Existence of Graphs with Prescribed Mean Distance and Local Connectivity

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Abstract. This paper concerns the existence of graphs and digraphs with prescribed mean distance and the existence of graphs with prescribed mean local connectivity.

Section 1 Mean Distance

1.1 Introduction

Suppose G is a graph on ν vertices and $S(G)$ is the sum of the distances $d(x, y)$ for all unordered pairs $\{x, y\}$ of distinct vertices of G. The *mean distance* of G is defined by $\mu(G) = S(G)/\binom{\nu}{2}$.

In a digraph G on ν vertices let $\vec{S}(G)$ be the sum of the distances $\vec{d}(x, y)$ for all ordered pairs (x, y) of distinct vertices of G. Define the *mean distance* of G by $\vec{\mu}(G) = \vec{S}(G)/(\nu(\nu-1))$.

Because in a disconnected graph (or non-diconnected graph), the mean distance is infinite, we will only deal with connected graphs (or diconnected graphs) in this section. In order to facilitate our counting of mean distance we first present two lemmas.

Lemma 1.1. ([5], Corollary 1.3) *If G is a connected graph with n vertices then $1 \le \mu(G) \le (n+1)/3$ with equality holding on the left iff G is complete and on the right iff G is a path.*

Lemma 1.2. *A complete graph on n vertices has the sum of distances of $\binom{n}{2}$ while a path has $\binom{n+1}{3}$. (The easy proof is omitted.)*

From the above lemmas, we know that among graphs with a certain number of vertices, a path has the largest mean distance while a complete graph has the smallest.

In a graph G, the *radius* $\text{rad}(G)$ of G is defined to be $\min(D(v))$ for all $v \in V(G)$ where $D(v)$ is the maximal distance from vertex v to other vertices in graph G. And in a digraph, the *radius* $\text{rad}(G)$ is defined to be $\min(\max(D^+(v), D^-(v)))$ for all $v \in V(G)$ where $D^+(v), D^-(v)$ are the maximal distances from vertex v to other vertices and from other vertices to vertex v, respectively.

In a graph (or digraph) G, the *diameter* $\text{diam}(G)$ of G is defined to be the maximal distance in G. The following theorem was by Plesník.

ARS COMBINATORIA 29(1990), pp.73-86.

Theorem 1.1. ([9], Theorem 9.(1)) *Let r and d be integers with $d/2 \leq r \leq d$, and t be a real number with $1 \leq t \leq d$. Given a real number $\sigma > 0$, there exists a graph (or digraph) G with* radius(G) = r *and* diameter(G) = d, *and such that* $|\mu(G) - t| < \sigma$.

Omitting the restraints on the diameter and radius of the graphs, we have the following nice theorem of Hendry and Truszczyński.

Theorem 1.2. ([7],[10]) *For each rational number $t \geq 1$ there exist infinitely many graphs G with $\mu(G) = t$.*

Readers are advised to see proofs by Hendry [7] and Truszczyński [10]. It is natural to wonder whether we can put restrictions on the graphs for which the theorem still holds. In the following theorem, we restrict the graphs to be of prescribed connectivity as it might seem to many people that connectivity could affect the mean distance by holding the graph closer. Throughout this paper, we adhere to the notation found in [2].

The following are the major results I have obtained

(1) Given a rational number $t > 1$ and positive integer k, there are infinitely many graphs with mean distance t and connectivity k.
(2) Given a rational number $t \geq 3/2$, there are infinitely many directed graphs without digons whose mean distance is t.
(3) Given a rational number $t \geq 0$, there are infinitely many disconnected graphs with mean local connectivity t.
(4) Given a rational number $t \geq 1$, there are infinitely many connected graphs with mean local connectivity t.

with two further problems in the last section.

1.2 Graphs

Theorem 1.3. *For each rational number $t > 1$ and positive integer k there exist infinitely many graphs with connectivity k and mean distance t. For $t = 1$ there is only one such graph with connectivity k.*

Proof: For $t = 1$, K_{k+1}, the complete graph on $(k + 1)$ vertices, is obviously a graph with mean distance 1 and connectivity k. Since in a graph with mean distance 1, any two vertices are adjacent, K_{k+1} is the only graph with connectivity k and mean distance 1.

For $t > 1$, we construct a graph H in the following way. Let G_p be a path on p vertices, K_k be the complete graph on k vertices and let G' be the graph on q isolated vertices. We denote the vertices of $G_p \times K_k$, the product of the two graphs, by $V(i,j)$ where $1 \leq i \leq p$ and $1 \leq j \leq k$ and the vertices of G' by Wi where $1 \leq i \leq q$.

Let $V(H) = V(G_p \times K_k) + V(G')$ and $E(H) = E(G_p \times K_k) + \{V_{(1,j)}W_i | 1 \leq i \leq q, 1 \leq j \leq k\}$ as shown in Figure 1.1. There the edges in the replicas of K_k have been suppressed for clarity and the double line means W_i is adjacent to each vertex $V_{(1,j)}$. This graph H is obviously of connectivity k.

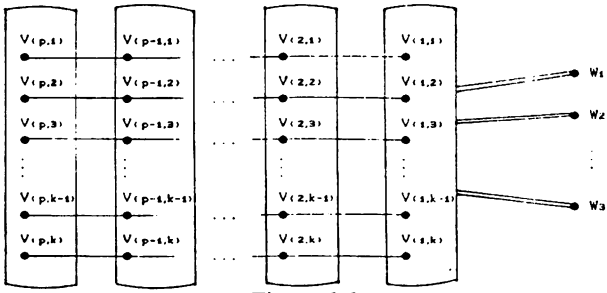

Figure 1.1

Now we show that $S(H) = 2\binom{q}{2} + qp(p+1)k/2 + p\binom{k}{2} + k^2(p-1)p(2p-1)/12 + k(3k/2-1)\binom{p}{2}$. Let $V_j = \{V_{(j,1)}, V_{(j,2)}, \dots, V_{(j,k)}\}, V = \{V_1, V_2, \dots, V_p\}$, $W = \{W_1, W_2, \dots, W_q\}$. If X and Y are subsets of the vertices of H, let $d(X,Y)$ denote $\sum\{d(x,y) : x \in X, y \in Y\}$, taking $d(z,z) = 0$. Then

(1) $d(W,W) = 2\binom{q}{2}$.

(2) $d(W,V) = qk\sum_{j=1}^{p} j$ because $d(W,V_j) = qjk$ and

(3) $d(V,V) = p\binom{k}{2} + \sum_{i=1}^{p-1}\sum_{j=1}^{p-1} k(j+(j+1)(k-1))$ because $d(V_i,V_i) = \binom{k}{2}$ and $d(V_i,V_{i+j}) = k(j+(j+1)(k-1))$ for $1 \leq j \leq p-i$.

Therefore $S(H) = 2\binom{q}{2} + qk\sum_{j=1}^{p} j + p\binom{k}{2} + \sum_{i=1}^{p-1}\sum_{j=1}^{p-1} k(j+(j+1)(k-1))$, and the equations follows directly.

Now we add edges, one by one, to any two distinct vertices in G'. For any two distinct vertices W_i, W_j, $d(W_i,W_j) = 2$ before edge W_iW_j is added and $d(W_i,W_j) = 1$ after the edge is added. Furthermore, adding edge W_iW_j does not affect the distance of any other two distinct vertices since any vertex adjacent to $W_i(W_j)$ has a distance of 1 or 2 to $W_i(W_j)$. So the total sum of distances of the graph decreases exactly by one. Let $L_{p,q} = S(H)$. We can add $\binom{q}{2}$ edges altogether to G' in such a way, so there is a graph $H' \supseteq H$ with $\nu(H') = kp + q$ and $S(H')$ can be any number between $L_{p,q}$ and $L_{p,q} - \binom{q}{2}$, inclusively. We denote $L_{p,q} - \binom{q}{2}$ by $S_{p,q}$. H' is obviously of connectivity k of $p \geq 2$.

For positive integer x and arbitrary c with $0 \leq c < p$, let $q = xp + c$. Then we have

$$L_{p-1,q+k} - S_{p,q} = \binom{xp+c}{2} + O(x), \text{ hence } \lim_{x\to\infty} (L_{p-1,q+k} - S_{p,q})/\binom{xp}{2} = 1,$$

so there is an X_0 such that for all $d \geq X_0$, if $q \geq dp$ then $L_{p-1,q+k} \geq S_{p,q}$. Suppose $t = m/n > 1$. Let $p = nr$ and $q = dnr$, where r is any positive integer, then $\nu = kp + q = (k+d)nr$. We have

$$[S_{2,\nu-2k}, \; L_{2,\nu-2k}] \cup [S_{3,\nu-3k}, L_{3,\nu-3k}] \cup \ldots \cup [S_{nr,dnr}, L_{nr,dnr}] \supseteq [S_{2,\nu-2k}, L_{nr,dnr}] .$$

So there is a graph $H' \supseteq H$ with $\nu(H') = (k+d)nr$ and $S(H')$ can be any integer in $[S_{2,\nu-2k}, L_{nr,dnr}]$. There is an integer R_0 such that $(S_{2,\nu-2k})/\binom{\nu}{2} \leq m/n \leq (L_{nr,dnr})/\binom{\nu}{2}$ for all $r \geq R_0$, because

$$\lim_{r\to\infty} \frac{S_{2,\nu-2k}}{\binom{\nu}{2}} = \lim_{r\to\infty} \frac{(\nu^2/2) + O(\nu)}{\binom{\nu}{2}} = 1$$

$$\lim_{r\to\infty} \frac{L_{nr,dnr}}{\binom{\nu}{2}} = \lim_{r\to\infty} \frac{O(r^3)}{O(r^2)} = \infty$$

and $m/n > 1$. Take $r = 2R_0$ then $S_{2,\nu-2k} \leq m/n\binom{\nu}{2} \leq L_{nr,dnr}$ and $(m/n)\binom{\nu}{2} = (\nu-1)R_0(k+d)n$ is an integer and $p \geq 2$, therefore there is a graph H' of connectivity k and $\nu(H') = (k+d)nr$ and $S(H') = \binom{\nu}{2}m/n$, i.e., $\mu(H') = t$.

If we take different multiples of $2R_0$ for r, then we get different graphs with mean distance t. So for $t > 1$ there are infinitely many graphs with mean distance t and connectivity k. ▮

1.3 Digraphs

We know that for any rational number $t > 1$ there are not only infinitely many graphs that have the prescribed mean distance but also infinitely many with a prescribed connectivity that have the prescribed mean distance. As to digraphs with digons, the corresponding problem is trivial if we change each edge of the graph in Theorem 1.1 into a digon thus changing the graph into a digraph. So in the following we only discuss digraphs without digons.

Lemma 1.3. *For any non-trivial digraph G without digons, the mean distance $\vec{\mu}(G) \geq 3/2$.*

Proof: For any two distinct vertices $x, y \in V(G)$, we have $\vec{d}(x,y) \geq 1, \vec{d}(y,x) \geq 1$ and they can not hold at the same time (otherwise G has a digon). So $\vec{d}(x,y) + \vec{d}(y,x) \geq 3$. Consequently

$$\vec{\mu}(G) = \frac{\vec{S}(G)}{\nu(\nu-1)} = \frac{\sum_{x,y\in V(G)} (\vec{d}(x,y) + \vec{d}(y,x))}{2\nu(\nu-1)} \geq \frac{3\nu(\nu-1)}{2\nu(\nu-1)} = \frac{3}{2}.$$

▮

Lemma 1.4. *Any complete graph K_n ($n \neq 1, 2, 4$) can be assigned an orientation to form a tournament in which the distance from any vertex to another vertex is no more than 2. Thus the mean distance of this tournament is $3/2$.*

Proof: By induction on the number of vertices of the graph. Let G be such a graph.

When $\nu(G) = 3$, we have a directed triangle. It can easily be seen that the distance from any vertex to another is no more than 2.

When $\nu(G) = 6$, there is a tournament whose adjacency matrix is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}, \text{ then } A^2 = \begin{bmatrix} 0 & 1 & 2 & 3 & 1 & 1 \\ 1 & 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 1 & 2 & 0 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 & 0 & 1 \\ 2 & 2 & 2 & 1 & 0 & 0 \end{bmatrix}$$

$$\text{and } A^2 + A = \begin{bmatrix} 0 & 2 & 2 & 3 & 2 & 2 \\ 1 & 0 & 2 & 2 & 1 & 1 \\ 1 & 2 & 0 & 1 & 1 & 2 \\ 2 & 2 & 1 & 0 & 1 & 1 \\ 2 & 2 & 2 & 2 & 0 & 1 \\ 2 & 2 & 3 & 2 & 1 & 0 \end{bmatrix}.$$

Thus in this digraph the distance from any vertex to another is no more than 2.

Suppose then when $\nu(G) = k$, the lemma is true. When $\nu(G) = k + 2$, we can assign the first k vertices such an orientation, then give the edges connecting V_{k+1} and the first k vertices the direction from V_{k+1} to each of them and give the edges connecting V_{k+2} and the first k vertices the direction from them to V_{k+2}. Finally give the edge $V_{k+2}V_{k+1}$ the direction from V_{k+2} to V_{k+1}. Of the first k vertices, the distances from any vertex to another is no more than 2 and any of the first k vertices, V_{k+1} and V_{k+2} form a directed triangle. So the distance from any vertex to another is no more than 2. Therefore the mean distance of this digraph is $3/2$. ∎

The induction argument above was introduced to me by D.A. Gregory and the same approach can be found in [9] (Corollary 3.).

Theorem 1.4. *For any rational $t \geq 3/2$ there are infinitely many digraphs with mean distance t.*

Proof: For $t = 3/2$, by Lemma 1.4, we have infinitely many digraphs with mean distance $3/2$.

For $t > 3/2$, let G_1 and G_2 be two digraphs on q vertices (q is a positive integer ≥ 6) whose mean distances are $3/2$ as described in Lemma 1.4.

Let $V(G_1) = \{W_{1i} | 1 \le i \le q\}$, $V(G_2) = \{W_{2i} | 1 \le i \le q\}$, and let p be an integer≥ 4. We construct a graph H in the following way, the vertex-set of H, $V(H) = V(G_1) \cup V(G_2) \cup \{V_i | 1 \le i \le p\} \cup \{U\}$; its edge-set $E(H) = E(G_1) \cup E(G_2) \cup \{V_iV_{i+1} | 1 \le i \le p-1\} \cup \{V_pV_1, V_2U\} \cup \{UW_{1i} | 1 \le i \le q\} \cup \{V_1W_{1i} | 1 \le i \le q\} \cup \{W_{1i}V_2 | 1 \le i \le q\} \cup \{V_2W_{2i} | 1 \le i \le q\} \cup \{W_{2i}V_1 | 1 \le i \le q\}$, as shown in Figure 1.2. There $P \Rightarrow G$ represents the fact that every vertex in G is reachable from every vertex in P. Then $\nu(H) = 2q + p + 1$.

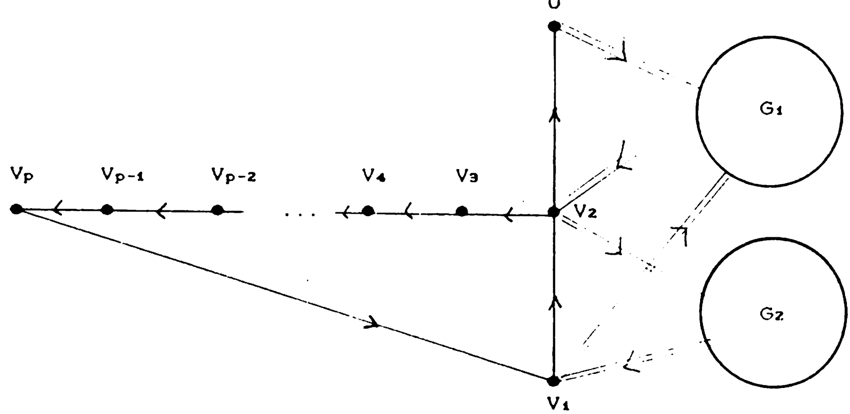

Figure 1.2

Now we show that $\vec{S}(H) = p^2(p+1)/2 + 2qp^2 + 6 + 11q + 7q^2$. First we denote this sum by $L_{p,q}$ for later use. Let $W_1 = V(G_1)$, $W_2 = V(G_2)$ and $V = \{V_1, V_2, \ldots, V_p\}$. The equation above can be easily inferred from the following table that gives $\vec{d}(X,Y)$ for sets of pairs of vertices in H.

$\vec{d}(X,Y)$	W_1	W_2	U	V
W_1	$3q(q-1)/2$	$2q^2$	$2q$	$q(3+3(p-1)/2)$
W_2	$2q^2$	$3q(q-1)/2$	$3q$	$qp(p+1)/2$
U	q	$3q$	0	$3+p(p+1)/2$
V	$q(2+p(p-1)/2)$	$qp(p+1)/2$	$p(p+1)/2$	$3-p+p^2(p-1)/2$

Now we add arcs $W_{2i}W_{1j}$, one by one, to H. Every time an arc is added $\vec{d}(W_{2i}, W_{1j})$ changes from 2 to 1, decreasing by one. Any other distance will change if and only if any of $\vec{d}(W_{2i}, X)$ where $W_{1j}X$ is an arc or any of $\vec{d}(X, W_{1j})$ where XW_{2i} is an arc is affected if and only if any of $\vec{d}(W_{2i}, X)$ where $X \in S \equiv \{V_2\} \cup \{W_{1a} | 1 \le a \le q \text{ and } a \ne j\}$ or any of $\vec{d}(X, W_{1j})$ where $X \in T \equiv \{V_2\} \cup \{W_{2a} | 1 \le a \le q \text{ and } a \ne i\}$ is affected. Thus none of them will change if $W_{2i}W_{1j}$ is added, since $\vec{d}(W_{2i}, X) = 1$ or 2 when $x \in S$ and $\vec{d}(X, W_{1j}) = 1$ or 2 when $X \in T$. So the total sum of distances of the graph decreases exactly by one.

We can add q^2 arcs together in this way. So there is a digraph H' on $\nu = 2q+p+1$ vertices and $S(H')$ can be any integer between $L_{p,q}$ and $L_{p,q} - q^2$. We denote the latter by $S_{p,q}$.

For positive integer x and arbitrary c with $0 \leq c < p$, let $q = xp + c$. Then we have

$$\lim_{x\to\infty} \left(L_{p-2,q+1} - S_{p,q}\right)/(xp)^2 = \lim_{x\to\infty} \left(q^2 + O(x)\right)/(xp)^2 = 1$$

So there exists an integer k such that for all $x \geq k$, if $q \geq xp$ then $L_{p-2,q+1} \geq S_{p,q}$. Suppose that $t = m/n > 3/2$. Now let $q = knr$, $p = nr$, where r is any even positive integer≥ 6, so $\nu = (2k+1)nr + 1$. Let $a = (\nu - 5)/2$, then

$$\left[S_{4,a}, L_{4,a}\right] \cup \left[S_{6,a-1}, L_{6,a-1}\right] \cup \ldots \cup \left[S_{nr-2,knr+1}, L_{nr-2,knr+1}\right] \cup \left[S_{nr,knr}, L_{nr,knr}\right] \supseteq \left[S_{4,a}, L_{nr,knr}\right] \ldots\ldots\ldots\ldots (*).$$

We have

$$\lim_{r\to\infty} \frac{S_{4,a}}{\nu(\nu-1)} = \lim_{r\to\infty} \frac{(7-1)(\nu/2)^2}{\nu(\nu-1)} = \frac{3}{2}$$
$$\lim_{r\to\infty} \frac{L_{nr,knr}}{\nu(\ nu-1)} = \lim_{r\to\infty} \frac{O(\nu^3)}{O(\nu^2)} = \infty.$$

Since $m/n > 3/2$, there is an integer R_0 such that for all $r \geq R_0$

$$\frac{S_{4,a}}{\nu(\nu-1)} \leq \frac{m}{n} \leq \frac{L_{nr,knr}}{\nu(\nu-1)}$$

Because $\nu(\nu-1)m/n = \nu(2k+1)rm$ is an integer, $(*)$ implies that there is a digraph $H' \supseteq H$ with $\nu(H') = \nu$ and $S(H') = \nu(\nu-1)m/n$, i.e. $\mu(H') = m/n$.

We get different graphs by letting r be different multiples of $6R_0$, so there are infinitely many graphs with mean distance t. ∎

In the above proofs, we employed the method of adding edges one by one to the graphs to get a sequence of graphs with consecutive sum of distances. This method was first used by G.R.T. Hendry [7] in his proof of Plesník's problem. Both of us only gave an existence proof. M. Truszczyński proved it by a different approach and he actually constructed a specific graph with the prescribed mean distance. That, at the same time, also leaves us further problems (see Section 3).

Section 2 Mean Local Connectivity

2.1 Concepts

In the previous section, we discussed the mean distance. We now look at another property of graphs, local connectivity.

In the graph G, the *local connectivity* $C_G(x,y)$ of two non-adjacent vertices is the minimum number of vertices separating x from y. If x and y are adjacent, their *local connectivity* $C_G(x,y)$ is defined as $C_H(x,y)+1$ where $H = G - xy$ [1]. We call $C(G) = \sum_{\{x,y\}\subseteq V(G)} C_G(x,y)$ the *coherence* of G. Define $\mu(G) = C(G)/\binom{\nu}{2}$ to be the *mean local connectivity* of G where ν is the number of vertices of the graph G.

Like the mean distance being a natural measure of "compactness" of the graph, the mean local connectivity can be looked on as a natural measure of invulnerability to disconnection of a corresponding network. The analogous problem for local connectivity is the existence of graph with prescribed mean local connectivity. Here I present the proofs for disconnected and connected graphs individually.

2.2 Disconnected Graphs

To highlight the main part of the theorem and make it more easily understood, we introduce a lemma.

Lemma 2.1. *Let G_0 be the edge-free graph with vertex set $\{V_1, V_2, \ldots, V_n\}$. We add edges, one by one, to form a complete graph at the end in such way that we make the first i vertices a complete graph before connecting V_k ($k > i$) to any of these first i vertices. Let G_p be the graph obtained from G_0 by adding p edges that way, then we have*

$$C(G_{p+1}) \leq C(G_p) + 2n \quad \text{for all } 0 \leq p \leq \binom{n}{2}.$$

Proof: For any $0 \leq p \leq \binom{n}{2}$, let the next edge to be added be $e = V_iV_j$ ($i > j$), let $P_1 = \{x \in V(G_p) | xV_i \in E(G_p)\}$, and $P_2 = \{V_1, V_2, \ldots, V_{i-1}\} \backslash P_1$. Then the induced subgraph $G(P_1 \cup P_2)$ is complete and $G_{p+1} = G_p + e$, as shown in Figure 2.1.

When adding the edge $e = V_iV_j$, $C(x,y)$ changes only if $x = V_i$ and $y \in P_1 \cup P_2$ or $x = V_j$ and $y \in P_1$. Therefore

$$C(G_{p+1}) - C(G_p) = |P_2| + 2|P_1| \leq 2(i-1) \leq 2n.$$

∎

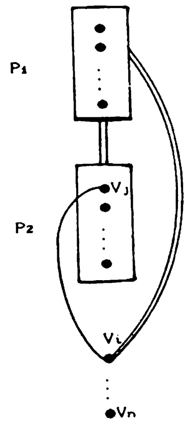

Figure 2.1

Theorem 2.1. *For any rational number $t = k/m \geq 0$, there are infinitely many disconnected graphs with mean local connectivity t.*

Proof: For each integer r, let $n = 2rm$.

Let G_p, where $0 \leq p \leq \binom{n}{2}$, be the graphs described in Lemma 2.1. Let M_0 be the edge-free graph with vertex-set

$$V(M_0) = \{a_i | 1 \leq i \leq 2n\} \cup \{b_i | 1 \leq i \leq 2n\}, \quad \text{and}$$

M_q be the graph obtained from M_0 by adding q edges $a_i b_i$ ($1 \leq i \leq q$).

Let $H_{p,q} = G_p + M_q$, then $H_{p,q}$ is a graph on $5n$ vertices as shown in Figure 2.2.

$H_{p,q}$

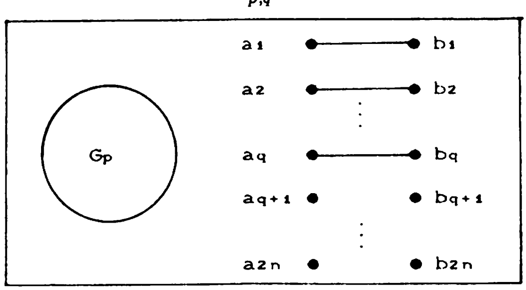

Figure 2.2

Because $H_{p,q}$ consists of components G_p and q replicas of K_2, we have $C(H_{p,q}) = C(G_p) + q$ for each $0 \leq q \leq 2n$. It follows from the Lemma 2.1 that $C(H_{p,q}) = C(G_p) + q$ for each $0 \leq q \leq C(G_{p+1}) - C(G_p) = C(H_{p+1,0}) - C(H_{p,0})$. So we have a graph on $5n$ vertices whose coherence can be any integer between $C(H_{p+1,0})$ and $C(H_{p,0})$.

For any integer $0 \leq N \leq \binom{n}{2}(n-1)$ there is a disconnected graph $H_{p,q}$ on $5n$ vertices for some p and q with $C(H_{p,q}) = N$, because $C(H_{o,o}) = 0$ and $C(H_{m,0}) = \binom{n}{2}(n-1)$ where $m = \binom{n}{2}$.

Since $\lim_{r\to\infty} \binom{n}{2}(n-1)/\binom{5n}{2} = \infty$, there is an integer R_0 such that $\binom{5n}{2}k/m \leq \binom{n}{2}(n-1)$ for all $r > R_0$. But $\binom{5n}{2}k/m = 5kr(5n-1)$ is integral, so there is a graph $H_{p,q}$, for some p and q, on $5n$ vertices such that $C(H_{p,q}) = \binom{5n}{2}k/m$, that is, $\mu(H_{p,q}) = C(H_{p,q})/\binom{5n}{2} = k/m = t$.

For different $r > R_0$ we have different graphs with mean local connectivity t, so there are infinitely many graphs with mean local connectivity t. ∎

2.3 Connected Graphs

For the same purpose as in 1.2, we now introduce some other lemmas.

Lemma 2.2. *The coherence of the n-cycle C_n is $2\binom{n}{2}$. The coherence of $C_n + e$, the graph obtained by adding an edge e to two non-adjacent vertices in C_n, is $2\binom{n}{2}+1$.*

Proof: Any pair of vertices in C_n has a local connectivity 2 and any pair of vertices in $C_n + e$ has a local connectivity 2 except that the pair of the two ends of e has a local connectivity 3. ∎

By Lemma 2.2, $C(C_4) = 2\binom{4}{2}$ and $C(C_4 + e) = 2\binom{4}{2} + 1$.

Lemma 2.3. *For a graph G_0 with vertex-set $V(G) = \{V_1, V_2, \ldots, V_n\}$ and edge-set $E(G)\{V_iV_n | 1 \leq i \leq n-1\}$, we add edges, one by one, to form a complete graph in such a way that we make the first i vertices a complete graph before adding an edge to connect V_j $(j > i)$ to one of the first i vertices. Let G_p be the graph after adding p edges to G_0, then $C(G_{p+1}) - C(G_p) \leq 2n$ for all $0 \leq p < \binom{n-1}{2}$.*

(The proof is similar to that of Lemma 2.1 and is omited.)

Theorem 2.2. *For any rational number $t = k/m \geq 1$, there are infinitely many connected graphs with mean local connectivity t. Furthermore $t = 1$ is best possible for non-trivial graphs.*

Proof: In the non-trivial connected graph G, $C_G(x, y) \geq 1$ for any distinct $x, y \in V(G)$. Therefore the mean local connectivity of G is at least 1.

For $t = 1$, P_r, the path of length r, is obviously a connected graph with mean local connectivity $\mu(P_r) = 1$ for each $r \geq 1$.

For $t > 1$, we construct a graph $H_{p,q}$ on $(9n+1)$ vertices ($n = 2rm$, r is an integer) in the following way.

Let M_0 be a graph consisting of $2n$ mutually disjoint 4-cycles and another vertex U which is connected to a vertex of each of the 4-cycles. Let M_q be the

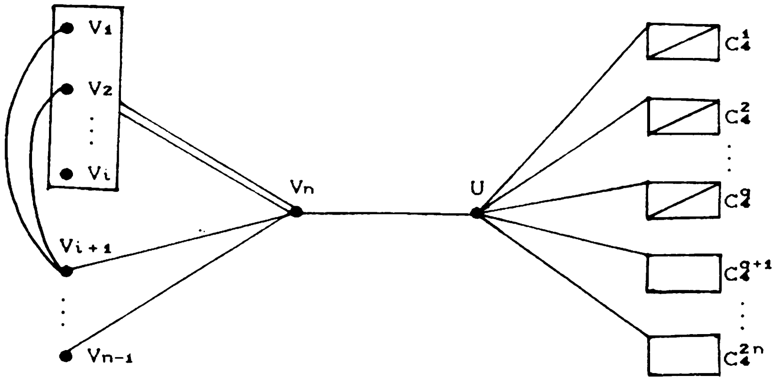

Figure 2.3

graph obtained from M_0 by adding one edge to each of the first q 4-cycles so that the maximum degree of each vertex in M_q (except U) is 3.

Let $H_{p,q} = G_p + M_q + V_nU$ (G_p is the graph mentioned in Lemma 2.3) as shown in Figure 2.3. Because any pair of vertices in $H_{0,0}$ have local connectivity 1, except that the pairs of vertices that are in one of the 4-cycles have local connectivity 2, $C(H_{0,0}) = \{\binom{9n+1}{2} + \binom{4}{2}2n\}$. It follows from Lemma 2.2 that $C(H_{p,q}) = C(H_{p,0}) + q$ for each $0 \le q \le C(G_{p+1}) - C(G_p) = C(H_{p+1,0}) - C(H_{p,0})$. To be exact, if we add an edge e_{p+1} to G_p to form G_{p+1} $(p < \binom{n-1}{2})$ we get a gap= $C(H_{p+1,0}) - C(H_{p,0}) \le 2n$ while if we add an edge to M_q $(Q \le 2n-1)$ we get an increase of exactly one. So if we add edges, one by one, to M_0 until the total increase is one less than the gap, then delete all the edges added to M_0 and add the edge e_{p+1} to G_p, we get graphs with consecutive coherences.

So we have a graph $H_{p,q}$ on $(9n+1)$ vertices for some p and q such that $C(H_{p,q})$ can be any integer between $C(H_{0,0})$ and

$$C(H_{p,0}) = \binom{9n+1}{2} + \binom{4}{2}2n + \binom{n}{2}(n-2) \quad \text{where } p = \binom{n-1}{2}.$$

Now $\{\binom{9n+1}{2} + \binom{4}{2}2n + \binom{n}{2}(n-2)\}/\binom{9n+1}{2} \longrightarrow \infty$ as $r \longrightarrow \infty$, and $\{\binom{9n+1}{2} + \binom{4}{2}2n\}/\binom{9n+1}{2} \longrightarrow 1$ as $r \longrightarrow \infty$, therefore $\{\binom{9n+1}{2} + \binom{4}{2}2n + \binom{n}{2}(n-2)\} \ge \binom{9n+1}{2}k/m \ge \{\binom{9n+1}{2} + \binom{4}{2}2n\}$ for r large enough. Because $\binom{9n+1}{2}k/m = 9(9n+1)rk$ is an integer, there is a graph $H_{p,q}$ for some p and q such that $C(H_{p,q}) = \binom{9n+1}{2}k/m$. Hence $\mu(H_{p,q}) = t$.

Therefore there are infinitely many such graphs because there are different graphs with $\mu(H_{p,q}) = t$ for different r. ∎

Remarks

Another concept analogous to local connectivity is local edge-connectivity. The *local edge-connectivity* between two vertices is the minimum number of edges separating the two vertices. Interestingly enough, the construction above is valid for local edge-connectivity. We can get the same results for local edge-connectivity and the proofs are similar.

Section 3 Further Problems

We say that a rational number t is *realizable* by a graph if there is a graph with mean distance t. The graph is called a *realization* of t, and a *best realization* if the graph has the minimum number of vertices.

We know that any rational number $t \geq 1$ is realizable by infinitely many graphs. Furthermore by Truszczyński's proof, we know that the rational number $t = a/b$ (a, b are relatively prime positive integers) is realizable by a graph with $2c(c+1)b+1$ vertices, where c is an integer such that $2/(c+2) \leq b/a \leq 2/(c+1)$. This suggests the following problem, which is mostly due to Dr. Pullman.

Problem 3.1. *How to minimize the number of vertices of a graph that has a prescribed mean distance, or furthermore what is the minimum number of vertices and best realization?*

Here I only present a solution to the problem for special kinds of t's. By Lemma 1.1, the following proposition is obvious.

Proposition 3.1. *For $t = d/3$ when d is a positive integer ≥ 3, then there is a best realization with $(d-1)$ vertices; the realization must be a path.*

We may find the minimum number of vertices for other special kinds of t's, but it does not solve the problem as a whole, hence we won't go any further.

We now know that $t \geq 1$ is not only realizable by infinitely many graphs but also by infinitely many k-connected graphs for any positive k. Is $t \geq t_0$ for some t_0 realizable by any other special kind of graphs, most temptingly trees?

Problem 3.2. *Is $t \geq 2$ realizable by trees?*

To facilitate the counting of mean distance of a tree, we present a neat way by J.K. Doyle and J.E. Graver [5], which is very effective for certain types of trees.

Theorem 3.1. ([5], Theorem 1.2) *Let T be a tree with n vertices. For any $v \in V(T)$, let $m_1, m_2, m_3, \ldots, m_q$ be the numbers of vertices in the connected components of the graph obtained by deleting v and let $h(v)$ be $\sum m_i m_j m_k$ for all $1 \leq i < j < k \leq q$, then*

$$\mu(T) = \frac{n+1}{3} - \sum \frac{h(v)}{\binom{n}{2}}.$$

So far we have been trying to lower the number of vertices needed for a realization. We can also approach the problem by ruling out some numbers.

Proposition 3.2. *If graph G with n vertices is a realization of $t = a/b$ (a, b are positive and relatively prime) then $n \geq 3a/b - 1$ and furthermore $\sqrt{bk + 1/4} + 1/2$ is an integer for some k.*

Proof: By Lemma 1.1, we have $a/b \leq (n+1)/3$, so $n \geq a/b - 1$.
Because $a/b = 2S(G)/(n(n-1))$, so $n(n-1) = bk$ for some k and $n = \sqrt{bk + 1/4} + 1/2$. Therefore $\sqrt{bk + 1/4} + 1/2$ is an integer. ∎

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