Blocking sets in the large Mathieu designs, III:the case $S(5,8,24)$

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Dedicated to Prof. Guido Zappa on his 70^{th} birthday.

Abstract. Blocking sets in little and large Mathieu designs, have all been characterized except the case $S(5,8,24)$. The aim of this paper is to give the complete classification of blocking sets in this remaining case.

1. Introduction

As it is well known, E. Mathieu (1835–1890) discovered five special simple groups, which do not appear as members of an infinity family. They are denoted by M_{11}, M_{12}, M_{22}, M_{23}, M_{24}. Nowadays, such simple groups are referred to as sporadic simple groups. Recently, all sporadic groups have been classified. Moreover, M_{24} and M_{12} are quintuply transitive groups, while M_{23} and M_{11} are 4-transitive groups. Other examples of t-transitive groups with $t \geq 4$ are not known. In 1938 E. Witt gave the so-called Witt's construction of the Mathieu groups using the existence and uniqueness of certain Steiner systems. Such systems are called respectively the *little* Mathieu designs $S(5,6,12)$, $S(4,5,11)$ and the *large* Mathieu designs $S(5,8,24)$, $S(4,7,23)$ and $S(3,6,22)$. The Mathieu groups are precisely the automorphism group of such systems. One pattern of $S(5,8,24)$ is given by Todd and this can be found in the appendix of [10].

Little and large Mathieu designs have been investigated from many points of view. One open geometric problem is the study of their blocking sets.

A set of points of a Steiner system is called a *blocking set* if it contains no block, but intersects every block. Clearly, if C is a blocking set, its complement is a blocking set too. Moreover, a blocking set C is said to be *irreducible* if $\forall x \in C$, the set $C - \{x\}$ is not a blocking set. In a Steiner system there is no blocking set contained in one block (cf. [1]). A blocking set is called of *index two* if it is contained in the union of two blocks. The study of the blocking sets in the little Mathieu designs is very simple and it appears in [1], [5]. On the other hand, the study of the blocking sets in the large designs is more complicated. Three papers — [2], [3], [4] — have been done on them.

Now we deal with the case $S(5,8,24)$. Denote by B, B' two blocks in $S(5,8,24)$ with $|B \cap B'| = 2$. Fix $u \in B \backslash B'$, $v \in B' \backslash B$, $a \in B \triangle B'$, $z \in B \cap B'$. Define the following sets:

(1.1) $\mathcal{M} := B\triangle B'; \mathcal{M}_0 := B\triangle B' - \{a\}$,

(1.2) $I := B \cup B' - \{u, v\}; R := B \cup B' - \{z, a\}$.

In Section 3 of this paper we shall prove the following:

1.1 Theorem. *Let C be a blocking set in $S(5,8,24)$. Then $11 \leq |C| \leq 13$. Moreover,*

(a) $|C| = 11$ *implies that* $C = \mathcal{M}_0, \mathcal{M}_0$ *being contained in the union of two blocks, i.e.* $\mathcal{M}_0$ *has index two.*

(b) $|C| = 12$ *and C irreducible imply that $C = I$, being I of index two.*

(c) $|C| = 12$ *and C reducible imply that* $C = \mathcal{M}_0 \cup \{x\}, x \notin \mathcal{M}_0$. *Moreover, if C has index two, then either $C = \mathcal{M}$ or $C = R$.*

(d) $|C| = 13$ *implies that C is reducible and C is the complement of* $\mathcal{M}_0$. *Moreover, if C is of index two, then $C = B \cup B' - \{a\}$, where B, B' are two blocks with $|B \cap B'| = 2$ and $a \in B \cap B'$.*

We recall that the *general blocking set problem* is the following. Can an n-colouring exist in a block design such that any block contains at least one point of each colour?

In Section 4 we deal with this problem, in view of papers [1] to [5]. So, in the case of Mathieu designs we have the following answer. In $S(2,5,21) \simeq PG(2,4)$ there is an n-colouring with $n = 2,3$. In $S(4,5,11)$ and $S(2,4,9)$ there are some monochromatic blocks in any 2-colouring. In the other cases there is only a 2-colouring.

2. Preliminaries and results

We recall that a Steiner system $S(t,k,v)$ is a pair $(S, \mathcal{B})$, where S is a v-set of elements called points, $\mathcal{B}$ is a family of k-sets called blocks, such that any fixed t-set is contained in exactly one element of $\mathcal{B}$. Denote by r_s ($s = 0,1,\dots,t$) the number of blocks containing a fixed s-set, then:

$$r_s = \binom{v-s}{t-s} \Big/ \binom{k-s}{t-s}.$$

Fix a point $x \in S$, set:

$$\mathcal{B}_x = \{B - \{x\} : x \in B, B \in \mathcal{B}\}.$$

The pair $(S - \{x\}, \mathcal{B}_x)$ is a Steiner system $S(t-1, k-1, v-1)$, which is said to be the contraction of $S(t,k,v)$ at the point x.

Let C be a c-set in a Steiner system $S(t,k,v)$. Denote by x_i the number of blocks having exactly i points in common with C. Let $M = \{m_1, m_2, \dots, m_h\}$ be a set of integers with $0 \leq m_1 < m_2 < \dots < m_h$. A set C is said to be of type $(m_1, m_2, \dots, m_h)$ if $x_i \neq 0$, if and only if $i \in M$. We have the following identities (cf. [1], [9]):

$$\text{(2.1)} \qquad \sum_{i=s}^{k} \binom{i}{s} x_i = r_s \binom{c}{s}, s = 0,1,\dots,t.$$

In the case of $S(5,8,24)$, if C is a blocking set, then (2.1) implies:

$$(2.2)\qquad \begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 759 \\ x_2 + 2x_3 + 3x_4 + 4x_5 + 5x_6 + 6x_7 = g_1 \\ x_3 + 3x_4 + 6x_5 + 10x_6 + 15x_7 = g_2 \\ x_4 + 4x_5 + 10x_6 + 20x_7 = g_3 \\ x_5 + 5x_6 + 15x_7 = g_4 \\ x_6 + 6x_7 = g_5 \end{cases}$$

where

$$(2.3)\qquad \begin{aligned} g_1 &= 253c - 759, \\ 2g_2 &= 77c(c-1) - 2g_1, \\ 6g_3 &= 21c(c-1)(c-2) - 6g_2, \\ 24g_4 &= 5c(c-1)(c-2)(c-3) - 24g_3, \\ 120g_5 &= c(c-1)(c-2)(c-3)(c-4) - 120g_4. \end{aligned}$$

By W. Jónsson [7] and some simple counting argument, we have

2.1 Lemma. *Let B, B' be two blocks in $S(5,8,24)$. Then*

(a) *The characters of one block are*

$$(2.4)\qquad x_8 = 1, x_4 = 280, x_2 = 444, x_0 = 30 \text{ and } x_i = 0 \text{ if } i = 1,3,5,6,7.$$

(b) *If $|B \cap B'| = 4$, then $B \triangle B'$ is a block.*

(c) *If $|B \cap B'| = 2$, then $\mathcal{M} := B \triangle B'$ is a set of type $(2,4,6)$ with*

$$(2.5)\qquad x_6 = x_2 = 132, x_4 = 495, x_i = 0 \ (i = 0,1,3,5,7,8).$$

(d) *If $|B \cap B'| = 0$, then $B \triangle B' = B \cup B'$ is a set of type $(0,4,6,8)$ with*

$$x_0 = 1, x_4 = 280, x_6 = 448, x_8 = 30, x_i = 0 \ (i = 1,2,3,5,7)$$

(e) *The complement of a set is a block iff the set is the union of two disjoint blocks.*

(f) *Denote by F a 4-set with $F \cap B = \emptyset$. There is at least one block B' containing F and having no point in common with B.*

3. Proof of the theorem

In this Section we prove Theorem 1.1. First of all, we prove

3.1 Lemma. *Let C be a blocking set in $S(5,8,24)$. Then* $11 \leq |C| \leq 13$.

Proof: We proved (cf. [1]) that the cardinality of a blocking set C of a Steiner system is such that $|C| > r_{t-1}$, so in $S(5,8,24)$ $|C| \geq 6$. It is simple to verify that if $6 \leq |C| \leq 8$, in (2.2) we have $g_5 < 0$, a contradiction.

Now suppose $|C| = 9$ or 10. By (2.2) $x_6 + 6x_7 = 6$ or 28 respectively. Then $(x_6, x_7) \neq (0,0)$. Denote by B a block that is at least 6-secant to C. The points of $C \backslash B$ are contained in a 4-set F. By 2.1 there is a block B' such that $F \subset B'$ and $B \cap B' = \emptyset$. Then C is contained in $B \cup B'$ and by (2.6) C has an external block, a contradiction. ▮

3.2 Remark: Consider the sets $\mathcal{M}$ and $\mathcal{M}_0$ defined in (1.1). By 2.1(c) the set $\mathcal{M}$ is a 12-set of type $(2,4,6)$, i.e. a reducible blocking set in x, $\forall x \in \mathcal{M}$. Consequently, $\mathcal{M}_0 = \mathcal{M} - \{x\}$ is a blocking 11-set of index two, which is irreducible, since 11 is the smallest cardinality.

Now we prove a little more than 1.1(a).

3.3 Proposition. *Denote by C a blocking 11-set in $S(5,8,24)$. Then $C = \mathcal{M}_0$ has no 7-secant block.*

Proof: Let C be a blocking 11-set. Suppose that B is a block 7-secant to C. Put $F = C \backslash B$. The same argument of the last point of 3.1 proves a contradiction. Then $x_7 = 0$ and by 2.1 we have $x_6 \neq 0$.

Let B be one block 6-secant to C. Denote by B' the block containing the 5-set $C \backslash B$. It holds that $|B \cap B'| = 2$. In fact, if $|B \cap B'| = 0$, then the block-complement of $B \cup B'$ would be external to C. Moreover, $|B \cap B'| \neq 4$, since B' has 5 points outside B. Now we prove that the two points of $B \cap B'$ lie outside C; this means $C = \mathcal{M}_0$. Suppose $B \cap B' \subset C$. Then B' is 7-secant to C. This is a contradiction, since $x_0 = 0$. Suppose $B \cap B' = \{x, y\}$ with $x \in C$, $y \notin C$. Then B and B' are 6-secant to C. Denote by u, v two points with $u \in B \backslash B'$, $v \in B' \backslash B$ and $u, v \notin C$. We consider the contraction $S(4,7,23)$ of $S(5,8,24)$ in y. In this contraction the sets $L_1 = B - \{y\}$ and $L_2 = B' - \{y\}$ are two blocks of $S(4,7,23)$ with one common point x. In view of Lemma 2.11 in [4] there is a block L_3 in $S(4,7,23)$ intersecting $L_1 \cap L_2$ only at $\{u, v\}$. So $B'' = L_3 \cup \{y\}$ is a block of $S(5,8,24)$ having no point in common with C, a contradiction. ▮

About Theorem 1.1.(b) we prove that

3.4 Proposition. *The 12-set I, defined by (1.2), is an irreducible blocking set.*

Proof: Put $I := B \cup B' - \{u, v\}$, where $u \in B \backslash B'$, $v \in B' \backslash B$, $B \cap B' = \{x, y\}$. Every block B'' ($\neq B, B'$) intersects $B \triangle B'$ in at least two points by 2.1(c). Two such points cannot be u and v by 2.1. Then every block intersects I.

Set I contains no block. Assume the contrary, suppose that B_0 is a block contained in I. Then B_0 cannot contain either point x or y, otherwise it would be at most 7-secant to I. So B_0 is contained in $I-\{x,y\}$, which is contained in $B\triangle B'$ of type $(2,4,6)$, a contradiction.

Finally, we prove that I is irreducible. Assume the contrary, suppose that there is a point $w\in I$ such that $I-\{w\}$ is a blocking set. Set $I-\{w\}$ necessarily coincides with $\mathcal{M}_0$. Since $\mathcal{M}_0$ has no 7-secant block, then $w\in B\cap B'$, necessarily. But if we consider the contraction in w, by Lemma 2.11 of [4] we have a contradiction. (cf. the last point of 3.3 Proof). ∎

3.5 Remark: The 12-set R defined in (1.2) is a reducible blocking set. Note that $R=\mathcal{M}_0\cup\{z\}$, where $z\in B\cap B'$. Since $\mathcal{M}_0$ has no 7-secant block, set R is clearly a reducible blocking set.

By (2.2), if $|C|=12$, we have:

$$(3.1)\qquad x_1=x_7, x_2=x_6=132-6x_7, x_3=x_5=15x_7, x_4=495-20x_7$$

Now we prove 1.1(b).

3.6 Proposition. *If C is an irreducible blocking 12-set, then $C=I$.*

Proof: Since C is irreducible, we have $x_1=x_7\geq 12$. Denote by B and B' two blocks 7-secant to C.

If $B\cap B'=\emptyset$ then $|C|\geq 14$, a contradiction.

Suppose $B\cap B'=\{x,y\}$. If $x\in C$, $y\notin C$, then $|C|\geq 13$, a contradiction. If $x,y\in C$, then $C=I$. Now suppose $|B\cap B'|=4$. If one point of $B\cap B'$ is outside C, then $B\triangle B'$ is a block contained in C, a contradiction. The last remaining case is $|B\cap B'|=4$ with $B\cap B'$ contained in C. Consider the 3 points of $B'\setminus C$ and the two points of $C\setminus(B\cup B')$. The block B'' containing these 5 points is necessarily 2-secant to B. In fact $|B\cap B''|\neq 4$, since B'' has 5 points outside B; moreover, $B\cap B'\neq\emptyset$ by 2.1. Hence $B\cap B'':=\{a,b\}$. If $a\in C$ and $b\notin C$, then $C=R$, a contradiction, since R is reducible. Consequently $a,b\in C$. This means $C=I$. ∎

Now we deal with case 1.1.(c). By 3.3 we have

3.7 Corollary. *If a blocking 12-set C of $S(5,8,24)$ is reducible, then $C=\mathcal{M}_0\cup\{w\},\forall w\notin\mathcal{M}_0$.*

3.8 Proposition. *Let C be a reducible blocking 12-set of $S(5,8,24)$. If C has index two, then either $C=\mathcal{M}$ or $C=R$.*

Proof: We have $C=\mathcal{M}_0\cup\{w\}$ by 3.7, where $\mathcal{M}_0=B\cup B'-\{x,y,a\}$ according to (1.1). If C is contained in $B\cup B'$, then either $w=a$ or $w\in\{x,y\}$ and the assertion is proved. ∎

Finally, we prove 1.1.(d).

3.9 Proposition. *If C is a blocking 13-set in $S(5,8,24)$, then C is reducible. Moreover, if C has index two, then $C = B \cup B' - \{a\}$, where B, B' are blocks with $|B \cap B'| = 2$ and $a \in B \cap B'$.*

Proof: The complement $\overline{C}$ of C is $\mathcal{M}_0$, necessarily. Since $\mathcal{M}_0$ has no 7-secant block, it follows that C has no 1-secant block, this means that C is a reducible blocking set.

Suppose C of index two. Clearly, C cannot contain clocking 12-set I, so C contains $\mathcal{M}_0$. Consequently, $C = \mathcal{M}_0 \cup \{x,y\}$, being $\{x,y\} \not\subset B \cap B'$ and B, B' as defined in (1.1). This proves the assertion. ∎

We conclude this Section with the following

3.10 Proposition. *Let A be one of the 12-sets $I, \mathcal{M}, R$ of $S(5,8,24)$. Then the complement $\overline{A}$ of A is isomorphic to A.*

Proof: We divide the proof in the following steps.

Step 1. Suppose that C is a blocking set with $|C| = 12$ and $x_7 = 0$, then $C = \mathcal{M}$. By (3.1) and $x_7 = 0$ it follows that C is of type $(2,4,6)$. Then $C \not\subset I$, since it is reducible, being $x_1 = 0$. So $C = \mathcal{M}_0 \cup \{z\}$, $z \notin \mathcal{M}_0$. We note that, if $\mathcal{M}_0 = B \cup B' - \{x,y,a\}$ is defined as in (1.1), then it cannot be $z \notin B \cup B'$, otherwise B (or B') would be 5-secant C, while C is of type $(2,4,6)$. Moreover, $z \in B \cap B'$ is impossible too, otherwise B (or B') would be 7-secant C. So $z = a$ and $C = B \triangle B'$.

Step 2. The complement of $\mathcal{M}$ is a 12-set with no 7-secant block by (3.1). So by Step 1 the assertion of Prop. follows.

Step 3. Suppose that C is a blocking set with $|C| = 12$ and $x_7 = 1$, then $C = R$. Denote by B the block 7-secant C, and denote by B' the block containing the 5 points of $C \backslash B$.

a) $B \cap B' = \emptyset$ is impossible, otherwise C would be contained in $B \cup B'$ having an external block by 2.1.
b) $|B \cap B'| = 4$ is impossible, since B' has 5 points outside B. Consequently, $|B \cap B'| = 2$.
c) If $|B \cap B'| = 2$, set $B \cap B'$ cannot be contained in C, otherwise B and B' are 7-secant to C, while $x_7 = 1$.

Consequently, B' intersects B only at one point of C. This means $C = R$.

Step 4. By (3.1) the complement of R has only one 7-secant block. So, the assertion of Prop. follows by step 3.

Step 5. Suppose that C is a blocking 12-set in $S(5,8,24)$ with $x_7 \geq 2$, then $C = I$. This is a consequence of the proof of 3.6.

Step 6. The complement of I is a 12-set having at least two 7-secant blocks by (3.1). So the assertion follows. ∎

4. The n-fold blocking sets of the Mathieu designs

In a block design $\mathcal{D}$ every family $\mathcal{B}$ of n disjoint blocking sets is said to be an *n-fold blocking set.* An n-fold blocking set $\mathcal{B}$ is said to be *maximal* if the set E of points outside the blocking sets of $\mathcal{B}$ contains no blocking set. The points of E can be adjoined to the components of $\mathcal{B}$ in an arbitrary way. So we can obtain a new n-fold blocking set B', which is a partition of $\mathcal{D}$. Such a partition is clearly an *n-colouring* of $\mathcal{D}$ with the property that every block of $\mathcal{D}$ intersects each set of $\mathcal{B}'$ in at least one point. In other words, each block contains at least one point of each colour.

In the sequel it is very useful to recall some consequences of the Lüneburg-construction of $S = S(3,6,22)$, cf. [3]. Let us consider the point-set of $\underline{P} = PG(2,4)$ and a new point ∞. In $S = \underline{P} \cup \{\infty\}$ we construct the structure of $S(3,6,22)$ with the following families of 77 blocks:

a) The 21 sets $\{L \cup \{\infty\}\}$, where L is a line of $PG(2,4)$.
b) A class of 56 hyperoval of $PG(2,4)$ (i.e. 6-arcs) constructed in the following way. Let $\mathcal{H}$ be the set of 168 hyperovals of $PG(2,4)$. If $H_1, H_2 \in \mathcal{H}$, we say that

$$H_1 \sim H_2 \iff |H_1 \cap H_2| = 0, 2, \text{ or } 6.$$

The relation $\sim$ is an equivalence relation. We have exactly 3 equivalence classes each of which contains 56 hyperovals.We can assume each of these 3 classes as the class of 56 blocks in $S(3,6,22)$.

In [3] we proves that in $S(3,6,22)$ the blocking sets with minimal cardinality are the so-called Fano sets. They are the 7-sets of $S(3,6,22)$ intersected by each block at 1 or 3 points (we call them of type $(1,3)$). Every Fano set F of $S = S(3,6,22)$ is a Fano subplane in the contraction of S at a point $x \notin F$. The converse of this is more complicated. In the projective plane $\underline{P}$ fix a Fano subplane $F(2)$ and one Lüneburg class consisting of a family of 56 hyperovals. The family $\{F(2) \triangle T\}$, where T describes the family of 3-secant lines of $F(2)$, is clearly a family of equivalent 6-arcs (in the sense used above). These 6-arcs are called 6-arcs associated with $F(2)$. In [3] it is proved that $F(2)$ is a Fano set in the extension S of $\underline{P}$ if and only if one (and then all) of the 6-arcs associated with $F(2)$ is not a block of S.

Finally, we recall that $PG(2,4)$ contains a 3-fold blocking set, which consists of a partition of $\underline{P}$ in 3 disjoint Baer subplanes that in this case are Fano subplanes. We recall that $PG(2,4)$ contains 360 Baer subplanes and 120 distinct partition into Baer subplanes (Baer-partitions).

Now we are ready to prove the following

4.1 Theorem. *Denote by $\mathcal{D}$ a Mathieu system with an n-fold blocking set $\mathcal{B}$ with $n \geq 3$. Then $\mathcal{D}$ is the projective plane of order four and $\mathcal{B}$ is a partition into Baer subplanes.*

Proof: If $\mathcal{D} = S(2,3,9)$ or $S(4,5,11)$, then in $\mathcal{D}$ there is no blocking set, as proved in [1]. If $\mathcal{D} = S(3,4,10)$, $S(5,6,12)$, $S(4,7,23)$ or $S(5,8,24)$, then in $\mathcal{D}$ there is not 3-fold blocking set, since the minimal cardinality of a blocking set is 5, 6, 11, 11, respectively, as proved in [5], [1], [4] and in Section 3 of this paper. So in $\mathcal{D}$ we do not have enough points to construct a 3-fold blocking set. A similar counting argument proves that an n-fold blocking set with $n \geq 4$ exists neither in $PG(2,4)$ nor in $S(3,6,22)$.

Now we prove that $S = S(3,6,22)$ contains no 3-fold blocking set. Suppose that $\mathcal{B}$ is a 3-fold blocking set of S. Since the blocking 8-sets of S are all reducible (cf. [2]), we can suppose that $\mathcal{B}$ is the union of 3 disjoint Fano sets F_1, F_2, F_3 and one point x outside them. Let us consider the contraction of S at the point x, then F_1, F_2, F_3 is a Baer partition of the projective plane $S - \{x\}$. Fix a line in $S - \{x\}$ (i.e. a block of S though x) which is 3-secant to F_1 at u, v, w and 1-secant to F_2 and F_3 at y and z respectively. There is a line through w which is 3-secant to F_2 at 3 points a, b, c. This line intersects F_3 at a point d. Consider the two sets C_1 and C_2 defined by

$$C_1 := F_1 = \{u,v,w\} \cup \{y,z\},\ C_2 := F_2 - \{a,b,c\} \cup \{w,d\}.$$

The sets C_1 and C_2 are 6-arcs associated with F_1 and F_2 respectively, with the property $|C_1 \cap C_2| = |\{w\}| = 1$. So, they are in two different Lüneberg classes. In other words, the families associated with F_1, F_2 and F_3 are in 3 different Lüneberg classes. Consequently, one of these F_i is not a Fano set, a contradiction.

Then only the projective plane of order four contains a 3-fold blocking set which, as proved in [2], is a partition into Baer subplanes. ∎

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