Perfect Mendelsohn designs with block size four

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Abstract. Let v, k, and λ be positive integers. A (v, k, λ)-Mendelsohn design (briefly (v, k, λ)-MD) is a pair $(X, \mathbf{B})$ where X is a v-set (of *points*) and $\mathbf{B}$ is a collection of cyclically ordered k-subsets of X (called *blocks*) such that every ordered pair of points of X are consecutive in exactly λ blocks of $\mathbf{B}$. A set of k distinct elements $\{a_1, a_2, \ldots, a_k\}$ is said to be cyclically ordered by $a_1 < a_2 < \ldots < a_k < a_1$ and the pair a_i, a_{i+t} are said to be t-apart in a cyclic k-tuple $(a_1, a_2, \ldots, a_k)$ where $i + t$ is taken modulo k. If for all $t = 1, 2, \ldots, k-1$, every ordered pair of points of X are t-apart in exactly λ blocks of $\mathbf{B}$, then the (v, k, λ)-MD is called a *perfect* design and is denoted briefly by (v, k, λ)-PMD. A necessary condition for the existence of a (v, k, λ)-PMD is $\lambda v(v-1) \equiv 0 \pmod{k}$. In this paper, we shall be concerned mainly with the case where $k = 4$. It will be shown that the necessary condition for the existence of a $(v, 4, \lambda)$-PMD, namely, $\lambda v(v-1) \equiv 0 \pmod{4}$, is also sufficient, except for $v = 4$ and λ odd, $v = 8$ and $\lambda = 1$, and possibly excepting $v = 12$ and $\lambda = 1$. Apart from the existence of a $(12, 4, 1)$-PMD, which remains very much in doubt, the problem of existence of $(v, 4, \lambda)$-PMDs is now completely settled.

1. Introduction

The notion of a perfect cyclic design was introduced by N.S. Mendelsohn **[15]**. This concept was further developed and studied in subsequent papers by various authors (see, for example, **[1–4, 11, 12, 17]**). In what follows, we shall adapt the terminology and notation of Hsu and Keedwell **[11]**, where the designs have been called Mendelsohn designs.

A set of k distinct elements $\{a_1, a_2, \ldots, a_k\}$ is said to be cyclically ordered by $a_1 < a_2 < \ldots < a_k < a_1$ and the pair a_i, a_{i+t} are said to be *t-apart* in a cyclic k-tuple $(a_1, a_2, \ldots, a_k)$ where $i + t$ is taken modulo k.

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Let v, k, and λ be positive integers. A (v,k,λ)*-Mendelsohn design* (briefly (v,k,λ)-MD) is a pair $(X, \mathbf{B})$ where X is a v-set (*of points*) and $\mathbf{B}$ is a collection of cyclically ordered k-subsets of X (called *blocks*) such that every ordered pair of points of X are consecutive in exactly λ blocks of $\mathbf{B}$. The (v,k,λ)-MD is called r*-fold perfect* if each ordered pair of points X appears t-apart in exactly λ blocks for all $t = 1,2,\ldots,r$. A $(k-1)$-fold perfect (v,k,λ)-MD is called *perfect* and is denoted briefly by (v,k,λ)-PMD. It is perhaps worth mentioning that a (v,k,λ)-MD is equivalent to the decomposition of the complete directed multigraph λK_v^* on v vertices into k-circuits.

It is easy to show that the number of blocks in a (v,k,λ)-PMD is $\lambda v(v-1)/k$, and hence an obvious necessary condition for its existence is $\lambda v(v-1) \equiv 0 \pmod{k}$. It is known [1, 13] that the necessary condition for the existence of a $(v,3,\lambda)$-PMD is also sufficient, except for $v = 6$ and $\lambda = 1$. In this paper, we shall be concerned mainly with the case where $k = 4$. It will be shown that the necessary condition for the existence of a $(v,4,\lambda)$-PMD, namely, $\lambda v(v-1) \equiv 0 \pmod{4}$, is also sufficient, except fort $v = 4$ and λ odd, $v = 8$ and $\lambda = 1$, and possibly excepting $v = 12$ and $\lambda = 1$. For practical purposes, the necessary condition for the existence of a $(v,4,\lambda)$-PMD can be reduced to the following:

Lemma 1.1. *A necessary condition for the existence of a $(v,4,\lambda)$-PMD is*

(1) $v \equiv 0$ *or* $1 \pmod{4}$ *for* λ *odd,*
(2) *any integer* $v \geq 4$ *for* λ *even.*

The problem of the existence of a $(v,4,1)$-PMD was initially studied by N.S. Mendelsohn [14] and remained open for quite sometime after. However, the results contained in [2, 17] now present us with an almost complete solution in the form of the following theorem.

Theorem 1.2. *A* $(v,4,1)$*-PMD exists for every positive integer* $v \equiv 0$ *or* $1 \pmod{4}$ *with the exception of* $v = 4$ *and the possible exception of* $v = 8, 12, 33$.

Remark 1.3: Katherine Heinrich [10] has informed the authors that the nonexistence of a $(8,4,1)$-PMD was established through an exhaustive computer search, and our independent investigation has confirmed this result. In this paper we shall construct a $(33,4,1)$-PMD, and consequently, only the existence of a $(12,4,1)$-PMD remains to be determined in Theorem 1.2.

2. Preliminaries

In order to establish our main result, we shall employ both direct and recursive constructions. Our recursive construction will involve the notion of pairwise balanced designs (PBDs), which we briefly describe below. For more information on PBDs and related designs, the interested reader is referred to [7, 9, 16].

Let K be a set of positive integers. A *pairwise balanced design* (PBD) of index unity $B(K, 1; v)$ is a pair $(X, \mathbf{B})$ where X is a v-set (of *points*) and $\mathbf{B}$ is a collection of subsets of X (called *blocks*) with sizes from K such that every pair of distinct points of X is contained in exactly one block of $\mathbf{B}$. We shall denote by $B(K)$ the set of all integers v for which there exists a PBD $B(K, 1; v)$. A PBD $B(\{k\}, 1; v)$ is essentially a *balanced incomplete block design* (BIBD) with parameters v, k and $\lambda = 1$.

The following result is fairly well-known (see, for example, [**5**, Theorem 4.5] or [**8**]) and it will be quite useful.

Lemma 2.1. *Let* $K_4 = \{4, 5, \ldots, 12, 14, 15, 18, 19, 23\}$. *For every integer* $v \geq 4$, $v \in B(K_4)$ *holds.*

We shall make use of this obvious result.

Lemma 2.2. *If a* (v, k, λ_1)*-PMD and a* (v, k, λ_2)*-PMD exist, then there exists a* $(v, k, m\lambda_1 + n\lambda_2)$*-PMD, where* m *and* n *are non-negative integers.*

The following recursive construction is a consequence of [**15**, Theorem 2.9] and Lemma 2.2.

Lemma 2.3. *Let* v, k, λ_1, *and* m *be positive integers. Suppose there exists a PBD* $B(\{k_1, k_2, \ldots, k_r\}, 1; v)$ *and for each* k_i *there exists a* (k_i, k, λ_1)*-PMD. Then there exists a* $(v, k, m\lambda_1)$*-PMD.*

For the most part, our direct method of construction will be a variation of the method using difference sets in the construction of BIBDs (see, for example, [7]). Instead of listing all the blocks of a design, it will be sufficient to give the group G acting on a set of base blocks. In this paper, the group G will always be the cyclic group Z_m. We shall adapt the following notation:

$$\operatorname{dev} \mathbf{B} = \{B + g : B \in \mathbf{B} \text{ and } g \in G\},$$

where $\mathbf{B}$ is the collection of base blocks of the design.

3. The Construction of $(v, 4, \lambda)$-PMD, λ even

In this section we shall show that the necessary condition for the existence of a $(v, 4, \lambda)$-PMD for λ even, namely, (2) of Lemma 1.1, is also sufficient. In view of Lemma 2.2, we need only establish the result for the case $\lambda = 2$. Moreover, by Lemmas 2.1 and 2.3, it will be sufficient to establish the existence of a $(v, 4, 2)$-PMD when $v \in K_4$ as defined in Lemma 2.1. For this purpose, we give some direct constructions using the difference method. It will be important to observe the obvious fact that a $(v, 4, \lambda)$-MD is perfect if it is 2-fold perfect.

Lemma 3.1. *If* $v \equiv 3 \pmod 4$ *and* $\gcd(v,3) = 1$, *then there exists a* $(v,4,2)$*-PMD.*

Proof: Let $v = 4t+3$. Let $X = Z_v$ and $G = Z_v$. Define the following collection of base blocks:

$$\mathbf{B} = \{(0,2k,k,-k) : k = 1,2,\ldots,2t+1\}.$$

Now the 1-apart and 2-apart differences from these base blocks are the following collections, respectively:

$$\{2k,-k,-2k,k : 1 \le k \le 2t+1\}, \text{ and } \{\pm k, \pm 3k : 1 \le k \le 2t+1\}.$$

Since $\gcd(v,6) = 1$, each of them contains twice the collection of elements on $G \backslash \{0\}$. Therefore, it is easy to see that $(X, \operatorname{dev} \mathbf{B})$ is a $(v,4,2)$-PMD.

Lemma 3.2. *There exists a* $(v,4,2)$*-PMD for* $v = 4,8,10,12,14$.

Proof: In each of the following five cases for v, we let $G = Z_{v-1}$ and $X = Z_{v-1} \cup \{\infty\}$. We then present a collection of base blocks $\mathbf{B}$, and it is readily checked that $(X, \operatorname{dev} \mathbf{B})$ is the required $(v,4,2)$-PMD.

(1) $v = 4, G = Z_3$,
$\mathbf{B} = \{(\infty,0,1,2),(\infty,0,2,1)\}$.

(2) $v = 8, G = Z_7$,
$\mathbf{B} = \{(\infty,0,1,3),(\infty,0,-1,-3),(0,1,-2,2),(0,-1,2,-2)\}$.

(3) $v = 10, G = Z_9$,
$\mathbf{B} = \{(\infty,0,3,1),(\infty,0,4,-1),(0,1,2,4),(0,3,-4,1),$
$(0,-3,3,1)\}$.

(4) $v = 12, G = Z_{11}$,
$\mathbf{B} = \{(\infty,0,1,5),(\infty,0,-1,-5),(0,1,3,8),(0,-1,-3,-8),$
$(0,-4,-2,-5),(0,3,1,5)\}$.

(5) $v = 14, G = Z_{13}$,
$\mathbf{B} = \{(\infty,0,-1,-2),(\infty,0,-4,2),(0,1,3,6),(0,4,1,-5),$
$(0,6,1,3),(0,1,6,-4),(0,-2,-4,4)\}$.

Lemma 3.3. *There exists a* $(6,4,2)$*-PMD.*

Proof: Let $X = \{1,2,3,4,5,6\}$ and let $\mathbf{B}$ be the following collection of blocks:

$$\begin{aligned}\mathbf{B} = \{&(1,2,3,4),(1,2,5,3),(1,6,3,2),(1,4,2,6),\\ &(1,6,2,5),(1,4,5,2),(1,5,6,3),(1,3,6,4),\\ &(1,3,4,5),(1,5,4,6),(2,3,5,4),(2,4,3,6),\\ &(2,6,5,3),(2,4,6,5),(3,5,6,4)\}.\end{aligned}$$

Then it is readily verified that $(X, \mathbf{B})$ is a $(6,4,2)$-PMD.

Lemma 3.4. *There exists a* $(v,4,2)$*-PMD for* $v = 15$ *and* 18.

Proof: For $v = 15$, let $G = Z_{11}$ and $X = Z_{11} \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$. Let $\mathbf{B}_1$ be the following base blocks:

$$\begin{aligned}\mathbf{B}_1 = \{&(\infty_1,0,4,1),(\infty_2,0,5,2),(\infty_3,0,-4,3),(\infty_4,0,-5,4),\\ &(\infty_1,0,4,5),(\infty_2,0,2,1),(\infty_3,0,-1,2),(\infty_4,0,-2,4),\\ &(0,1,3,-5)\}.\end{aligned}$$

Let $\mathbf{B}_2$ be the blocks of a $(4,4,2)$-PMD based on the set $\{\infty_1, \infty_2, \infty_3, \infty_4\}$, the existence of which is guaranteed by Lemma 3.2. Then it is readily checked that $(X, \mathbf{B}_2 \cup \operatorname{dev} \mathbf{B}_1)$ is a $(15,4,2)$-PMD.

For $v = 18$, take $G = Z_{13}$ and $X = Z_{13} \cup \{\infty_i : 1 \le i \le 5\}$. Let $\mathbf{B}_1$ be the following base blocks:

$$\begin{aligned}\mathbf{B}_1 = \{&(\infty_1,0,2,1),(\infty_2,0,4,2),(\infty_3,0,6,3),(\infty_4,0,1,-4),\\ &(\infty_1,0,-2,-6),(\infty_2,0,4,1),(\infty_3,0,6,-2),(\infty_4,0,-4,4),\\ &(\infty_5,0,3,-5),(\infty_5,0,-1,6),(0,1,3,6)\}.\end{aligned}$$

Let $\mathbf{B}_2$ be the blocks of a $(5,4,2)$-PMD based on the set of $\{\infty_i : 1 \le i \le 5\}$, the existence of which is guaranteed by Theorem 1.2 and Lemma 2.2. Then it is easy to verify that $(X, \mathbf{B}_2 \cup \operatorname{dev} \mathbf{B}_1)$ is a $(18,4,2)$-PMD.

Lemma 3.5. *A* $(v,4,2)$*-PMD exists for any integer* $v \ge 4$.

Proof: From Lemmas 3.1–3.4, we are guaranteed the existence of a $(v,4,2)$-PMD for any v in K_4 except when $v = 5$ and 9. However, for these two values of v, the existence of a $(v,4,1)$-PMD in Theorem 1.2 implies the existence of a $(v,4,2)$-PMD by Lemma 2.2. Thus a $(v,4,2)$-PMD exists for any integer v in K_4. From Lemmas 2.1 and 2.3, the conclusion follows.

By applying Lemma 2.2 to the result of Lemma 3.5, we have essentially proved the following theorem.

Theorem 3.6. *If* λ *is even, then a* $(v,4,\lambda)$*-PMD exists for any integer* $v \ge 4$.

4. The Construction of $(v,4,\lambda)$-PMD, λ odd

In this section, we shall prove the existence of a $(33,4,1)$-PMD and thereby remove $v = 33$ as a possible exception in Theorem 1.2. As already mentioned in Remark 1.3, there does not exist a $(8,4,1)$-PMD. We shall prove the nonexistence of a $(4,4,\lambda)$-MD for any odd λ. Then we shall establish the following theorem, which addresses the necessary condition (1) of Lemma 1.1.

Theorem 4.1. *Let λ be an odd integer. The necessary condition for the existence of a $(v,4,\lambda)$-PMD, namely, $v \equiv 0$ or $1 \pmod 4$, is also sufficient, except for $v = 4$, $v = 8$ and $\lambda = 1$, and possibly excepting $v = 12$ and $\lambda = 1$.*

Lemma 4.2. *There exists a $(33,4,1)$-PMD.*

Proof: Take $G = Z_{24}$. Let $X = Z_{24} \cup \{\infty_i : 1 \le i \le 9\}$. Let $\mathbf{B}_1$ be the blocks of a $(9,4,1)$-PMD based on the set $\{\infty_i : 1 \le i \le 9\}$, the existence of which is known from Theorem 1.2. Let $\mathbf{B}_2$ be the following blocks:

$$\mathbf{B}_2 = \{(0+i, 6+i, 12+i, 18+i) : 0 \le i \le 5\}.$$

Let $\mathbf{B}_3$ be the following base blocks:

$$\begin{aligned}\mathbf{B}_3 = \{&(\infty_1,0,-1,-6),(\infty_2,0,-2,-8),(\infty_3,0,9,-1),(\infty_4,0,10,-2),\\ &(\infty_5,0,8,5),(\infty_6,0,11,-9),(\infty_7,0,7,10),(\infty_8,0,-4,-11),\\ &(\infty_9,0,-9,4),(0,1,3,8)\}.\end{aligned}$$

Then it is readily checked that $(X, \mathbf{B}_1 \cup \mathbf{B}_2 \cup \operatorname{dev} \mathbf{B}_3)$ is a $(33,4,1)$-PMD.

The result of the following lemma is essentially contained in [**6**].

Lemma 4.3. *There does not exist a $(4,4,\lambda)$-MD for any odd λ.*

Proof: Let $X = \{1,2,3,4\}$ and λ be an odd integer. Suppose that there exists a $(4,4,\lambda)$-MD based on X. Without loss of generality, we can assume that the ordered pair $(1,2)$ occurs in blocks of the type $B_1 = (1,2,3,4)$ of multiplicity m_1 and type $B_2 = (1,2,4,3)$ of multiplicity m_2, so that $m_1 + m_2 = \lambda$. Similarly, the ordered pair $(2,3)$ occurs in blocks of the type B_1 and type $B_3 = (1,4,2,3)$ of multiplicity m_3, so that $m_1 + m_3 = \lambda$. Hence we have $m_2 = m_3$. But the ordered pair $(3,1)$ occurs only in a block of type B_2 or B_3. Consequently, $\lambda = m_2 + m_3 = 2m_2$ is even, which is a contradiction. Thus a $(4,4,\lambda)$-MD cannot exist for any odd λ.

We need some direct constructions for the proof of Theorem 4.1.

Lemma 4.4. *There exists a $(v,4,3)$-PMD for $v = 8$ and 12.*

Proof: For $v = 8$, let $G = Z_7$ and $X = Z_7 \cup \{\infty\}$. Let $\mathbf{B}$ be the following base blocks:

$$\begin{aligned}\mathbf{B} = \{&(\infty,0,2,1),(\infty,0,4,3),(\infty,0,5,3),(0,2,5,1),\\ &(0,3,5,6),(0,4,5,3)\}.\end{aligned}$$

Then it can be easily verified that $(X, \operatorname{dev} \mathbf{B})$ is a $(8,4,3)$-PMD.

For $v = 12$, let $G = Z_{11}$ and $X = Z_{11} \cup \{\infty\}$. Let **B** be the following base blocks:

$$\mathbf{B} = \{(\infty,0,1,9),(\infty,0,4,10),(\infty,0,6,3),(0,1,4,6),\\ (0,3,2,4),(0,4,6,2),(0,4,7,1),(0,5,6,4),\\ (0,6,3,2)\}.$$

Then it is readily checked that $(X, \text{dev}\,\mathbf{B})$ is indeed a $(12,4,3)$-PMD.

Combining Lemma 4.4 with the results of Theorem 3.6 for $v = 8$ and 12 and applying Lemma 2.2, we readily obtain the following result.

Lemma 4.5. *There exists a $(v,4,\lambda)$-PMD for $v = 8, 12$ and all odd $\lambda > 1$.*

We are now in position to establish the main result.

Proof of Theorem 4.1: When $\lambda = 1$ and $v \neq 4,8,12$, we know from Theorem 1.2 and Lemma 4.2 that the conclusion is true. Then, by applying Lemma 2.2, we have the conclusion for any odd λ and $v \neq 4,8,12$. The case $v = 4$ is taken care of in Lemma 4.3. The cases $v = 8$ and 12 are handled in Lemma 4.5 and Remark 1.3. This completes the proof.

5. Conclusion

Combining Theorems 3.6 and 4.1, the main result of the paper can be summarized in the following theorem.

Theorem 5.1. *The necessary condition for the existence of a $(v,4,\lambda)$-PMD, namely, $\lambda v(v-1) \equiv 0 \pmod 4$, is also sufficient, except for $v = 4$ and λ odd, $v = 8$ and $\lambda = 1$, and possibly excepting $v = 12$ and $\lambda = 1$.*

In conclusion, we wish to remark that, from our investigations, the existence of a $(12,4,1)$-PMD appears to be unlikely.

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