Introduction.

Various authors including Saha [1], Sarvate [2] and Sinha [3, 4] have constructed series of balanced incomplete block designs (BIBDs) and partially balanced incomplete block designs (PBIBDs) using t-designs.

The present construction is a generalization of a construction by Sarvate [2]. It is of interest as it is based on occurrences of unordered s-triples (rather than pairs) in the blocks of a t-design.

2. Constructions.

Theorem 2.1. *Let* $X = BIBD(v, b, r, k, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)$ *be a 6-design. Then* $Y = PBIBD\big(V = \binom{v}{3},\ B = b,$

$$\begin{aligned} R &= b - 3r + 3\lambda_2, \\ K &= \binom{k}{3} + \binom{v-k}{3}, \\ \Lambda_1 &= b - 4r + 6\lambda_2 - 4\lambda_3 + 2\lambda_4, \\ \Lambda_2 &= b - 5r + 10\lambda_2 - 10\lambda_3 + 5\lambda_4, \\ \Lambda_3 &= b - 6r + 15\lambda_2 - 18\lambda_3 + 9\lambda_4\big) \end{aligned}$$

exists.

Proof: Let the points of the PBIBD Y be the 3-sets (triples) of the points of X. Each block B of X gives a block B' of Y, constructed as follows:

The points in B' are the triples of the points in B and the triples of the points in the complement of B. (Complement of B consists of all the points of X not in B).

Parameter checking:

R : A triple (a, b, c) occurs λ_3 times in X, each of the three pairs of the triple, but not the complete triple, occurs $\lambda_2 - \lambda_3$ times in X. Each of the points a, b, c, but not any other point of the triple, occurs

$$r - 2(\lambda_2 - \lambda_3) - \lambda_3 = r - 2\lambda_2 + \lambda_3$$

times in X. Therefore, the number of times the triple (a, b, c) is in the complement of a block of X is

$$b - \lambda_3 - 3(\lambda_2 - \lambda_3) - 3(r - 2\lambda_2 + \lambda_3) = b - 3r + 3\lambda_2 - \lambda_3.$$

Therefore, the triple (a, b, c) is in $R = b - 3r + 3\lambda_2$ blocks of Y.

Λ_1 : The first associates are the triples having two points in common. Consider a 4-set $\{a, b, c, d\}$, it occurs in λ_4 blocks of X and none of a, b, c and d occurs in $b - 4r + 6\lambda_2 - 4\lambda_3 + \lambda_4$ blocks. Therefore, the pairs of the type $((a, b, c), (a, b, d))$ occur $b - 4r + 6\lambda_2 - 4\lambda_3 + 2\lambda_4$ times in Y.

Λ_2 : The second associates are the triples having one point in common. We can verify that $\Lambda_2 = b - 5r + 10\lambda_2 - 10\lambda_3 + 5\lambda_4$ by noting that such pairs of triples arise from 5-sets, each of which occurs in λ_5 blocks of X.

Λ_3 : The third associates are the triples with none of the points in common. We can verify that $\Lambda_3 = b - 6r + 15\lambda_2 - 18\lambda_3 + 9\lambda_4$ by noting that such pairs of triples arise from 6-sets, each of which occurs in λ_6 blocks of X. ∎

When Λ_1, Λ_2 and Λ_3 are equated it can be seen that $v = k$. Therefore, we can not get a BIBD from the construction.

Theorem 2.2. *Let* $X = BIBD(v, b, r, k, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8)$ *be an 8-design. Then there exists* $Y = PBIBD\left(V = \binom{v}{4},\ B = b,\right.$

$$R = b - 4r + 6\lambda_2 - 4\lambda_3 + 2\lambda_4,$$
$$K = \binom{k}{4} + \binom{v-k}{4},$$
$$\Lambda_1 = b - 5r + 10\lambda_1 - 10\lambda_3 + 5\lambda_4,$$
$$\Lambda_2 = b - 6r + 15\lambda_2 - 20\lambda_3 + 15\lambda_4 - 6\lambda_5 + 2\lambda_6,$$
$$\Lambda_3 = b - 7r + 21\lambda_2 - 35\lambda_3 + 35\lambda_4 - 21\lambda_5 + 7\lambda_6,$$
$$\left.\Lambda_4 = b - 8r + 28\lambda_2 - 56\lambda_3 + 72\lambda_4 - 64\lambda_5 + 40\lambda_6 - 16\lambda_7 + 4\lambda_8\right).$$

Proof: Let the points of the PBIBD Y be the 4-sets (quadruples) of the points of X. Each block B of X gives a block B' of Y, constructed as follows:

The points in B' are the quadruples of the points in B and the quadruples of the points in the complement of B. The proof is based on a similar but lengthier counting argument. The first associates are the quadruples (a, b, c, d) and (a, b, c, e) that is, where only one element is different. The second, third and fourth associates of a quadruple are, respectively, the quadruples with two, three and all of the four elements different from the given quadruple. ∎

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References

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