The Transitive Graphs with at Most 26 Vertices

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Abstract. We complete the construction of all the simple graphs with at most 26 vertices and transitive automorphism group. The transitive graphs with up to 19 vertices were earlier constructed by McKay, and the transitive graphs with 24 vertices by Praeger and Royle. Although most of the construction was done by computer, a substantial preparation was necessary. Some of this theory may be on independent interest.

1. Introduction

Let G be a finite simple graph with automorphism group Aut(G). If Aut(G) acts transitively on $V(G)$, then we say that G is *transitive*. The aim of this paper is to describe the methods by which the complete set of transitive graphs of order at most 26 has been generated.

The transitive graphs on a prime number p of vertices are the graphs whose automorphism groups contain a p-cycle. The isomorphism classes were determined by Elspas and Turner [5].

For the case when the number of vertices is $2p$, p prime, Alspach and Sutcliffe [1] described a particular family of transitive graphs and conjectured that there were no others. The truth of their conjecture follows from results of Masrušič [15] in conjunction with a corollary of the classification of the finite simple groups (that there are no simply-transitive primitive permutation groups of degree $2p$ for $p \neq 5$).

For other orders, few general results are known. H.P. Yap made the first significant attempt at a catalogue; he found all the transitive graphs up to 11 vertices, and many classes of them on 12 vertices. A complete list of transitive graphs up to 19 vertices was compiled by McKay [18] and published in [17]. The method of construction was not described in [17], however; that will be the subject of our Sections 2 and 3. The transitive graphs on 20–23 vertices were found by McKay

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and Royle [24]; we will describe this construction in Section 4. Section 4 also describes, for the first time, the construction of the transitive graphs on 25 or 26 vertices. Finally, the transitive graphs on 24 vertices were found by Royle and Praeger [24, 25]; we will not repeat this construction here.

A few related compilations can be mentioned here. The circulant graphs (those on n vertices whose automorphism group contains an n-cycle) were found up to order 37 by the first author in 1977 (unpublished). Graphs of order up to 11 with isomorphic vertex neighbourhoods were found by J. Hall [11]. D.H. Rees [23] determined all the cubic symmetric graphs of order up to 40 (G is symmetric if $\mathrm{Aut}(G)$ acts transitively on the directed edges of G); more extensive classifications or compilations of cubic transitive graphs were performed by Coxeter, Frucht and Powers [4] and Lorimer [12, 14]. A classification of symmetric graphs of prime degree was made by Lorimer [13]. The transitive planar graphs were completely classified by Fleischner and Imrich [6]. The complete list of Cayley graphs to 23 vertices was constructed in 1977 by the first author (unpublished) and to 31 vertices in 1986 by the second author [24]. Finally, R. Mathon [16] found all transitive self-complementary graphs with less than 50 vertices.

2. Theoretical Background

We will assume that the reader is conversant with the elementary terminology of graph theory and group theory. Only simple graphs will be considered. We will denote an edge $\{x, y\}$ of a graph as xy for brevity. $E(G)$ is the edge-set of G and $\overline{G}$ is the complement of G. The set of neighbours of v in G will be denoted by $N(v, G)$, and $V(G)\backslash(\{v\} \cup N(v, G))$ will be denoted by $\overline{N}(v, G)$.

Suppose that Λ is a set of permutations (not necessarily a group) acting on a set V. The *support* $\mathrm{supp}(\Lambda)$ of Λ is the set of elements of V moved by some element of Λ, while the *fixed-point set* $\mathrm{fix}(\Lambda)$ of Λ is the set of elements of V fixed by every element of Λ. Obviously, $\mathrm{supp}(\Lambda) \cup \mathrm{fix}(\Lambda) = V$.

If G is any graph, then the *switching graph* of G, denoted $\mathrm{Sw}(G)$, has $V(\mathrm{Sw}(G)) = V(G) \times \{0, 1\}$ and $E(\mathrm{Sw}(G)) = \{(x,i)(y,j) \mid i = j \text{ and } xy \in E(G), \text{ or } i \neq j \text{ and } xy \in E(\overline{G})\}$. Switching graphs have relevance to the *switching classes* of [26]; in particular, two graphs are in the same switching class if and only if their switching graphs are isomorphic [8].

If G and H are graphs, the *lexicographic product* $G[H]$ has $V(G[H]) = V(G) \times V(H)$ and $E(G[H]) = \{(x_1, y_1)(x_2, y_2) \mid x_1x_2 \in E(G) \text{ or } x_1 = x_2 \text{ and } y_1y_2 \in E(H)\}$. We will say that G is a *non-trivial lexicographic product* (NTLP) if $G = H[J]$ for some graphs H and J with at least two vertices. The importance of NTLPs to us comes from the following lemma. A subset $W \subseteq V(G)$ is called *externally related* (ER) in G if each pair of vertices in W are adjacent to exactly the same vertices in $V(G)\backslash W$. W is a *non-trivial* ER subset if $2 \leq |W| \leq V(G) - 1$.

Theorem 2.1. *Let G be a transitive graph which is neither empty nor complete. Then the following are equivalent.*

(a) *G is an NTLP.*

(b) *$G = H[J]$ for some transitive graphs H and J with at least two vertices.*

(c) *G has a non-trivial ER subset.*

(d) $\mathrm{Aut}(G)$ *has a non-trivial ER block.*

(e) $\mathrm{Aut}(G)$ *has an intransitive subgroup with exactly one orbit of length greater than one.*

Proof: Obviously, (b)⇒(a)⇒(c) and (d)⇒(e)⇒(c), so that it will suffice to prove that (c)⇒(d)⇒(b).

Suppose that condition (c) is satisfied. Let W be a non-trivial ER subset of the least possible size. If $\mathrm{Aut}(G)$ constains no transpositions, then $|W| \geq 3$. Now, for each $\gamma \in \mathrm{Aut}(G)$, if $W \cap W^{\gamma} \neq \emptyset$ then $W^{\gamma} = W$ since otherwise one of $W \cap W^{\gamma}$ and $W \backslash W^{\gamma}$ would be a non-trivial ER subset smaller than W. Suppose alternatively that $\mathrm{Aut}(G)$ contains a transposition $(x\ y)$. By replacing G by $\overline{G}$ if necessary, we have $N(x,G) = N(y,G)$. Then $\{v \in V(G) \,|\, N(v,G) = N(x,G)\}$ is a non-trivial ER block of $\mathrm{Aut}(G)$ or else G is empty.

Suppose that condition (d) is satisfied and let $B_1, B_2, \ldots, B_r$ be the corresponding complete block system. Since $\mathrm{Aut}(G)$ acts transitively on the blocks, each B_i is ER and induces an isomorphic subgraph of G. Thus, each distinct pair B_i and B_j are joined either by no edges of G or by all possible edges. Condition (b) is thus satisfied. ∎

The implications (a)⇔(e) were first proved by C. Godsil. As sample applications of Theorem 2.1, we have the following theorems.

Theorem 2.2. *Let G be a non-complete connected transitive graph. If $\overline{N}(v,G)$ is disconnected for some $v \in V(G)$, then G is an NTLP.*

Proof: By Gardiner [7] or Ashbacher [2], either $N(v,g) = N(w,G)$ for some $v \neq w$ (implying that $(v\ w) \in \mathrm{Aut}(G)$) or G has a non-trivial ER block. Theorem 2.1 applies immediately in either case. ∎

Theorem 2.3. *Let G be a connected non-complete transitive graph with odd order $n \geq 7$. If* $\mathrm{Aut}(G)$ *contains a non-trivial subgroup Λ which moves at most 7 vertices, then G is an NTLP.*

Proof: By considering all the possibilities for Λ, we see that Λ contains a subgroup satifying part (e) of Theorem 2.1 or else a subgroup of the form $\langle (a\ b)(c\ d) \rangle$ or $\langle (a\ b\ c)(d\ e\ f) \rangle$. In the latter case, consider all the possibilities for the subgraph induced by $\{a,b,c,d,e,f\}$; in every case we find that $(a\ b)(d\ e) \in \Lambda$. Now suppose $\mathcal{D}(G) \neq \emptyset$, where $\mathcal{D}(G)$ is the set of all elements of $\mathrm{Aut}(G)$ of the form $(a\ b)(c\ d)$. Since n is odd, and $\mathrm{Aut}(G)$ is transitive, there are distinct

$\gamma, \delta \in \mathcal{D}(G)$ such that $\mathrm{supp}(\gamma) \cap \mathrm{supp}(\delta) \neq \emptyset$. Now consider all the ways that γ and δ can overlap. In most cases, $\langle \gamma, \delta \rangle$ contains a subgraph satisfying part (e) of Theorem 2.1. [For example, take $\gamma = (a\ b)(c\ d)$ and $\lambda = (a\ e)(c\ f)$, where all these vertices are distinct. Then $\langle \gamma, \delta\gamma\delta \rangle$ contains exactly one orbit.] The only exception is when γ and δ overlap as do $(a\ b)(c\ d)$ and $(a\ e)(c\ f)$, so assume that all non-trivial overlaps between elements of $\mathcal{D}(G)$ have this form. Define a relation "$\sim$" on $V(G)$:

(i) $x \sim x$ for all x.

(ii) If $x \neq y$, then $x \sim y$ if and only if there are elements $\gamma = (x\ a)(y\ b)$ and $\delta = (x\ c)(y\ d)$ of $\mathcal{D}(G)$ such that $a \neq c$ and $b \neq d$.

It is easily seen that "$\sim$" is an equivalence relation with classes of size 2, contradicting the assumption that n is odd. ∎

If Λ_1 and Λ_2 are permutation groups acting on a set V, and $\mathrm{supp}(\Lambda_1) \cap \mathrm{supp}(\Lambda_2) = \emptyset$, then we will write $\Lambda_1 \oplus \Lambda_2$ for the group $\langle \Lambda_1, \Lambda_2 \rangle$. Clearly, $\Lambda_1 \oplus \Lambda_2$ is isomorphic as an abstract group to the direct product of Λ_1 and Λ_2, but the permutation representation is important to us here. If Λ is a non-trivial permutation group acting on V, then Λ has a unique representation

$$\Lambda = \Lambda^{(1)} \oplus \Lambda^{(2)} \oplus \cdots \oplus \Lambda^{(r)},$$

where the supports of the $\Lambda^{(i)}$ are non-empty and disjoint, and r is maximum. We will refer to the groups $\Lambda^{(i)}$ as the *fragments* of Λ.

For a group Λ and a prime p, let $\mathrm{Syl}_p(\Lambda)$ be the set of all Sylow p-subgroups of Λ.

Lemma 2.4. *Let G be a transitive graph, and let $v \in V(G)$. Suppose that $P \in \mathrm{Syl}_p(\Gamma_v)$, where $\Gamma = \mathrm{Aut}(G)$, Γ_v denotes the point stabiliser of v in Γ, and p is a prime dividing $|\Gamma_v|$. Define a graph $H = H(G,P)$ as follows. $V(H)$ is the set of non-trivial orbits of P. Two distinct vertices of H are adjacent if and only if the corresponding orbits of P are joined by some edges of G but not completely joined in G. Then the supports of the fragments of P correspond to the components of H.*

Proof: Let $P^{(1)}, P^{(2)}, \ldots, P^{(r)}$ be the fragments of P. Since the action of P on orbits in different fragments is independent, fragments correspond to unions of orbits. Now suppose that the support of a fragment $P^{(i)}$ is $V_1 \cup V_2$, where V_1 and V_2 are disjoint non-empty sets of orbits each of which corresponds to a union of components of H. Then the restrictions $P|_{V_1}$ and $P|_{V_2}$ are each in Γ by the structure of G. However, $P|_{V_1} \oplus P|_{V_2}$ is not in $P^{(i)}$ since $P^{(i)}$ is a fragment. Thus $P^{(1)} \oplus \cdots \oplus P^{(i-1)} \oplus P|_{V_1} \oplus P|_{V_2} \oplus P^{(i+1)} \oplus \cdots \oplus P^{(r)}$ is a p-subgroup of Γ_W larger than P, contradicting the assumption that $P \in \mathrm{Syl}_p(\Gamma_v)$. ∎

If $\Lambda \leq \Phi \leq \Gamma$ are groups, we say that Λ is *weakly-closed* in Φ with respect to Γ if, for each $\gamma \in \Gamma$, $\Lambda^\gamma \leq \Phi$ if and only if $\Lambda^\gamma = \Lambda$.

Lemma 2.5. *Suppose that Γ is a group acting transitively on $V = \{1, 2, \ldots, n\}$, and let $1 < P \in \mathrm{Syl}_p(\Gamma_1)$ for some prime p. If $1 < \Lambda \leq P$ and Λ is weakly closed in P with respect to Γ, then $|\mathrm{fix}(\Lambda)| \leq n/2$.*

Proof: Suppose that $|\mathrm{fix}(\Lambda)| > n/2$. Let $\gamma \in \Gamma$ and $\Phi \in \langle \Lambda, \Lambda^\gamma \rangle$. Then $|\mathrm{fix}(\Phi)| \geq 1$, so that $\Phi \leq \Gamma_x$ for some $x \in V$. By Sylow's Theorem, there are $Q \in \mathrm{Syl}_p(\Phi)$ and $\phi \in \Phi$ such that $\Lambda \leq Q$ and $\Lambda^\gamma \leq Q^\phi$. But then Λ^ϕ and Λ^γ are both in Q^ϕ and hence in any conjugate of P which contains Q^ϕ. Therefore $\Lambda^\phi = \Lambda^\gamma$ by the weak closure condition. But then $|\mathrm{fix}(\Lambda)| \leq (n-1)/2$ by a result of C. Praeger [22], contradicting our assumption. ∎

Theorem 2.6. *Assume the definitions of Lemma 2.4. Suppose that some fragment Φ of P is uniquely identified amongst the fragments of P by the sizes of its orbits and that, for every $\gamma \in \Gamma$, $\Phi \leq P$ only if the non-trivial orbits of Φ^γ are orbits of P. Then $|\mathrm{supp}(\Phi)| \geq n/2$.*

Proof: Let $\gamma \in \Gamma$. If $\Phi^\gamma \leq P$ then, by assumption, $\mathrm{supp}(\Phi)$ is a union of orbits of P. Since $\mathrm{supp}(\Phi^\gamma)$ is a component of $H(G, P^\gamma)$, $\mathrm{supp}(\Phi^\gamma) = \mathrm{supp}(\Phi')$ for some fragments Φ' of P. But then $\Phi^\gamma \leq \Phi'$ since $\Phi^\gamma \leq P$ and so $\Phi^\gamma = \Phi'$, since both Φ^γ and Φ' are Sylow p-subgroups of the subgroup of Γ which fixes the orbits of Φ' setwise.

It follows that Φ is weakly closed in P with respect to Γ, so Lemma 2.5 applies. ∎

As an example of the use of Theorem 2.6, consider the automorphism group of a transitive graph with 15 vertices. A Sylow 2-subgroup cannot have the form $\langle (2\ 3)(4\ 5)(6\ 7), (8\ 9)(10\ 11)(12\ 13)(14\ 15) \rangle$, since the fragment $\langle (2\ 3)(4\ 5)(6\ 7) \rangle$ has a support which is too small.

Next we classify some types of subgroups of Γ_w where Γ is a group acting transitively on V, $w \in V(G)$ and p is a prime dividing $|\Gamma_w|$:

(a) Γ_w itself is a *type-1* subgroup.
(b) Any $\Lambda \in \mathrm{Syl}_p(\Gamma_w)$ is a *type-2* subgroup.
(c) The subgroup $\langle \mathrm{Syl}_p(\Gamma_w) \rangle$ generated by all the Sylow p-subgroups of Γ_w is a *type-3* subgroup.
(d) Suppose that $\Lambda \in \mathrm{Syl}_p(\Gamma_w)$ has $\mathrm{fix}(\Lambda) = \{w\}$. If Λ has an orbit X of size p and $|\Lambda_x| > 1$ for $x \in X$, then Λ_x is a *type-4* subgroup.

Theorem 2.7.

(a) *If Λ is a subgroup of Γ_w of type 1, 2, 3 or 4, then the normaliser $N_\Gamma(\Lambda)$ acts transitively on* $\mathrm{fix}(\Lambda)$.
(b) *If Λ is a subgroup of Γ_w of type 1 or 3, then* $\mathrm{fix}(\Lambda)$ *is a block of* Γ.

Proof: In (a), Λ is a conjugate in Γ_w to any of its conjugates in Γ which lie in Γ_w. (For type-4 subgroups, Lemma 7.4.7 of [10] is required.) We can thus apply Jordan's theorem (Theorem 3.5 of [29]). Claim (b) is an elementary exercise. ∎

We now turn to some applications of linear algebra. Let G be a transitive graph with $V(G) = \{1, 2, \ldots, n\}$. Let A be the $(0, 1)$ adjacency matrix of G. Suppose that $\Lambda \leq \mathrm{Aut}(G)$ and let $V_1, V_2, \ldots, V_m$ be the orbits of Λ in lexicographical order. The $m \times m$ matrix $Q(G, \Lambda) = (q_{ij})$ is defined by

$$q_{ij} = \sqrt{|V_j|/|V_i|}\, e_{ji},$$

where e_{ij} is the number of vertices in V_j to which each vertex in V_i is adjacent in G. It is not completely obvious, but true, that $Q(G, \Lambda)$ is symmetric.

For any real symmetric matrix M and real number λ, define $\mu_M(\lambda)$ to be zero if λ is not an eigenvalue of M and the multiplicity of λ as an eigenvalue of M otherwise.

Theorem 2.8. *Suppose that V_t contains a single vertex for some t. Let $Q = Q(G, \Lambda)$. For any real number λ, define $\rho(Q, \lambda, t)$ as follows.*

(a) *If λ is not an eigenvalue of Q, $\rho(Q, \lambda, t) = 0$.*
(b) *If λ is an eigenvalue of Q, let $x_1, x_2, \ldots, x_s$ be a complete orthonormal set of eigenvectors of Q for λ. Then define $\rho(Q, \lambda, t) = \sum_{i=1}^{s} (x_i)_t^2$, where the summand is the square of the t-th entry of x_i.*

Then $\mu_A(\lambda) = n\rho(Q, \lambda, t)$.

Proof: This is a special case of Theorem 3.4 of [9]. ∎

Corollary 2.9. *$\rho(Q, \lambda, t)$ is independent of t so long as $|V_t| = 1$.* ∎

Corollary 2.10. *$Q(G, \Lambda)$ and A have the same eigenvalues up to multiplicities.* ∎

We will also have use for the following facts about simple eigenvalues of A.

Theorem 2.11. *Let A be the $(0, 1)$-adjacency matrix of a vertex transitive graph G of order n and degree k.*

(a) *If λ is a simple eigenvalue of A, then λ is an integer of the form $k - 2\alpha$, for integer α.*
(b) *n is even if A has at least two simple eigenvalues, and divisible by four if A has at least three simple eigenvalues.*

Proof: Part (a) was first proved by Petersdorff and Sachs [21]. A proof of part (b) can be found in [9]. ∎

3. Construction of transitive graphs up to 19 vertices

This construction was very involved, and many steps required computations whose intermediate steps were too numerous to list here. We will confine ourselves to a brief overview; a more detailed description can be found in [18].

Throughout this section G will be a transitive graph of degree k with vertex set $V = \{1,2,\ldots,n\}$ and automorphism group $\Gamma = \mathrm{Aut}(G)$.

Our basic approach to constructing the graphs was to investigate the subgroups of Γ_1. To make this a little easier, we generated some simple families of transitive graphs separately. Define $\mathcal{G}$ to be the family of all transitive graphs G such that

(i) $n \in \{8,9,10,12,14,15,16,18\}$,
(ii) $3 \leq k \leq (n-1)/2$,
(iii) G is not an NTLP,
(iv) G is not a switching graph,
(v) Γ is not regular,
(vi) G has connectivity k, and
(vii) G is not strongly regular.

We will first describe how to generate the transitive graphs *not* in $\mathcal{G}$. Those with prime order have a p-cycle as an automorphism, by Sylow's theorem. This enabled rapid generation using the isomorphim program described in [20]. Those with degree at most two, or order at most six, are easily determined by hand; those which have degree greater than $(n-1)/2$ are complements of those which don't.

All the transitive switching graphs and NTLPs were found with the help of the catalogue of 9-vertex graphs made by Baker, Dewdney and Szilard [3]. Note that it is only necessary to form the switching graph of one graph from each switching class. Similarly, transitive NTLPs are NTLPs of transitive graphs, by Theorem 2.1(b). The transitive stongly regular graphs were extracted from Weisfeiler's list [28].

The transitive graphs with connectivity less than their degree are studied by Watkins [27]. With the help of his theory, it can be shown that there is only one such graph satisfying (i), (ii) and (iii), namely the graph in Figure 1. See [18] for a proof of this claim.

To obtain the transitive graphs with regular automorphism groups, we generated all the Cayley graphs of groups of order up to 18 and determined their isomorphism types using the program described in [20]. Such graphs were found in 12, 14, 16 and 18 vertices. We should note here that we could have excluded all Cayley graphs from $\mathcal{G}$, but we could not see how this could help us detemine $\mathcal{G}$ (even though it would be very much smaller). In any case, the fact that all the Cayley graphs in $\mathcal{G}$ were found by the general method constitutes a good check.

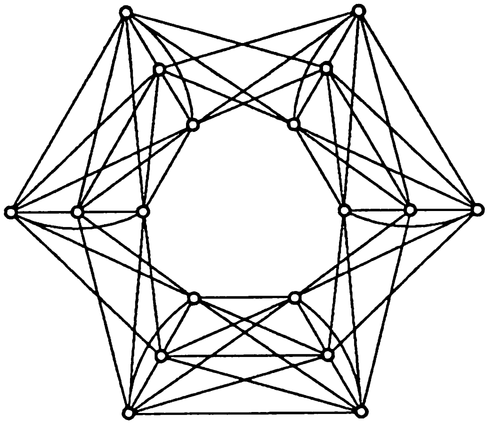

Figure 1.

We can now turn to the construction of $\mathcal{G}$. A *numerical partition* of n is a sequence σ of the form $(n; n_1^{m_1}, n_2^{m_2}, \ldots, n_r^{m_r})$ such that $1 \leq n_1 < n_2 < \ldots < n_r$, $m_i > 0$ for $1 \leq i \leq r$, and $n = \sum_{i=1}^{r} m_i n_i$. Superscripts equal to one are usually omitted. Further, define $r_{n_i}(\sigma) = m_i$ for $1 \leq i \leq r$ and $r_j(\sigma) = 0$ if $j \notin \{n_1, n_2, \ldots, n_r\}$. Also define $R(\sigma) = \{j | j \geq 2, r_j(\sigma) \neq 0\}$. If Λ is a permutation group of degree n, then $\sigma(\Lambda)$ is the numerical partition $(n; n_1^{m_1}, n_2^{m_2}, \ldots, n_r^{m_r})$ with m_i equal to the number of orbits of length n_i for each i.

Lemma 3.1. *For a numerical partition σ of n, define $m(\sigma) = \max\{r_i | i \in R(\sigma)\}$ and $t(\sigma) = \sum_{i \in R(\sigma)}$. Let Σ be the set of numerical partitions σ of $n \in \{8, 9, 10, 12, 14, 15, 16, 18\}$ such that $1 \leq r_1(\sigma) < n$, $r_1(\sigma)$ is a divisor of n, and σ satisfies none of the following conditions.*

(a) $r_1 \geq 2$ *and* $m(\sigma) = 1$.
(b) $t(\sigma) = 1$.
(c) $r_1 = 1$ *and* $t(\sigma) = 2$.
(d) *For some* $i \geq 2, r_i = 1$ *and* $(i, j) = 1$ *for all* $i \neq j \in R(\sigma)$.
(e) *For some prime* $p, r_1 = p$ *and* $m(\sigma) < p$.
(f) $\max\{j | r_j(\sigma) \neq 0\} > 10$.

Then, for every $G \in \mathcal{G}, \sigma(\Gamma_1) \in \Sigma$, *where* $\Gamma = \mathrm{Aut}(G)$.

Proof: Let $G \in \mathcal{G}$, $\Gamma = \mathrm{Aut}(G)$, and $\sigma = \sigma(\Gamma_1)$. $r_1(\sigma)$ is a divisor of n by Theorem 2.7(b).

By Theorem 2.1, we know that G cannot have any non-trivial ER subsets. If (a) is satisfied, $\mathrm{fix}(\Gamma_1)$ is ER, since $N_\Gamma(\Gamma_1)$ fixes each of the non-trivial ER subsets. If (b) is satisfied, the non-trivial orbit of Γ_1 is ER. If (c) is satisfied, G is strongly regular. If (d) is satisfied, the orbit of size i is ER, since the coprimality condition ensures that it is either not joined or completely joined to each other orbit.

Suppose (e) is satisfied. Then, by Theorem 2.7(a), a Sylow p-subgroup P of $N_\Gamma(\Gamma_1)$ acts transitively on fix(P). Also, P permutes the orbits of Γ_1, and so fixes the non-trivial orbits set-wise because there are fewer than p of each size. Therefore, fix(Γ_1) is ER.

Suppose (f) is satisfied, and let l be the length of the longest orbit of Γ_1. Since $n \leq 18$, condition (d) is satisfied if $l \in \{11, 13, 15, 16, 17\}$. If $l = 14$, either (a) or (d) is satisfied. This leaves only the possibilities $(18; 1^2, 2^2, 12)$ and $(18; 1, 2, 3, 12)$. In the first case the 12-orbit is ER, while in the second case the 2-orbit is ER. ∎

The set Σ contains 154 numerical partitions and, as the preceding theorem states, includes the orbit structure of all type-1 subgroups for elements of $\mathcal{G}$. Some of these partitions can be easily shown not to occur. Furthermore, if we allow subgroups of type 2, 3 or 4 as well, we can reduce the number of partitions even more (since each graph only needs to have one subgroup represented). Eventually, we come to the following conclusion.

Theorem 3.2. *There is a set Σ' of 57 numerical partitions such that, for any $G \in \mathcal{G}$,* Aut$(G)_1$ *contains a subgroup Λ of type 2 or type 4 such that $\sigma(\Lambda) \in \Sigma'$.*

Proof: This occupies eight pages of [18] as well as some computations; we will be content here to give a few examples. Type-1 and type-3 subgroups don't need inclusion in the theorem because all the entries of each $\sigma \in \Sigma'$ are powers of the same prime. However, type-1 and type-3 subgroups play an important part in the proof. Let $G \in \mathcal{G}$ and $\Gamma = \text{Aut}(G)$.

Example 1. Suppose that Γ_1 has all its non-trival orbits even, but not all powers of 2. Consider $P \in \text{Syl}_2(\Gamma_1)$. Clearly $\sigma(P) \neq \sigma(\Gamma_1)$, and $\sigma(P) \in \Sigma$ by its definition. Thus, we can eliminate $\sigma(\Gamma_1)$ without losing G. [Such eliminations need to be done with great care less we remove partitions required for the elimination of other partitions.]

Example 2. Suppose that Γ_1 has at least one orbit of size 2 and at least one orbit of length divisible by 3. Let $\Lambda = \langle \text{Syl}_3(\Gamma_1) \rangle$. Then $\sigma(\Lambda) \neq \sigma(\Gamma_1)$ and $\sigma(\Lambda) \in \Sigma$. With some care, this enables us to eliminate $\sigma(\Gamma_1)$ from Σ without losing G.

Example 3. Suppose $\sigma(\Lambda) = (18; 1, 3, 6, 8)$ for some $\Lambda \leq \Gamma_1$. Then G has degree 3, 6 or 8. By considering the numbers of edges between each of the orbits, it turns out that at least one of the non-trivial orbits must be ER. Thus, this case doesn't occur at all.

Example 4. Suppose $\sigma(\Lambda)$ has the form $(2r+2; 1^2, r^2)$ for some $\Lambda \leq \Gamma_1$. Since $G \in \mathcal{G}$, G has degree r and the two fixed points are not adjacent. If they are adjacent to the same r-orbit, fix(Λ) is ER. If they are adjacent to different r-orbits, G is a switching graph. Thus, these cases cannot occur.
Example 5. Suppose $\sigma(P) = (9; 1, 2^2, 4)$ for some $P \in \mathrm{Syl}_2(\Gamma_1)$. Consider $\Lambda = P_w$, where w lies in orbit of size 2. If $\sigma(P_w)$ was $(9; 1^3, 2, 4)$, $(9; 1^5, 4)$ or $(9; 1^5, 2^2)$, fix(P_w) would be ER, so we must have $\sigma(P_w) = (9; 1^3, 2^3)$. This is present in Σ', so we can eliminate $(9; 1, 2^2, 4)$. ∎

The numerical partitions of $n = 12$ which appear in Σ' are $(12; 1^2, 2, 4^2)$, $(12; 1^2, 2^3, 4)$, $(12; 1^2, 2^5)$, $(12; 1^4, 4^2)$, $(12; 1^4, 2^2, 4)$, $(12; 1^4, 2^4)$, $(12; 1^6, 2^3)$, $(12; 1^3, 3^3)$ and $(12; 1^6, 3^2)$.

Our next task will be to determine a family $\mathcal{Q}$ of matrices such that for every $G \in \mathcal{G}$, Aut$(G)_1$ contains a subgroup Λ of type 2 or type 4 such that $Q(G, \Lambda) \in \mathcal{Q}$. An important tool is Theorem 2.7(a), which implies that Aut$(Q(G,\Lambda))$ acts transitively on fix(Λ).

Choose an arbitrary $G \in \mathcal{G}$ and let Λ be a type-2 or type-4 subgroup of Aut(G) which is represented in Σ'. Let k be the degree of G. To begin with, we must choose the orbits to which vertex 1 is joined. Let $\sigma'(G, \Lambda)$ be the numerical partition of k associated with the sizes of these orbits. Note that Theorem 2.7(a) implies that we would the same numerical partition if we chose a fixed point other than 1.

Lemma 3.2. *Let* $\sigma = \sigma(\Lambda)$ *and* $\sigma' = \sigma'(G, \Lambda)$. *Then*

(a) *for all* i, $r_i(\sigma') \leq r_i(\sigma)$,
(b) $r_1(\sigma') < r_1(\sigma)$,
(c) $r_1(\sigma') < k$,
(d) $r_1(\sigma) r_1(\sigma')$ *is even, and*
(e) *if* $r_1(\sigma) \geq 2$, *there is some* $i \geq 2$ *such that* $0 \leq r_i(\sigma') < r_i(\sigma)$.

Proof: Conditions (a) and (b) are obvious. Condition (c) is necessary because G is connected. Condition (d) follows from the fact that the subgraph fix(Λ) is regular. Finally, if condition (e) was not satisfied, fix(Λ) would be ER. ∎

The next step is to detemine which orbits of Λ are completely joined in G. Define a graph $K = K(G, \Lambda)$ as follows. $V(G)$ is the set of orbits of Λ. Two vertices are adjacent if the corresponding orbits are completely joined in G. This includes all edges between fixed points, as well as a loop in each vertex associated with a non-trivial orbit containing a complete subgraph of G.

Clearly, Aut(K) must act transitively on fix(Λ). The subgraph of K induced by fix(Λ) must therefore be transitive; the possibilities can be extracted from the catalogue of [3]. The possibilities for the other edges of K were then computed by a sequence of backtrack searches which made extensive use of Theorem 2.7(a). Also, ER subsets were avoided, and some possibilities were rejected on a variety of connectivity grounds. Eventually a family of 199601 was produced, after about 4 hours computation.

Addition of the non-complete joins was done by similar means, and led to our first solution for $\mathcal{Q}$, containing 962131 matrices. These matrices were then subjected to a battery of necessary conditions, of which we describe a few. Let $Q = Q(G,\Lambda) \in \mathcal{Q}$.

(a) The number of triangles (3-cycles) of G incident with each vertex can be found by examining the neighbours of a fixed point Q. This number t must be independent of which fixed point is used. Also, upper and lower bounds for the number of incident triangles can be obtained for the vertices lying in larger orbits as well. These bounds must include t. Finally, $nt/3$ is the total number of triangles in G and so must be an integer. These tests removed all by 62818 matrices.

(b) The orbits of the fragments of Λ can be determined using Lemma 2.4. A fragment which moves 7 or fewer vertices cannot occur for odd n, by Theorem 2.3. Many other sets of fragments are rules out by Theorem 2.6.

(c) Define $\rho(Q,\lambda,t)$ as in Theorem 2.8. Then, whenever λ is an eigenvalue of Q, $n\rho(Q,\lambda,t)$ must be the same positive integer for all $t \in \text{fix}(\Lambda)$. If this integer is 1, we have a simple eigenvalue of A; the conclusions of Theorem 2.11 must hold. Since these calculations require floating point arithmetic, rounding error must be taken seriously. All eigensystem computations were verified to at least ten significant digits and then assumed accurate to only 3 digits. Fortunately, the tests are not invalidated if we mistake two very close eigenvalues for a single eigenvalue. These tests were spectacularly successful: of the 58454 matrices tested, only 709 passed.

We can now complete the determination of $\mathcal{G}$ by searching for realisations of each $Q \in \mathcal{Q}$. There are only 7 matrices corresponding to groups with an orbit of length 5 or more; these were processed by hand. All transitive realisations of the other 702 matrices were found by the computer in about 12 minutes. Only 120 matrices produced no transitive realisations, attesting to the power of the necessary conditions described above.

After merging $\mathcal{G}$ with the transitive graphs not in $\mathcal{G}$, all isomorphs were removed. The resulting graphs appear in [17], together with a large number of their properties and relationships.

4. Construction of transitive groups with 20–23 or 25–26 vertices

The transitive graphs of order 23 contain a 23-cycle in their automorphism groups; this makes their generation easy. For 20, 21, 22 or 26 vertices, we use the following Lemma.

Lemma 4.1. *Let* Γ *be a transitive with degree* $n = pm$*, where* p *is a prime and* $1 \leq m \leq p$*. Then* Γ *has an element of order* p *without fixed points.*

Proof: Apply Burnside's Lemma to a Sylow p-subgroup of Γ. ∎

We will now describe the construction of the 20-vertex transitive graphs. The others are similar but easier.

Let G be a transitive graph with $V(G) = \{v_{ij} | 1 \leq i \leq 4, 1 \leq j \leq 5\}$ and degree k. By Lemma 4.1, we can suppose that Aut(G) contains the permutation $\gamma = \gamma_1 \gamma_2 \gamma_3 \gamma_4$, where γ_i is the cycle $(v_{i1}\ v_{i2}\ v_{i3}\ v_{i4}\ v_{i5})$. Let X_i denote the support of γ_i, and also the subgraph of G induced by it. Let X_{ij} denote the subgraph with vertices $X_i \cup X_j$ and all the edges with one end in X_i and the other in X_j.

It is clear that each X_i is transitive, although different X_i need not be isomorphic. Similarly, X_{ij} is a regular graph for which $\gamma_i \gamma_j$ is an automorphism.

Define the matrix $Q = Q(G, \langle\gamma\rangle)$ as in Section 2. Since all the orbits are the same length, we see that q_{ii} is the degree of X_i and q_{ij} $(i \neq j)$ is the degree of X_{ij}. Our first task is to determine all potential values of Q. Some necessary conditions on Q are

- (a) Q is a 4×4 symmetric matrix,
- (b) for $1 \leq i \leq 4, q_{ii} \in \{0, 2, 4\}$,
- (c) for $1 \leq i < j \leq 4, q_{ij} \in \{0, 1, 2, 3, 4, 5\}$, and
- (d) for $1 \leq i \leq 4, \sum_{j=1}^{4} q_{ij} = k$.

Condition (d) follows from the regularity of G.

We used a backtrack procedure to determine all solutions of (a)–(d) and then applied the progam *nauty* [19] to eliminate isomorphs.

Next, we determined all transitive realisations of each matrix. Suppose Q is one such matrix. We can begin by filling in the subgraphs X_i, as there is only one possibility for each matrix. We can begin by filling in the subgraphs X_i, as there is only one possibility for each degree. (This is not true for $n = 21$ and $n = 22$; in any case just try all the possibilities.) Then we can try all the possibilities for the graphs X_{ij} in the order $X_{12}, X_{13}, X_{14}, X_{23}, X_{24}, X_{34}$.

It is unnecessarily expensive to try all the available subgraphs in each case. Consider X_{12}, for example. If two possibilities for X_{12} are equivalent under the group $\langle\gamma_2\rangle$, they clearly lead to the same possibilities for G. If X_2 is empty or complete, we can also consider equivalence under the group $\langle(v_{21}\, v_{22}\, v_{24}\, v_{23})\rangle$, since application of that group preserves the property that $\gamma_1 \gamma_2$ is an automorphism of X_{12}. We can us the same reductions for X_{13} and X_{14}, but not for X_{23}, X_{24} and X_{34}.

Finally, the resulting graphs were tested for transitivity using *nauty*, and isomorphs were eliminated. The complete computation took about 30 minutes on a VAX for $n = 20$, and less than 10 minutes each for $n = 21, 22, 26$.

Lemma 4.1 could also be used to construct the transitive graphs with 25 vertices, but it is easier to notice that all these graphs must be Cayley graphs. In fact, we have the following more general theorem.

Theorem 4.1. *Let p be prime. Then any transitive graph with p^2 vertices is a Cayley graph.*

Proof: Let G be a transitive graph with $n = p^2$ vertices, and let P be a Sylow p-subgroup of Aut(G). Clearly, P is transitive. Now consider P_v, for some $v \in V(G)$. If P_v is trivial, then G is a Cayley graph of P. Otherwise, P_v must have p fixed points and $p-1$ orbits of size p. Thus, by Theorem 2.7(a), the normaliser $N_P(P_v)$ acts transitively on fix(P) and fixes the larger orbits set-wise. This implies that fix(P) is an ER subset, which in turn implies (by Theorem 2.1) that G in an NTLP $G_1[G_2]$, where G_1 and G_2 are transitive graphs, and hence Cayley graphs, with p vertices. Thus, G is a Cayley graph of $C_p \times C_p$. ∎

The construction of 25-vertex transitive graphs is now easily completed by generating all Cayley graphs of C_{25} and $C_5 \times C_5$ and eliminating isomorphs.

5. Summary of Results

In Table 1 we give the number of transitive graphs of each order and degree. For convenience, we also restate the numbers of 24-vertex transitive graphs as found in [25]. To obtain the counts for degrees not in the table, simply look up the complementary degrees.

Since a disconnected transitive graph is just a collection of isomorphic connected transitive graphs, we can easily obtain Table 2, in which only connected graphs are counted. In this case, counts for degrees not shown are obtained by looking up the complementary degrees in Table 1.

It turns out that the great majority of transitive graphs are Cayley graphs. In fact, non-Cayley transitive graphs don't occur for $n \leq 25$ except for $n \in \{10, 15, 16, 18, 20, 24, 26\}$. The examples on 10 vertices are, of course, the Peterson graph and its complement. The counts of non-Cayley transitive graphs are shown in Table 3. Use the complementary degree for degrees not shown. The values for $n = 24$ are taken from [25]. The others were found by computing all the Cayley graphs and comparing this list with our list of all transitive graphs.

	degree													
n	0	1	2	3	4	5	6	7	8	9	10	11	12	total
1	1													1
2	1	1												2
3	1		1											2
4	1	1	1	1										4
5	1		1		1									3
6	1	1	2	2	1	1								8
7	1		1		1		1							4
8	1	1	2	3	3	2	1	1						14
9	1		2		3		2		1					9
10	1	1	2	3	4	4	3	2	1	1				22
11	1		1		2		2		1		1			8
12	1	1	4	7	11	13	13	11	7	4	1	1		74
13	1		1		3		4		3		1		1	14
14	1	1	2	3	6	6	9	9	6	6	3	2	1	56
15	1		3		8		12		12		8		3	48
16	1	1	3	7	16	27	40	48	48	40	27	16	7	286
17	1		1		4		7		10		7		4	36
18	1	1	4	7	16	24	38	45	54	54	45	38	24	380
19	1		1		4		10		14		14		10	60
20	1	1	4	11	28	47	83	115	149	168	168	149	115	1214
21	1		3		11		29		48		56		48	240
22	1	1	2	3	11	18	38	52	79	94	109	109	94	816
23	1		1		5		15		30		42		42	188
24	1	1	6	20	74	167	373	652	1064	1473	1858	2064	2064	15506
25	1		2		9		25		57		86		104	464
26	1	1	2	5	16	29	71	117	204	286	397	466	523	4236

Table 1. The number of transitive graphs.

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	degree													
n	0	1	2	3	4	5	6	7	8	9	10	11	12	total
1	1													1
2		1												1
3			1											1
4			1	1										2
5			1		1									2
6			1	2	1	1								5
7			1		1		1							3
8			1	2	3	2	1	1						10
9			1		3		2		1					7
10			1	3	3	4	3	2	1	1				18
11			1		2		2		1		1			7
12			1	4	10	12	13	11	7	4	1	1		64
13			1		3		4		3		1		1	13
14			1	3	5	6	8	9	6	6	3	2	1	51
15			1		7		12		12		8		3	44
16			1	4	13	25	39	47	48	40	27	16	7	272
17			1		4		7		10		7		4	35
18			1	5	12	23	36	45	53	54	45	38	24	365
19			1		4		10		14		14		10	59
20			1	7	24	43	80	113	148	167	168	149	115	1190
21			1		10		28		48		56		48	235
22			1	3	9	18	36	52	78	94	108	109	94	807
23			1		5		15		30		42		42	187
24			1	11	60	152	359	640	1057	1469	1857	2063	2064	15422
25			1		8		25		57		86		104	461
26			1	5	13	29	67	117	201	286	396	466	522	4221

Table 2. The number of connected transitive graphs.

	degree										
n	3	4	5	6	7	8	9	10	11	12	total
10	1			1							2
15		1		1		1		1			4
16		1	1	1	1	1	1	1	1		8
18			1			1	1			1	4
20	3	4	4	7	8	7	8	8	7	8	82
24			1	5	7	9	11	11	12	12	112
26	1		2	3	7	6	8	13	14	12	132

Table 3. The number of transitive graphs which are not Cayley graphs.

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