On Random Walks In A Plane

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Abstract. Consider a random walk in a plane in which a particle at any stage moves one unit in any one of the four directions, namely, north, south, east, west with equal probability. The problem of finding the distribution of any characteristic of the above random walk when the particle reaches a fixed point (a, b) after d steps reduces to the counting of lattice paths in a plane in which the path can move one unit in any of the four directions. In this paper path counting results related to the boundaries $y - x = k_1$ and $y + x = k_2$ such as touchings, crossings, etc., are obtained by using either combinatorial or probabilistic methods. Some extensions to higher dimensions are indicated.

1. Introduction.

Random walks on a line have been studied extensively and most of the results are reported in various books (for example, Feller [4], Mohanty [7], Spitzer [8]). Let us consider a random walk in a plane in which a particle starting from the origin moves at any stage one unit in any of the four directions, namely, north, south, east and west with equal probability (in short, it is the 2-dimensional simple symmetric walk). In the random walk, since every path of length d in the plane has the probability $(1/4)^d$, the determination of the distribution of any characteristic of the walk when the particle starting from $(0,0)$ reaches a fixed point (a,b) after d steps needs the knowledge of the number of paths corresponding to the characteristic under consideration and the number of all paths of length d from $(0,0)$ to (a,b). Thus, our attention is focused to certain lattice path counting problems.

In this direction, some results of DeTemple and Robertson [2] which are of use in the sequel are quoted below.

(i) The total number of paths of length d from $(0,0)$ to (a,b) which is denoted by $N(a, b; d)$, is given by

$$N(a, b; d) = \binom{d}{\frac{d-a+b}{2}} \binom{d}{\frac{d-a-b}{2}}. \tag{1}$$

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(ii) The number of paths of length d from $(0,0)$ to (a,b), $a > b-k$, not touching the line $y-x = k$ $(k > 0)$ is given by

$$N(a,b;d) - N(a+k,b-k;d) = \binom{d}{\frac{d-a-b}{2}}\left[\binom{d}{\frac{d-a+b}{2}} - \binom{d}{\frac{d-a+b-2k}{2}}\right]. \quad (2)$$

Note that (2) is also the number of paths not crossing the line $y-x = k-1$. A special case of interest arises when $k = 1$. The number of paths of length d from $(0,0)$ to (a,b), $a \geq b$, not crossing the line $x = y$ and lying below it is

$$\frac{a-b+1}{d+1}\binom{d+1}{\frac{d-a+b}{2}}\binom{d}{\frac{d-a-b}{2}}. \quad (3)$$

(iii) The number of paths of length d from $(0,0)$ to (a,b), $a > b$, which remain entirely below the line $y = x$ except at the origin is given by

$$\frac{a-b}{d} N(a,b;d). \quad (4)$$

For (iii) see also DeTemple, Jones and Robertson [3]. In an alternative proof of (4), one may consider a typical path starting from $(0,0)$ which necessarily passes either through $(1,0)$ or $(0,-1)$. Therefore by using (2) for $k = 1$, the expression for the required number of paths can be seen to simplify to (4).

Observe that (2) and (4) are similar to those for two-directional lattice paths.

An application of (4) and the argument used in the alternative proof leads to the following result which will be used subsequently:

(iv) The number of paths of length d from $(0,0)$ to (a,a) lying entirely below the line $y = x$ except at the end points, denoted by $N(a;d;0)$ is given by

$$\frac{1}{d-1}\binom{d-1}{\frac{d-2}{2}}\binom{d}{\frac{d-2a}{2}}. \quad (5)$$

In Section 2, as examples of deriving distributions by combinatorial methods, we enumerate paths of length d from $(0,0)$ to (a,b) which (i) touch (ii) reach and (iii) cross the line $y-x = k$ or $y+x = k$ a fixed number of times. Also the paths not crossing the lines $y-x = k_1$ and $y-x = -k_2$ or the lines $x+y = k_1$ and $x+y = -k_2$, $k_1 > 0$, $k_2 > 0$ are counted. Here k_1, k_2 and k are integers.

In Section 3, we first establish an elegant probabilistic result which enables us to rederive distributions of Section 2 in an independent manner. In addition, a few new distributions are obtained with ease by this approach. Finally some extensions to higher dimensions are discussed.

2. Distributions by combinatorial method.

Because of the comment in Section 1 and the expression (1) what remains to be done for finding the conditional distribution of a characteristic, say, the distribution of touches with the line $y - x = k$, given the random walk at (a, b) in d steps is to determine the number of paths of length d from $(0, 0)$ to (a, b) which touch the line $y - x = k$ exactly r times and lying below the line $y - x = k$, $r = 0, 1, \ldots$. Denote this number by $N(a, b; d; k, r)$. In this notation, what is counted in (iii) can be denoted by $N(b, a; d; a - b, 1)$.

Theorem 1. *For* $a \geq b - k$, $k \geq 0$ *and* $r \geq 1$

$$N(a, b; d; k, r) = \frac{a - b + 2k + r - 1}{d - r + 1} \binom{d - r + 1}{\frac{d-a+b-2k-2r+2}{2}} \binom{d}{\frac{d-a-b}{2}}. \qquad (6)$$

When $k = 0$*, the starting point is counted as a touch. Also when* $k = 0$ *and* $a = b$*, the path lies below the line* $x = y$.

Proof: Consider $k > 0$. It is instructive to first examine the case $r = 1$. Two proofs will be presented, one by geometric construction and the other being analytic.

Let a typical path touch the line $y - x = k$ at the point $(x, k + x)$ in d_1 $(< d)$ steps. Transform this path into a new path by reflecting the segment from $(0, 0)$ to $(x, k + x)$ about $y = x$ and then attaching to it the reversed path of the remaining segment. The new path is one of length d from $(0, 0)$ to $(a + k, b - k)$ not touching the line $y - x = -(a - b + 2k)$ except at the end point. This mapping being one-to-one, the number of latter paths in reverse order by (4) equals to (6). This provides a proof by construction.

For an alternative proof, we see that

$$N(a, b; d; k, 1) = \begin{cases} \displaystyle\sum_{d_1 = k}^{d-a+b-k} \sum_{x=0}^{(d_1 - k)/2} A & \text{if } x \geq 0, \\ \displaystyle\sum_{d_1 = k}^{d-a-b+k-2} \sum_{x=-(k-1)}^{-1} A & \text{if } -(k-1) \leq x \leq -1, \\ \displaystyle\sum_{d_1 = k}^{d-a-b-k} \sum_{x=-(d_1+k)/2}^{-k} A & \text{if } x \leq -k. \end{cases} \qquad (7)$$

where

$$d_1 - k \equiv 0 \pmod 2,$$

and

$$A = N(x, k + x; d_1; k, 1)\, N(b - k - x, a - x; d - d_1; a - b + k, 1).$$

But (7) can be rewritten as

$$N(a,b;d;k,1) = \sum_{\substack{d_1=k \\ d_1-k\equiv 0 \pmod 2}}^{d-a+b-k} \sum_{x=-(d_1+k)/2}^{(d_1-k)/2} A \tag{8}$$

by observing that the extra terms are all zeros. Letting $u = \frac{d_1-k}{2}$, expression (8) becomes

$$\sum_{u=0}^{(d-a+b-2k)/2} \frac{k}{k+2u}\binom{k+2u}{u}\frac{a-b+k}{d-k-2u}\binom{d-k-2u}{\frac{d-a+b-2k-2u}{2}} \sum_{x=-(u+k)}^{u}\binom{k+2u}{u-x}\binom{d-k-2u}{\frac{d-a-b}{2}-u+x}. \tag{9}$$

Making the substitution $v = u - x$, the second summation simplifies to

$$\binom{d}{\frac{d-a-b}{2}}.$$

Now, it can be checked that expression (9) equals (6) by using the convolution identity in Mohanty [7], p. 15.

Let us come back to the general case and initially determine $N(x, k+x; d; k, r)$. For $r = 1$, the number is given by (4) as

$$\frac{k}{d}\binom{d}{\frac{d-k-2x}{2}}\binom{d}{\frac{d-k}{2}}$$

which verifies (6). We can see that

$$N(x,k+x;d;k,r)$$
$$= \sum_{\substack{d_1=k+2r-4 \\ d_1-k\equiv 0 \pmod 2}}^{d-2} \sum_{x_1=-(d_1+k)/2}^{(d_1-k)/2} N(x_1,k+x_1;d_1;k,r-1)N(x-x_1;d-d_1;0)$$

where the first factor is given by (6) by inductive reasoning and the second by (5). Further simplification in the lines that (8) has been done ultimately leads to (6) for $a = x, b = k + x$.

Finally observe that

$$N(a,b;d;k,r)$$
$$= \sum_{\substack{d_1=k+2r-4 \\ d_1-k\equiv 0 \pmod 2}}^{d-a+b-k} \sum_{x=-(d_1+k)/2}^{(d_1-k)/2} N(x,k+x;d_1;k,r)\ N(b-k-x,a-x;d-d_1;a-b+k,1)$$

which on simplification as done earlier becomes (6).

Proceeding in a similar manner as for $k > 0$, one can easily establish (6) for $k = 0$. This completes the proof. ∎

We remark that on summing over $r \geq 1$ and adding to it the expression for $r = 0$, we get the number of paths which do not cross the line $y - x = k$ and this expression checks with the number of those paths which do not touch the line $y - x = k + 1$.

Next consider the distribution of arrivals for which we need the number of paths of length d from $(0,0)$ to (a,b) reaching the line $y - x = k$ exactly r times, to be denoted by $N^{*}(a,b;d;k,r)$.

Theorem 2. *For* $a \geq b - k$, $k \geq 0$ *and* $r \geq 1$

$$N^{*}(a,b;d;k,r) = 2^{r-1}\, N(a,b;d;k,r). \tag{10}$$

When $k = 0$, *the starting point is counted as an arrival.*

Proof: Observe that every path with r touches and lying below the line $y - x = k$ by itself is a path with r arrivals and secondly becomes a path with r arrivals when any segment of the path between any two consecutive touches is reflected about the line $y - x = k$ and this construction generates without repetition all paths with r arrivals. Since there are $r - 1$ such segments, the result follows. ∎

Now we move to determine the distribution of crossings and let us denote by $N^{+}(a,b;d;k,r)$ the number of paths of length d from $(0,0)$ to (a,b) crossing the line $y - x = k$ exactly r times.

Theorem 3.

(a) *For* $a > b - k$, $k > 0$ *and* $r \geq 1$

$$N^{+}(a,b;d;k,r) = \begin{cases} \frac{a-b+2k+2r}{d+2}\binom{d+2}{\frac{d-a+b-2k-2r+2}{2}}\binom{d}{\frac{d-a-b}{2}} & \text{when } r \text{ is even,} \\ 0 & \text{when } r \text{ is odd.} \end{cases} \tag{11}$$

(b) *For* $k > 0$ *and* $r \geq 1$

$$N^{+}(a,a+k;d;k,r) = \frac{k+1+2r}{d+1}\binom{d+1}{\frac{d-k-2r}{2}}\binom{d}{\frac{d-k-2a}{2}}. \tag{12}$$

(c) *For* $a > b$, *and* $r \geq 1$

$$N^{+}(a,b;d;0,r) = \frac{a-b+1+2r}{d+1}\binom{d+1}{\frac{d-a+b-2r}{2}}\binom{d}{\frac{d-a-b}{2}}. \tag{13}$$

(d) *For* $r \geq 1$

$$N^{+}(a,a;d;0,r) = \frac{4(r+1)}{d}\binom{d}{\frac{d-2(r+1)}{2}}\binom{d}{\frac{d-2a}{2}}. \tag{14}$$

The argument is routine and similar to the previous situations except that we have to use the number of paths not crossing a boundary line, the expression for which is given in (2) or (3). Sometimes the simplification becomes tedious. This provides an indication of the proof.

Distributions of these types for two directional walks have been considered by Csáki and Vincze [1], Kanwar Sen [5] and Mohanty [6].

So far all distributions have been derived only in relation to the boundary $y-x=k,\ k\geq 0$. However, because of symmetry we can obtain the same for boundaries $y-x=-k,\ y+x=k$ and $y+x=-k$ directly from the derived results for the line $y-x=k,\ k\geq 0$, by some simple changes. The changes are as follows: change (a,b) to (b,a) or $(-a,-b)$, $(-a,b)$ and $(a,-b)$ if the boundary is $y-x=-k$, $y+x=k$ and $y+x=-k$, respectively.

Another simple generalization of path counting problem is to find the number of paths between two boundaries.

Theorem 4. *The number of paths of length d from $(0,0)$ to (a,b), $b-k_1<a<b+k_2$, $k_1>0$, $k_2>0$ which do not touch the lines $y-x=k_1$ and $y-x=-k_2$ is given by*

$$\binom{d}{\frac{d-a-b}{2}}\sum_{j=-\infty}^{\infty}\left[\binom{d}{\frac{d-a+b}{2}+j(k_1+k_2)}-\binom{d}{\frac{d-a+b}{2}+j(k_1+k_2)-k_1}\right]. \tag{15}$$

The proof is omitted since it is analogous to the two-directional lattice path counting problem which involves the use of the inclusion-exclusion method and the repeated use of the reflection principle (see Mohanty [7] p. 6, Wilks [9] p. 457). In our case what is needed is to utilize (1) whenever necessary.

Note that if the boundaries are $x+y=k_1$ and $x+y=-k_2$, $k_1>0$, $k_2>0$ the corresponding result is obtained by replacing a by $-a$ in (15).

3. Distributions by probabilistic method.

We may have observed that in every result there are two factors one having $a+b$ (such as in $\binom{d}{\frac{d-a-b}{2}}$) and the other having $a-b$. This suggests that if (X,Y) is the position of the particle after d steps then $X+Y$ and $X-Y$ are independent. In fact we prove a general result to this effect in 2^m dimension.

Consider the n-dimensional simple symmetric walk and let $(X_d^{(1)},\dots,X_d^{(n)})$ denote the position after d steps. Assume that the walk started from the origin. Let $n=2^m$ for some m and denote by

$$\left(\varepsilon_1^{(i)},\dots,\varepsilon_m^{(i)}\right),\quad i=1,\dots,n$$

all possible combinations of +1's and −1's, that is, $\varepsilon_j^{(i)}$ is either $a+1$ or $a-1$ for every i and j and $(\varepsilon_1^{(\alpha)}, \ldots, \varepsilon_m^{(\alpha)}) \neq (\varepsilon_1^{(\beta)}, \ldots, \varepsilon_m^{(\beta)})$ if $\alpha \neq \beta$. Define

$$S_d^{(0)} = \sum_{i=1}^{n} X_d^{(i)}, \ S_d^{(j)} = \sum_{i=1}^{n} \varepsilon_j^{(i)} X_d^{(i)}, \quad j = 1, \ldots, m.$$

Theorem 5. *The processes* $\{S_d^{(i)},\ d = 0, 1, \ldots\}$, $i = 0, 1, \ldots, m$ *are independent one-dimensional simple symmetric random walks.*

Proof: Obviously $S_d^{(i)} - S_{d-1}^{(i)}$ $i = 0, 1, \ldots, m$ are either +1's or −1's with probability 1/2 each. Hence $\{S_d^{(i)},\ d = 0, 1, \ldots\}$, $i = 0, 1, \ldots, m$ are simple symmetric random walks. For independence, observe that $(X_d^{(1)} - X_{d-1}^{(1)}, \ldots, X_d^{(n)} - X_{d-1}^{(n)})$ are independent vectors for every d and therefore $(S_d^{(0)} - S_{d-1}^{(0)}, S_d^{(1)} - S_{d-1}^{(1)}, \ldots, S_d^{(m)} - S_{d-1}^{(m)})$ are independent vectors for every d. What remains is to show the independence within components of the vectors. Since the distributions do not depend on d, it suffices to prove independence of components only for $d = 1$, that is, for

$$\left(S_1^{(0)}, S_1^{(1)}, \ldots, S_1^{(m)}\right) = \left(\sum_{i=1}^{n} X_1^{(i)}, \sum_{i=1}^{n} \varepsilon_1^{(i)} X_1^{(i)}, \ldots, \sum_{i=1}^{n} \varepsilon_m^{(i)} X_1^{(i)}\right).$$

Note that for the n-dimensional simple symmetric walk we have

$$P\left(X_1^{(j)} = 1, X_1^{(i)} = 0 \ \text{ for each } i \neq j\right) = P\left(X_1^{(j)} = -1, X_1^{(i)} = 0 \ \text{ for each } i \neq j\right)$$
$$= \frac{1}{2n} = \frac{1}{2^{m+1}}, \quad j = 1, \ldots, n.$$

Thus

$$P\left(S_1^{(0)} = 1, S_1^{(1)} = \varepsilon_1^{(j)}, \ldots, S_1^{(m)} = \varepsilon_m^{(j)}\right)$$
$$= P\left(S_1^{(0)} = -1, S_1^{(1)} = -\varepsilon_1^{(j)}, \ldots, S_1^{(m)} = -\varepsilon_m^{(j)}\right) = \frac{1}{2^{m+1}}, \quad j = 1, \ldots, n.$$

This shows independence of $S_1^{(0)}, S_1^{(1)}, \ldots, S_1^{(m)}$ since $(1, \varepsilon_1^{(j)}, \ldots, \varepsilon_m^{(j)})$ and $(-1, -\varepsilon_1^{(j)}, \ldots, -\varepsilon_m^{(j)})$, $j = 1, \ldots, n$ give all possible combinations of +1's and −1's of size $m+1$. Hence the proof is completed. ∎

Remarks.

1. For $n = 2$ the theorem yields that $\{S_d^{(0)} = X_d^{(1)} + X_d^{(2)}\}$ and $\{S_d^{(1)} = X_d^{(1)} - X_d^{(2)}\}$ are independent one-dimensional simple symmetric random walks. For $n = 4$, these are $\{S_d^{(0)} = X_d^{(1)} + X_d^{(2)} + X_d^{(3)} + X_d^{(4)}\}$,

$\{S_d^{(1)} = X_d^{(1)} + X_d^{(2)} - X_d^{(3)} - X_d^{(4)}\}$, and $\{S_d^{(2)} = X_d^{(1)} - X_d^{(2)} + X_d^{(3)} - X_d^{(4)}\}$.

2. The assumption that n is a power of 2 is crucial since if $n \neq 2^m$ then this procedure does not work, and we cannot claim independence of simple symmetric random walks $S_d^{(0)}, S_d^{(1)}, \dots$.

From Theorem 5 we derive the following corollary which is remarkably useful in deriving distributions in a simple manner.

Corollary. *Let $F_d^{(j)}$ denote the σ-algebra of events generated by $S_1^{(j)}, \dots, S_d^{(j)}$, $j = 0, 1, \dots, m$ and assume that $A_j \in F_d^{(j)}, j = 0, 1, \dots, m$. Then*

$$P(A_1, \dots, A_j) = P(A_1)P(A_2)\dots P(A_j).$$

By applying this corollary we can derive results concerning higher dimensional random walks from the corresponding results for one-dimensional walk. For example, when $n = 2$ we have

$$P\left(X_d^{(1)} = a, X_d^{(2)} = b\right) = P\left(S_d^{(0)} = a + b, S_d^{(1)} = a - b\right)$$
$$= P\left(S_d^{(0)} = a + b\right)P\left(S_d^{(1)} = a - b\right) = \frac{1}{4^d}\binom{d}{\frac{d-a-b}{2}}\binom{d}{\frac{d-a+b}{2}}$$

which is essentially Theorem 2 of DeTemple and Robertson [2]. Moreover for $a > b, d \geq a + b, 0 < k \leq b + \frac{d-a-b}{2}$,

$$P\left(X_i^{(2)} - X_i^{(1)} < k,\ i = 1, \dots, d, X_d^{(1)} = a, X_d^{(2)} = b\right)$$
$$= P\left(S_i^{(1)} > -k,\ i = 1, \dots, d, S_d^{(0)} = a + b, S_d^{(1)} = a - b\right)$$
$$= P\left(S_i^{(1)} > -k,\ i = 1, \dots, d, S_d^{(1)} = a - b\right)P\left(S_d^{(0)} = a + b\right)$$
$$= \frac{1}{4^d}\left[\binom{d}{\frac{d-a+b}{2}} - \binom{d}{\frac{d-a+b-2k}{2}}\right]\binom{d}{\frac{d-a-b}{2}}$$

which is equivalent to a result in DeTemple and Robertson [2].

Likewise our results in Section 2 can be obtained. We only rederive (6). Let $k > 0, a \geq b - k$ and let $\rho = |\{i\colon 1 \leq i \leq d, X_i^{(2)} - X_i^{(1)} = k\}|$. Then for $r \geq 1$ we have

$$P\left(X_i^{(2)} - X_i^{(1)} \leq k,\ i = 1, 2, \dots, d, \rho = r, X_d^{(1)} = a, X_d^{(2)} = b\right)$$
$$= P\left(S_i^{(1)} \geq -k,\ i = 1, 2, \dots, d, \rho = r, S_d^{(1)} = a - b\right)P\left(S_d^{(0)} = a + b\right)$$
$$= \frac{1}{4^d}\frac{a - b + 2k + r - 1}{d - r + 1}\binom{d - r + 1}{\frac{d-a+b-2k-2r+2}{2}}\binom{d}{\frac{d-a-b}{2}}$$

by using a result of Kanwar Sen [5].

We realize that quite a few distributions can be derived by this approach. We will illustrate it by obtaining two new ones.

Let

$$M_d = \max_{1\le i\le d}\left(X_2^{(i)} - X_1^{(i)}\right), \quad \lambda_d = |\{i\colon X_2^{(i)} - X_1^{(i)} = M_{d'},\ 1 \le i \le d\}|,$$

and

$$\Delta_d^{(j)} = \text{ index where } M_d \text{ is achieved for the } j\text{th time.}$$

Then simple arguments concerning one-dimensional random walk lead us to the following joint distribution:

Theorem 6. *For* $a > b - k$, $1 \le j \le \ell$ *and* $k > 0$,

$$P\left(M_d = k, \lambda_d = \ell, \Delta_d^{(j)} = d_1, X_d^{(1)} = a, X_d^{(2)} = b\right)$$
$$= \frac{1}{4^d}\frac{k+j-1}{d_1-j+1}\binom{d_1-j+1}{\frac{d_1-k-2j+2}{2}}\frac{a-b+k+\ell-j}{d-d_1-\ell+j}\binom{d-d_1-\ell+j}{\frac{d-d_1-a+b-k-2\ell+2j}{2}}\binom{d}{\frac{d-a-b}{2}}. \tag{16}$$

The next one is a result relating two types of boundaries of the form $X_i^{(1)} + X_i^{(2)} < k_1$ and $X_i^{(1)} - X_i^{(2)} < k_2$.

Theorem 7.

(a) *For* $\alpha_1 > 0, \alpha_2 > 0$, $a + b < \alpha_1$ *and* $a - b < \alpha_2$

$$P\left(X_i^{(1)} + X_i^{(2)} < \alpha_1, X_i^{(1)} - X_i^{(2)} < \alpha_2,\ i = 1, \dots, d, X_d^{(1)} = a, X_d^{(2)} = b\right)$$
$$= \frac{1}{4^d}\left[\binom{d}{\frac{d-a-b}{2}} - \binom{d}{\frac{d-a-b-2\alpha_1}{2}}\right]\left[\binom{d}{\frac{d-a+b}{2}} - \binom{d}{\frac{d-a+b-2\alpha_2}{2}}\right] \tag{17}$$

(b) *For* $\beta_1 < 0 < \alpha_1$, $\beta_2 < 0 < \alpha_2$, $\beta_1 < a + b < \alpha_1$ *and* $\beta_2 < a - b < \alpha_2$

$$P\left(\beta_1 < X_i^{(1)} + X_i^{(2)} < \alpha_1, \beta_2 < X_i^{(1)} - X_i^{(2)} < \alpha_2,\ i = 1, \dots, d, X_d^{(1)} = a, X_d^{(2)} = b\right)$$
$$= \frac{1}{4^d}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\left[\binom{d}{\frac{d-a-b-2j(\alpha_1-\beta_1)}{2}} - \binom{d}{\frac{d-a-b-2j(\alpha_1-\beta_1)+2\beta_1}{2}}\right] \tag{18}$$
$$\times\left[\binom{d}{\frac{d-a+b+2k(\alpha_2-\beta_2)}{2}} - \binom{d}{\frac{d-a+b+2k(\alpha_2-\beta_2)+2\beta_2}{2}}\right].$$

In the last section only one type of boundary has been dealt with. For both types the counting does not seem to be straightforward, although knowing the result in the above theorem one may be able to device a counting procedure.

What has been demonstrated above is that whenever there is a one-dimensional result it can be extended to two-dimensional walk. We can think of another example, say, of having the distribution of number of times the particle stays on a particular side of the line $y - x = k$ (see Csáki and Vincze [1] Kanwar Sen [5]). Similarly some limiting distributions can also be obtained.

We conclude this section by giving a remark on higher dimensional distributions. If $m > 1$, even though there are $m + 1$ independent one-dimensional random walks, the final point $(X_d^{(1)}, \ldots, X_d^{(n)})$ cannot be specified by fixing $(S_d^{(0)}, S_d^{(1)}, \ldots, S_d^{(m)})$. For example, when $m = 2$ or $n = 4$ there are only $S_d^{(0)}, S_d^{(1)}$ and $S_d^{(2)}$ fixing which would not uniquely determine $(X_d^{(1)}, X_d^{(2)}, X_d^{(3)}, X_d^{(4)})$. Thus the elegant procedure adopted for $m = 1$, cannot be utilized for $m \geq 2$. However, if the final position is not fixed, the property of independence can be used for deriving appropriate distributions.

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