Short Proofs of Some Fan-Type Results

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Abstract. For a graph G, define $\phi(G) = \min\{\max\{d(u), d(v)\} \mid d(u,v) = 2\}$ if G contains two vertices at distance 2, and $\phi(G) = \infty$ otherwise. Fan proved that every 2-connected graph on n vertices with $\phi(G) \geq \frac{1}{2}n$ is hamiltonian. Short proofs of this result and a number of analogues, some known, some new, are presented. Also, it is shown that if G is 2-connected, $\phi(G) \geq \frac{1}{2}(n-i)$ and $G - \{v \in V(G) \mid d(v) \geq \frac{1}{2}(n-i)\}$ has at least three components with more than i vertices, then G is hamiltonian $(i \geq 1)$.

We consider only finite undirected graphs without loops or multiple edges. For notation and terminology not defined here we refer to [4].

For a graph G, we define

$$\phi(G) = \begin{cases} \infty \text{ if every component of } G \text{ is complete} \\ \min\{\max\{d(u), d(v)\} | d(u,v) = 2\} \text{ otherwise,} \end{cases}$$

where $d(x)$ denotes the degree of the vertex x in G and $d(x,y)$ the distance of the vertices x and y in G. Note that $\phi(G) \geq \delta(G)$ for any graph G, where $\delta(G)$ denotes the minimum degree of vertices in G. Geng-Hua Fan established the following result.

Theorem 1 **[5].** *If G is a 2-connected graph on n vertices with $\phi(G) \geq \frac{1}{2}n$, then G is hamiltonian.*

Here we obtain Theorem 1 as a special case of a generic result, Theorem 2 below. Before stating Theorem 2 we give a number of preliminary definitions.

Bondy and Chvátal [3] defined a graph property P to be **$f(n)$-stable** if a graph G on n vertices has property P whenever $G+uv$ has property P and $d(u)+d(v) \geq f(n)$. We call a collection $\mathcal{G}$ of graphs **invariant** if, whenever $G \in \mathcal{G}$, every spanning supergraph of G is in $\mathcal{G}$. For a graph G and a real number r, we define $V_r(G) = \{v \in V(G) | d(v) \geq r\}$. If $S \subseteq V(G)$, then $N(S)$ denotes the set of all vertices of $G - S$ adjacent to at least one vertex of S. If H is a subgraph of G, then $N(H)$ means $N(V(H))$. We now define a collection $\mathcal{F}$ of graphs as follows: $G \in \mathcal{F}$ if $V(G)$ contains a subset U such that

(i) the induced subgraph $G[U]$ is complete, and
(ii) every component of $G - U$ is complete, and
(iii) if G_1 and G_2 are distinct components of $G-U$, then $N(G_1) \cap N(G_2) = \phi$.

Theorem 2. *Assume $\mathcal{G}$ is an invariant collection of graphs and P an $f(n)$-stable property shared by all graphs in $\mathcal{F} \cap \mathcal{G}$. If G is a graph in $\mathcal{G}$ on n vertices with $\phi(G) \geq \frac{1}{2}f(n)$, then G has property P.*

Proof: Assuming the theorem is not true, let G be a maximal graph in $\mathcal{G}$ such that G has n vertices, $\phi(G) \geq \frac{1}{2}f(n)$ and G does not have property P. Set $U = V_{\frac{1}{2}f(n)}(G)$. Since $\phi(G) \geq \frac{1}{2}f(n)$, no two vertices in $G-U$ are at distance 2, implying that every component of $G-U$ is complete and $N(G_1) \cap N(G_2) = \phi$ whenever G_1 and G_2 are distinct components of $G-U$. Since all graphs in $\mathcal{F} \cap \mathcal{G}$ enjoy P, we have $G \notin \mathcal{F}$. It follows that U contains two nonadjacent vertices u and v. We have $\phi(G+uv) \geq \frac{1}{2}f(n)$ and, by the fact that $\mathcal{G}$ is invariant, $G+uv \in \mathcal{G}$. Using the maximality of G we conclude that $G+uv$ has property P. Also, $d(u)+d(v) \geq f(n)$, since $u, v \in U$. However, P is $f(n)$-stable, implying that G itself has property P, a contradiction. ∎

Ore [6] was the first to observe that the property of being hamiltonian is n-stable. Since the collection of all 2-connected graphs is invariant and every 2-connected graph in $\mathcal{F}$ is clearly hamiltonian, we obtain Theorem 1 as a special case of Theorem 2.

Theorem 2 can be used to generate several other specific Fan-type results. As examples we present Theorems 6, 7 and 8 below. Before doing so we develop some additional terminology and notations and make a few observations.

If G is a graph, then $\eta(G)$ denotes the number of cut vertices of G and $\mu(G)$ the smallest number of pairwise disjoint paths covering all vertices of G. For a nonnegative integer s, the graph G is **s-hamiltonian** if $G-S$ is hamiltonian for every subset S of $V(G)$ with $0 \leq |S| \leq s$. Clearly, G is s-hamiltonian only if G is $(s+2)$-connected. G is **Hamilton-connected** if every pair of distinct vertices of G is connected by a Hamilton path. Clearly, G is Hamilton-connected only if G is 3-connected or $|V(G)| \leq 3$.

The following lemma is obvious.

Lemma 3. *Let k be a nonnegative integer.*

(a) *The collection $\{G|G$ is k-connected$\}$ is invariant.*
(b) *The collection $\{G|G$ is connected and $\eta(G) \leq k\}$ is invariant.*

The observations in Lemma 4 below are also easily checked. (For example, note with respect to Lemma 4(a) that if $G \in \mathcal{F}$ and S is any subset of $V(G)$, then $G-S \in \mathcal{F}$).

Lemma 4. *Let G be a connected graph in $\mathcal{F}$.*

(a) *G is s-hamiltonian if and only if G is $(s+2)$-connected $(0 \leq s \leq n-3)$.*
(b) *If $\eta(G) \leq 2s$, then $\mu(G) \leq s$ $(1 \leq s \leq n)$.*
(c) *G is Hamilton-connected if and only if either G is 3-connected or $G \in \{K_1, K_2, K_3\}$.*

(Short) proofs of the following assertions occur in [3].

Lemma 5 [3].

(a) *The property of being* s*-hamiltonian is* $(n+s)$*-stable* $(0 \le s \le n-3)$.
(b) *The property* "$\mu(G) \le s$" *is* $(n-s)$*-stable* $(1 \le s \le n)$.
(c) *The property of being Hamilton-connected is* $(n+1)$*-stable.*

We are now ready to state the announced results. Note that Theorem 6 generalizes Theorem 1, Theorem 7 contains a sufficient condition for a graph to have a Hamilton path as a special case $(s = 1)$, and Theorem 8 was first proved by Benhocine and Wojda [2].

Theorem 6. *If* G *is an* $(s+2)$*-connected graph on* n *vertices with* $\phi(G) \ge \frac{1}{2}(n+s)$, *then* G *is* s*-hamiltonian* $(0 \le s \le n-3)$.

Proof: Combine Theorem 2 with Lemmas 3(a), 4(a) and 5(a). ∎

If $n \ge s+5$, then the graph $G = K_{\lceil\frac{1}{2}(n-s)-1\rceil,\lfloor\frac{1}{2}(n-s)+1\rfloor} \vee K_s$, where $\vee$ denotes join, is $(s+2)$-connected and not s-hamiltonian while $\phi(G) = \delta(G) = \lceil\frac{1}{2}(n+s)\rceil - 1$. Hence Theorem 6 is best possible.

Theorem 7. *If* G *is a connected graph on* n *vertices with* $\eta(G) \le 2s$ *and* $\phi(G) \ge \frac{1}{2}(n-s)$, *then* $\mu(G) \le s$ $(1 \le s \le n)$.

Proof: Combine Theorem 2 with Lemmas 3(b), 4(b) and 5(b). ∎

Theorem 7 is best possible, as shown by the graph $K_{\lceil\frac{1}{2}(n-s)-1\rceil,\lfloor\frac{1}{2}(n+s)+1\rfloor}$ $(n \ge s+3)$.

Theorem 8 [2]. *If* G *is a 3-connected graph on* n *vertices with* $\phi(G) \ge \frac{1}{2}(n+1)$, *then* G *is Hamilton-connected.*

Proof: Combine Theorem 2 with Lemmas 3(a), 4(c) and 5(c). ∎

The graph $K_{\lceil\frac{1}{2}(n-1)\rceil,\lfloor\frac{1}{2}(n+1)\rfloor}$ $(n \ge 6)$ shows that Theorem 8 is best possible.

We turn to an improvement of Theorem 1 occurring in [2]. We define the collection $\mathcal{B}$ of 2-connected non-hamiltonian graphs as $\{H\} \cup \bigcup_{n\ge 7} \mathcal{G}_n$, where H and $\mathcal{G}_n$ are defined according to [2].

Theorem 9 [2]. *Let* G *be a 2-connected graph on* n *vertices with independence number* $\alpha(G) \le \frac{1}{2}n$. *If* $\phi(G) \ge \frac{1}{2}(n-1)$, *then either* G *is hamiltonian or* $G \in \mathcal{B}$.

For a graph G and a positive integer i, define $\omega_i(G)$ as the number of components of G with at least i vertices. The feature of interest here of the graphs in $\mathcal{B}$ is that $\omega(G - V_{\frac{1}{2}(n-1)}(G)) \le 2$ whenever $G \in \mathcal{B}$ and $|V(G)| = n$. We thus have the following consequence of Theorem 9.

Corollary 10. *If G is a 2-connected graph on n vertices with $\alpha(G) \leq \frac{1}{2}n$, $\phi(G) \geq \frac{1}{2}(n-1)$ and $\omega_2(G - V_{\frac{1}{2}(n-1)}(G)) \geq 3$, then G is hamiltonian.*

We prove a generalization of Corollary 10, using a lemma recently established in [1].

Lemma 11 [1]. *Let G be a graph on n vertices and S a vertex cut of G. Suppose some component of $G - S$ is complete and has vertex set A. If u and v are nonadjacent vertices in $V(G) - (S \cup A)$ such that $d(u) + d(v) \geq n - |A| + 1$, then G is hamiltonian if and only if $G + uv$ is hamiltonian.*

Theorem 12. *Let G be a 2-connected graph and i a positive integer. If $\phi(G) \geq \frac{1}{2}(n-i)$ and $\omega_{i+1}(G - V_{\frac{1}{2}(n-i)}(G)) \geq 3$, then G is hamiltonian.*

Proof: Assuming the theorem is not true, let G be a maximal 2-connected graph on n vertices such that $\phi(G) \geq \frac{1}{2}(n-i)$, $\omega_{i+1}(G - V_{\frac{1}{2}(n-i)}(G)) \geq 3$ and G is non-hamiltonian. Set $U = V_{\frac{1}{2}(n-i)}(G)$ and let G_1, G_2, G_3 be three distinct components of $G - U$ with at least $i + 1$ vertices. As in the proof of Theorem 2, every component of $G - U$ is complete and $N(H_1) \cap N(H_2) = \phi$ whenever H_1 and H_2 are distinct components of $G - U$. Furthermore, U contains two nonadjacent vertices u and v, otherwise $G \in \mathcal{F}$ and G would be hamiltonian by Lemma 4(a) (for $s = 0$). The maximality of G implies that $G + uv$ is hamiltonian. Since $N(G_1)$, $N(G_2)$ and $N(G_3)$ are pairwise disjoint, there exists an integer $j \in \{1,2,3\}$ such that $u, v \notin N(G_j)$. Set $S = N(G_j)$ and $A = V(G_j)$. Then G_j is a complete component of $G - S$ with vertex set A while u and v are nonadjacent vertices in $V(G) - (S \cup A)$ with $d(u) + d(v) \geq n - i \geq n - |A| + 1$. From Lemma 11 it now follows that G is hamiltonian, a contradiction. ∎

Note that by Theorem 12 the condition $\alpha(G) \leq \frac{1}{2}n$ in Corollary 10 can be dropped. Also note that a graph G satisfies the hypothesis of Theorem 12 only if $n \geq 3i + 9$.

We now show that Theorem 12 is best possible in two senses. For $i \geq 1$ and $n \geq 3i + 8$ we define the graph $G_{n,i}$ on n vertices by the following requirements:

(1) $V(G_{n,i}) = A_1 \cup A_2 \cup B_1 \cup B_2 \cup D \cup \{a_1, a_2\}$.

(2) The vertex sets of the components of $G_{n,i} - \{a_1, a_2\}$ are $A_1 \cup B_1$, $A_2 \cup B_2$ and D; each of these components is complete.

(3) $N(a_1) = N(a_2) = A_1 \cup A_2 \cup D$.

(4) $|A_1 \cup B_1| = \lceil \frac{1}{2}(n - i - 2) \rceil$ and $|A_2 \cup B_2| = \lfloor \frac{1}{2}(n - i - 2) \rfloor$, whence $|D| = i$.

(5) $|A_1 \cup A_2| = \lfloor \frac{1}{2}(n - 3i) \rfloor$ and $0 \leq |A_1| - |A_2| \leq 1$.

In particular, $G_{n,i}$ is 2-connected and non-hamiltonian, while $\phi(G_{n,i}) = \lfloor \frac{1}{2}(n-i) \rfloor$ and $V_{\lfloor \frac{1}{2}(n-i) \rfloor}(G_{n,i}) = A_1 \cup A_2 \cup \{a_1, a_2\}$.

For $i \geq 1$ and $n \geq 3i + 11$, the graph $G_{n,i+1}$ has $\phi(G_{n,i+1}) = \lceil \frac{1}{2}(n-i) \rceil - 1$ and $\omega_{i+1}(G_{n,i+1} - V_{\lceil \frac{1}{2}(n-i) \rceil - 1}(G_{n,i+1})) = 3$, showing that the lower bound for

$\phi(G)$ in the hypothesis of Theorem 12 cannot be relaxed.

We finally show that the lower bound for $\omega_{i+1}(G - V_{\frac{1}{2}(n-i)}(G))$ in the hypothesis of Theorem 12 cannot be relaxed either. If $i \geq 1, n \geq 3i + 8$ and $n - i$ is even, then $\phi(G_{n,i}) = \frac{1}{2}(n - i)$ and $\omega_{i+1}(G_{n,i} - V_{\frac{1}{2}(n-i)}(G_{n,i})) = 2$. If $i \geq 2, n \geq 3i + 9$ and $n - i$ is odd, then $\phi(G_{n,i-1}) = \lceil\frac{1}{2}(n - i)\rceil$ and $\omega_{i+1}(G_{n,i-1} - V_{\frac{1}{2}(n-i)}(G_{n,i-1})) = 2$. Note that for odd values of $n-i$ an example of a 2-connected non-hamiltonian graph G with $|V(G)| = n$, $\phi(G) = \lceil\frac{1}{2}(n-i)\rceil$ and $\omega_{i+1}(G - V_{\frac{1}{2}(n-i)}(G)) = 2$ exists only if $i \geq 2$ in view of Theorem 1. We also remark that, as it should be in view of Theorem 9, $G_{n,1} \in \mathcal{B}$ if n is odd.

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