Eulerian Subgraphs in Graphs with Short Cycles

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Abstract

P. Paulraja recently showed that if every edge of a graph G lies in a cycle of length at most 5 and if G has no induced $K_{1,3}$ as a subgraph, then G has a spanning closed trail. We use a weaker hypothesis to obtain a stronger conclusion. We also give a related sufficient condition for the existence of a closed trail in G that contains at least one end of each edge of G.

We shall use the notation of Bondy and Murty [2], but we assume that graphs have no loops. For $k \geq 2$, the 2-regular connected graph of order k is called a <u>k-cycle</u> and is denoted C_k. We denote the symmetric difference of sets X and Y by $X \triangle Y$. A graph is called <u>eulerian</u> if it is connected and its vertices have even degree.

For any graph G and any edge $e \in E(G)$, we let G/e denote the graph obtained from G by contracting e and by deleting any resulting loops. If H is a connected subgraph of G, then G/H denotes the graph obtained by contracting all edges of $E(H)$ and by deleting any resulting loops.

A family of graphs will be called a <u>family</u>. A family $\mathcal{C}$ of graphs is said to be <u>closed under contraction</u> if

$$G \in \mathcal{C},\ e \in E(G) \implies G/e \in \mathcal{C}.$$

We call a family $\mathcal{C}$ of connected graphs <u>complete</u> if it satisfies these three conditions:

(C1) $K_1 \in \mathcal{C}$;
(C2) $\mathcal{C}$ is closed under contraction;
(C3) $H \subseteq G,\ H \in \mathcal{C},\ G/H \in \mathcal{C} \implies G \in \mathcal{C}$.

For any family $\mathcal{S}$ of graphs, define the <u>kernel</u> of $\mathcal{S}$ to be

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(1) $\mathcal{S}^O = \{H \mid \text{For all supergraphs } G \text{ of } H,\ G \in \mathcal{S} \iff G/H \in \mathcal{S}\}$.

For example, if $\mathcal{S}$ is the family of graphs with exactly 3 cut edges, say, then its kernel $\mathcal{S}^O$ is the family of 2-edge-connected graphs. The kernel of the family of k-edge-connected graphs equals itself, for any $k \in \mathbf{N}$. Any kernel always contains K_1, and often contains no other graph. In [4], the first author proved that if $\mathcal{S}$ is a family closed under contraction then its kernel is a complete family. Also:

Theorem 1 [4] For a family $\mathcal{C}$, closed under contraction, these are equivalent:
(a) $\mathcal{C}$ is a kernel of some family closed under contraction;
(b) $\mathcal{C}$ is a complete family;
(c) $\mathcal{C} = \mathcal{C}^O$ ($\mathcal{C}$ is the kernel of itself), and $\mathcal{C}$ is closed under contraction.

Let $k \in \mathbf{N}$, and let $\mathcal{S}$ be a family. A graph G is said to be at most k edges short of being in $\mathcal{S}$ if there is a graph G' in $\mathcal{S}$ such that $G' - E_k = G$ for some set E_k of at most k edges of G'. The second author [9] proved that every complete family is the kernel of a family that is not itself:

Theorem 2 [9] Let $\mathcal{C}$ be a complete family and for any $k \geq 0$ let

$$\mathcal{C}_k = \{G \mid G \text{ is at most } k \text{ edges short of being in } \mathcal{C}\}.$$

Then the kernel of $\mathcal{C}_k$ is $\mathcal{C}$.

Let $\mathcal{SL}$ denote the family of graphs with a spanning closed trail, where K_1 is regarded as being in $\mathcal{SL}$. We call a graph in $\mathcal{SL}$ supereulerian. Say that a graph G is collapsible if for every even subset $S \subseteq V(G)$, G has a spanning connected subgraph $\Gamma(S)$ such that S is the set of odd-degree vertices of $\Gamma(S)$. Let $\mathcal{CL}$ denote the family of collapsible graphs. Clearly,

(2) $$\mathcal{CL} \subseteq \mathcal{SL},$$

since we can let S be empty in the definition of $\mathcal{CL}$. Also, $K_1 \in \mathcal{CL}$, and all other collapsible graphs are 2-edge-connected.

The first author ([3] and [4]) proved that

(3) $$\mathcal{CL}^O = \mathcal{CL} \subseteq \mathcal{SL}^O$$

and conjectured that $\mathcal{CL} = \mathcal{SL}^O$. By (1) and (3), for any graph G and subgraph $H \subseteq G$, if $H \in \mathcal{CL}$ or if $H \in \mathcal{SL}^O$ then

(4) $$G \in \mathcal{SL} \iff G/H \in \mathcal{SL};$$

and if $H \in \mathcal{CL}$ then

(5) $$G \in \mathcal{CL} \iff G/H \in \mathcal{CL}.$$

By (3) and by (c) $\Rightarrow$ (b) of Theorem 1, the family $\mathcal{CL}$ of collapsible graphs is a complete family. The next result shows that it is substantial:

Theorem 3 [3] If the graph G is at most one edge short of having two edge-disjoint spanning trees, then exactly one of these holds:

(a) G has a cut edge; or

(b) $G \in \mathcal{CL}$.

Corollary 3A The 2-cycle and 3-cycle are in $\mathcal{CL}$.

Corollary 3B (Jaeger [7]) If a graph G has two edge-disjoint spanning trees, then $G \in \mathcal{SL}$.

The smallest graph of girth 4 in $\mathcal{CL}$ is $K_{3,3} - e$; the 4-cycle is not in $\mathcal{CL}$.

P. Paulraja [12] obtained sufficient conditions (Corollary 8B) for a graph G, whose edges each lie in cycles of length at most 5, to be supereulerian (i.e., for $G \in \mathcal{SL}$). In this paper, we generalize his conclusion from $\mathcal{SL}$ to any family $\mathcal{S}$ whose kernel $\mathcal{S}^O$ is closed under contraction and contains K_3 but not K_2 (by Corollary 3A and (3), $K_3 \in \mathcal{SL}^O$; by inspection, $K_2 \notin \mathcal{SL}^O$). We also relax Paulraja's hypothesis considerably. Following our first generalization (Theorem 8), we give another relevant example of a family with a large kernel. A second generalization (Theorem 9) involving eulerian subgraphs is also given.

Definition (1) is the basis for a reduction method to determine whether a particular graph is in a given family. We need the following related results.

Theorem 4 [4] Let G be a graph, let $\mathcal{C}$ be a complete family, and let H_1 and H_2 be subgraphs of G in $\mathcal{C}$. If $V(H_1) \cap V(H_2) \neq \emptyset$, then $H_1 \cup H_2 \in \mathcal{C}$.

For a complete family $\mathcal{C}$ and a graph G, a subgraph H of G with $H \in \mathcal{C}$ is called maximal if H is not a proper subgraph of another subgraph of G in $\mathcal{C}$.

Theorem 5 [4] Let $\mathcal{C}$ be a complete family. For a graph G, define

(6) $$E' = E'(\mathcal{C}) = \{e \in E(G) \mid e \text{ is in no subgraph of } G \text{ in } \mathcal{C}\}.$$

Then each component of $G - E'$ is in $\mathcal{C}$, and these components of $G - E'$ are the maximal subgraphs of G in $\mathcal{C}$.

Let G be a graph and let $\mathcal{C}$ be a complete family. The $\mathcal{C}$-reduction of G is the graph obtained from G by contracting the maximal subgraphs of G in $\mathcal{C}$ to distinct vertices. A graph G is called $\mathcal{C}$-reduced if every subgraph of G in $\mathcal{C}$ is K_1.

Theorem 6 [4] For any complete family $\mathcal{C}$, the $\mathcal{C}$-reduction of a graph is $\mathcal{C}$-reduced.

Corollary 6A If $\mathcal{C}$ is a complete family with $K_3 \in \mathcal{C}$, then the $\mathcal{C}$-reduction of any graph has girth at least 4.

Proof: By (C2), $C_2 \in \mathcal{C}$, since $K_3 \in \mathcal{C}$, and so by Theorem 6, the $\mathcal{C}$-reduction of any graph has girth at least 4. □

By Corollaries 3A and 6A, and since $\mathcal{CL}$ is a complete family,

(7) The $\mathcal{CL}$-reduction of a graph has girth at least 4.

Let G be a graph, let $\mathcal{C}$ be a complete family, and let G' be the $\mathcal{C}$-reduction of G. By Theorem 5, G' is well-defined. By (a) $\Rightarrow$ (b) of Theorem 1, if $\mathcal{S}$ is a family closed under contraction, then $\mathcal{S}^O$ is complete, and hence $\mathcal{S}^O$-reductions are defined. Hence, by repeated applications of (1), if G' is the $\mathcal{S}^O$-reduction of G, then

(8) $$G \in \mathcal{S} \iff G' \in \mathcal{S}.$$

By Theorem 5 and the definition of G'

(9) $$E(G') = E'.$$

Two conjectures due to Paulraja ([10], [11]; also [12]) have previously been proved by the second author.

Theorem 7 (H.-J. Lai [8], [9]) Let G be a 2-connected graph in which every edge is in a cycle of length at most 4. Then:

(a) G has an eulerian subgraph containing at least one vertex of every edge of G; and

(b) If $\delta(G) \geq 3$, then $G \in \mathcal{SL}$.

Furthermore, the second author ([8] and [9]) proved the stronger result that in part (b) of Theorem 7, G is also in $\mathcal{CL}$.

Definition of $\mathcal{F}$ Let $\mathcal{C}$ be a complete family that contains K_3 but not K_2. Let $\mathcal{F} = \mathcal{F}(\mathcal{C})$ be the family of connected graphs G such that every vertex of degree at least 3 in G is incident with at most one edge $e \in E'(\mathcal{C})$ (see (6)).

For example, if G has no induced $K_{1,3}$ as a subgraph, then $G \in \mathcal{F}$ follows from $K_3 \in \mathcal{C}$. Our main results all concern graphs in $\mathcal{F}$.

Let G be a graph, let H be a subgraph of G, and let G' be the $\mathcal{C}$-reduction of G. By (6) and (9),

(10) $$E(G') \subseteq E(G).$$

We call $G'[E(H) \cap E(G')]$ the subgraph of G' induced by H. We also say that H induces $G'[E(H) \cap E(G')]$. We denote by H' the subgraph of G' induced by H, but such a subgraph H' is not in general the $\mathcal{C}$-reduction of H.

Lemma 1 Let G' be the $\mathcal{C}$-reduction of G, where $\mathcal{C}$ is a complete family, and let E' be defined by (1). Then

(i) Every cycle of G' is induced by a cycle of G; and

(ii) Every cycle C of G induces a connected eulerian subgraph in G', and if $K_3 \in \mathcal{C}$, then any cycle in G' has length at least 4.

(iii) A cycle $C \subseteq G$ induces a K_1 in G' if and only if $E(C) \cap E' = \emptyset$.

(iv) If $K_3 \in \mathcal{C}$ then every cycle of length at most 7 in G induces either a K_1 or a cycle in G' of length at least 4.

Proof: By Theorem 5, G has a unique collection of maximal subgraphs, say $H_1, H_2, ..., H_c$, in $\mathcal{C}$. Since $K_1 \in \mathcal{C}$, each vertex of G is in some $V(H_i)$ $(1 \leq i \leq c)$. Recall that G' is formed from G by contracting each H_i $(1 \leq i \leq c)$ to a distinct vertex of $V(G_1)$. The first three parts of the lemma are just straightforward applications of this definition of G'. Each H_i $(1 \leq i \leq c)$ is connected, and so (i) holds. Note that a cycle C of G induces a K_1 in G' whenever C is contained in some H_i $(1 \leq i \leq c)$. By $K_3 \in \mathcal{C}$ and Corollary 6A, any cycle in G' has length at least 4, and we use this next to prove (iv). By both parts of (ii) and $K_3 \in \mathcal{C}$, if $|E(C)| \leq 7$ then the eulerian subgraph of G' induced by C cannot contain a cycle of length 2 or 3, and hence must be K_1 or a cycle of length at least 4. □

Lemma 2 Let $\mathcal{C}$ be a complete family not containing K_2. Let G be a graph in $\mathcal{F}(\mathcal{C})$, let H be a maximal nontrivial subgraph of G in $\mathcal{C}$, and let C be a cycle in G. Then either $V(H) \cap V(C) = \emptyset$ or $E(H) \cap E(C) \neq \emptyset$.

Proof: Suppose that $\mathcal{C}$, G, $\mathcal{F}$, H, and C satisfy the hypothesis, and suppose that $E' = E'(\mathcal{C})$ satisfies (6). By way of contradiction, suppose that Lemma 2 is false. Then there is a vertex $v \in V(C) \cap V(H)$ such that the two edges of C incident with v (e_1 and e_2, say) are not in H, and hence are in E'. But since $H \in \mathcal{C}$ is nontrivial, v is incident with at least two edges of $E(H)$ (because a graph in $\mathcal{C}$ is 2-edge-connected, for otherwise $K_2 \in \mathcal{C}$, by (C2)), and since $e_1, e_2 \notin E(H)$, $d(v) \geq 4$. But $G \in \mathcal{F}$, and so there can be only one edge of G in E' that is incident with v. This contradicts $e_1, e_2 \in E'$. □

For a graph G and a complete family $\mathcal{C}$, let G' be the $\mathcal{C}$-reduction of G. A vertex v of G' is called a trivial vertex if its preimage under the $\mathcal{C}$-reduction-mapping $G \longrightarrow G'$ is K_1. Otherwise, v is called nontrivial.

Lemma 3 Let $\mathcal{C}$ be a complete family and let G' be the $\mathcal{C}$-reduction of the graph G. If $G \in \mathcal{F}(\mathcal{C})$, then any trivial vertex of G' has degree at most 2 in both G and G'.

Proof: Let G' denote the $\mathcal{C}$-reduction of $G \in \mathcal{F}$. By Theorem 5, there are maximal subgraphs $H_1, H_2, ..., H_c$ of G in $\mathcal{C}$, and G' is obtained from G by contracting each H_i $(1 \leq i \leq c)$ to a vertex v_i. By way of contradiction, suppose v_i is a trivial vertex with degree at least 3. Since the degree is at least 3, $G \in \mathcal{F}$ implies that the preimage of v_i is a nontrivial subgraph $H_i \in \mathcal{C}$, and hence v_i is not trivial, a contradiction. □

Lemma 4 Let $\mathcal{C}$ be a complete family not containing K_2. Let $G \in \mathcal{F}(\mathcal{C})$, let G' be the $\mathcal{C}$-reduction of G, let C be a cycle in G that induces a cycle C' (say) in G', and define

$$X' = \{x \in V(C') \mid x \text{ is a nontrivial vertex of } G'\}.$$

Then

$$|E(C)| \geq |X'| + |E(C')|.$$

Proof: Let $\mathcal{C}$ be a complete family, let $G \in \mathcal{F}$, let G' denote the reduction of G, and let C, C', and X' satisfy the hypothesis of Lemma 4.

By Theorem 5, there is a unique collection of maximal subgraphs of G in $\mathcal{C}$, and since $K_1 \in \mathcal{C}$, every vertex of G is in some subgraph in this collection. Let $H_1, H_2, ..., H_t$ be the maximal subgraphs of G in $\mathcal{C}$ that each meet C in at least one vertex. When G is contracted to form its $\mathcal{C}$-reduction G', each H_i $(1 \leq i \leq t)$ is contracted to form a distinct vertex of C', the subgraph of G' induced by C. (By Lemma 1, C' may be a cycle, as assumed in the hypothesis of Lemma 4.) Thus,

$$t = |V(C')| = |E(C')|,$$

since each vertex of C is in some H_i $(1 \leq i \leq t)$. Without loss of generality, let $H_1, H_2, ..., H_s$ be the nontrivial members of $\{H_1, H_2, ..., H_t\}$, where necessarily $s \leq t$. Then the set X' of s distinct vertices of $V(G')$ onto which $H_1, H_2, ..., H_s$ are contracted are the nontrivial vertices of $V(C')$. Thus,

$$s = |X'|. \tag{11}$$

By Lemma 2, $E(C) \cap E(H_i)$ contains at least one edge $(1 \leq i \leq s)$, and so by (11),

$$\begin{aligned} |E(C)| &= |E(C) \cap \dot{\bigcup_{i=1}^{s}} E(H_i)| + |E(C')| \\ &\geq s + |E(C')| = |X'| + |E(C')|, \end{aligned}$$

as claimed. □

Theorem 8 Let $\mathcal{S}$ be a family whose kernel $\mathcal{S}^O$ is closed under contraction and contains K_3 but not K_2. Let $G \in \mathcal{F}(\mathcal{S}^O)$ and let $E' = E'(\mathcal{S}^O)$ be the set defined in (6). Suppose that each edge of E' is in a cycle of G of length at most 5. Then exactly one of the following holds.

(a) $G \in \mathcal{S}^O$;

(b) $G \in \{C_4, C_5\} \setminus \mathcal{S}^O$;

(c) Both $C_4 \notin \mathcal{S}^O$ and G has a nontrivial subgraph $H \in \mathcal{S}^O$ such that G/H is the union at least two 4-cycles whose only common vertex is v_H, the vertex of G/H corresponding to H. (See Figure 1.)

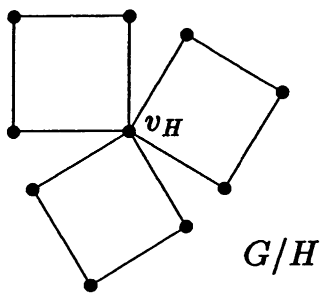

Figure 1

Proof: Suppose $\mathcal{S}$, $\mathcal{F}$, G, and E' satisfy the hypothesis of Theorem 8.

We first note that the conclusions of Theorem 8 are mutually exclusive. If (b) holds, then (a) fails, and since $K_2 \notin \mathcal{S}^O$, (1) implies that (c) fails. Suppose that both (a) and (c) hold. By (a) $\Rightarrow$ (b) of Theorem 1, $\mathcal{S}^O$ is a complete family, and so by (C2), every contraction of G is in $\mathcal{S}^O$. By (c) of Theorem 8, C_4 is a contraction of G. Hence, $C_4 \in \mathcal{S}^O$, contrary to (c).

Let $G \in \mathcal{F}(\mathcal{S}^O)$, and let G' be the $\mathcal{S}^O$-reduction of G. By the hypothesis of Theorem 8, for each edge $e \in E'$, we can find in G a cycle C_e (say) of length at most 5. If C_e has length at most 3, then $C_e \in \mathcal{S}^O$, and so $e \notin E'$, a contradiction. Thus, for all $e \in E'$,

(12) $$4 \leq |E(C_e)| \leq 5.$$

If $E' = \emptyset$, then $G \in \mathcal{S}^O$, and (a) holds. Otherwise, let $e \in E'$. Then the subgraph C'_e of G' induced by C_e contains e, and hence is a cycle of length at least 4, by (iv) of Lemma 1. By Lemma 4,

$$(13) \qquad |E(C_e)| \geq |X'| + |E(C'_e)|,$$

where C'_e is the cycle in G' induced by C_e and where

$$(14) \qquad X' = \{x \in V(C'_e) \mid x \text{ is a nontrivial vertex of } G'\}.$$

(Since $e \in E(C_e) \cap E'$, (ii) and (iii) of Lemma 1 imply that C'_e is a cycle in G' of girth at least 4.) Therefore, by (12) and (13),

$$5 \geq |E(C_e)| \geq |X'| + 4,$$

and so $|X'| \leq 1$.

If $X' = \emptyset$ for any $e \in E'$, then all vertices of C'_e are trivial. Then by Lemma 3 and since $G \in \mathcal{F}$, all vertices of C_e have degree at most 2 in G. Hence, $G = C_e$, and so (12) implies (b) of Theorem 8.

Otherwise, $|X'| = 1$ for every $e \in E'$. Therefore, each cycle C_e $(e \in E')$ meets a single nontrivial maximal subgraph $H_e \in \mathcal{C}$. Since G is connected, H_e must be the same for all $e \in E'$, and we denote this single nontrivial maximal subgraph in $\mathcal{C}$ by H. By Lemma 2, $|E(H) \cap E(C_e)| \geq 1$, and so C_e induces a cycle in G' of length at most 4, since $|E(C_e)| \leq 5$. By (ii) of Lemma 1, this induced cycle has length at least 4. Therefore, each C_e induces a 4-cycle of G', all such 4-cycles of G' meet only at the vertex v_H of G' corresponding to H (see Figure 1). □

Let

$$\eta(G) = \min_{E \subseteq E(G)} \frac{|E|}{\omega(G - E) - 1},$$

where the minimum is over all edge sets $E \subseteq E(G)$ such that $\omega(G - E)$, the number of components of $G - E$, is at least 2. Cunningham [5] proved that for any numbers $s, t \in \mathrm{N}$, $\eta(G) \geq s/t$ if and only if G has a family $\mathcal{T}$ of s spanning trees such that each edge of G is in at most t trees in $\mathcal{T}$.

The family $\mathcal{S} = \{G \mid \eta(G) \geq r\} \cup \{K_1\}$ satisfies the hypothesis of Theorem 8, if $r \geq 3/2$. Since $\mathcal{S} = \mathcal{S}^O$ (see [4]) in this case, we have $C_4, C_5 \notin \mathcal{S}^O$ in the conclusion of the theorem, if $r \geq 3/2$.

In [4], the first author proved that

$$\mathcal{CL} \subset \{G \mid \eta(G) \geq 3/2\} \cup \{K_1\},$$

and thus by (3) and the paragraph preceding (3), both $\mathcal{SL}$ and $\mathcal{CL}$ may be substituted for $\mathcal{S}$ in Theorem 8. In the conclusion of Theorem 8, neither C_4 nor

C_5 is in $\mathcal{SL}^O$ nor in $\mathcal{CL}^O = \mathcal{CL}$. Therefore, we have:

Corollary 8A Let $G \in \mathcal{F}(\mathcal{CL})$ and let $E'(\mathcal{CL})$ be the subset of $E(G)$ defined in (6). If each edge of $E'(\mathcal{CL})$ is in a cycle of G of length at most 5, then exactly one of the following holds:

(a) $G \in \mathcal{CL}$;

(b) $G \in \{C_4, C_5\}$;

(c) G has a nontrivial subgraph $H \in \mathcal{CL}$ such that G/H is a union of at least two 4-cycles having only one common vertex v_H (see Figure 1), where v_H is the vertex of G/H corresponding to H.

Corollary 8B (Paulraja [12]) Let G be a connected graph having no induced $K_{1,3}$ as a subgraph. If each edge of G is in a cycle of length at most 5, then $G \in \mathcal{SL}$.

Proof: Since G has no induced $K_{1,3}$, each vertex $v \in V(G)$ of degree at least 3 is incident with at most one edge not in a K_3. But $K_3 \in \mathcal{CL}$, and hence $G \in \mathcal{F}(\mathcal{CL})$, by definition. This and the hypothesis of Corollary 8B fulfill the hypothesis of Corollary 8A, and so G satisfies a conclusion of Corollary 8A. By (2), (a) of Corollary 8A implies $G \in \mathcal{SL}$, and by inspection, (b) and (c) of Corollary 8A imply $G \in \mathcal{SL}$. □

Let $\mathcal{S} = \mathcal{CL}$ or $\mathcal{S} = \mathcal{SL}$ in Theorem 8, and suppose that $\mathcal{F} = \mathcal{F}(\mathcal{CL})$ or $\mathcal{F} = \mathcal{F}(\mathcal{SL}^O)$, respectively. The graph G_1 of Figure 2 violates the hypothesis of Theorem 8 only because v, a vertex of degree 3, is incident with more than one edge lying in no collapsible subgraph. The graph G_2 lies in $\mathcal{F}$, and its shortest cycles containing certain edges of $E(G_2)$ have length 6, contrary to the hypothesis of Theorem 8. Both G_1 and G_2 have the nonsupereulerian graph $K_{2,3}$ as their reductions, and so the hypotheses of Theorem 8 and its corollaries are best-possible. (The only nontrivial subgraphs of G_1 or G_2 in $\mathcal{CL}$ or $\mathcal{SL}^O$ are 3-cycles.)

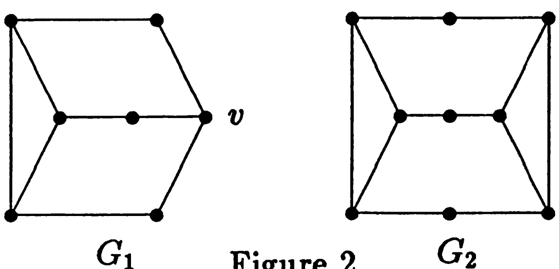

Figure 2

The graphs G_3 and G_4 (Figure 3) violate the hypotheses of Theorem 9 and Corollary 9A in the manner that G_1 and G_2 (Figure 2) violate the hypothesis of Theorem 8. Their only nontrivial subgraphs in $\mathcal{SL}^O$ are K_3's, and both graphs violate the conclusions of Theorem 9 and Corollary 9A.

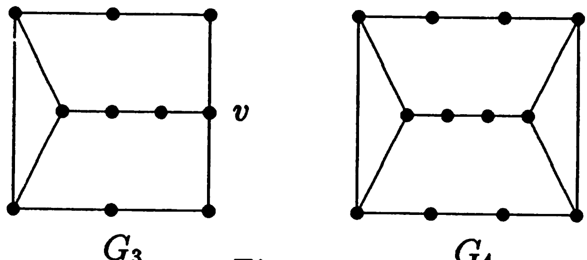

Figure 3

<u>Theorem 9</u> Let $G \in \mathcal{F}(\mathcal{SL}^O)$, define E' by (6), and suppose that every edge of E' is in a cycle of G having length at most 7. Then G has a connected eulerian subgraph Γ satisfying both of these properties:

(i) Each edge of G has at least one end in $V(\Gamma)$; and

(ii) $V(\Gamma)$ contains each vertex of G having degree at least 3.

<u>Proof</u>: Let $G \in \mathcal{F}(\mathcal{SL}^O)$ and let G' be the $\mathcal{SL}^O$-reduction of G. The family $\mathcal{SL}^O$ contains K_3 but not K_2, and so the prior lemmas hold. Let Γ' be a connected eulerian subgraph of G' such that

(15) $$|V(\Gamma')| \text{ is maximized}$$

and, subject to (15),

(16) $$\Gamma' \text{ has as many nontrivial vertices as possible.}$$

First we show that Γ' satisfies (26) and (27). Suppose, by way of contradiction, that either Γ' does not contain every nontrivial vertex of G' or that there is an edge $w'x' \in E(G')$ such that $w', x' \notin V(\Gamma')$, i.e., suppose that either (26) or (27) is false. Then there is an edge $vw \in E(G')$ with

(17) $$v \in V(\Gamma'), \quad w \notin V(\Gamma'),$$

and such that either

(18) $$w \text{ is a nontrivial vertex,}$$

or w has exactly two neighbors in G', say v and x, where

(19) $$x \notin V(\Gamma').$$

Denote $e = vw$. Since $e \in E(G') = E'$, the hypothesis of Theorem 9 implies that e is in a cycle C_e of G, with

(20) $$|E(C_e)| \leq 7.$$

Let C'_e be the subgraph in G' induced by C_e. By $e \in E'$, we have $e \in E(C'_e)$. Since $K_3 \in \mathcal{SL}^O$, (iv) of Lemma 1 and (20) imply C'_e is a cycle in G' with

$$|E(C'_e)| \geq 4.$$

This can be combined with (20) and Lemma 4 to give

$$7 \geq |E(C_e)| \geq |X'| + |E(C'_e)| \geq |X'| + 4, \tag{21}$$

where

$$X' = \{x \in V(C'_e) \mid x \text{ is a nontrivial vertex of } G'\}.$$

Hence,

$$|X'| \leq 3. \tag{22}$$

with equality only if $|E(C'_e)| = 4$.

If Γ' contains at most one edge of $E(C'_e)$, then the subgraph $\Gamma'' = G'[E(\Gamma') \triangle E(C'_e)]$ is also an eulerian subgraph of G', and since

$$V(\Gamma') \subseteq V(\Gamma') \cup \{w\} \subseteq V(\Gamma''),$$

Γ'' contradicts (15). Therefore,

$$|E(\Gamma') \cap E(C'_e)| \geq 2. \tag{23}$$

Lemma 5 Any vertex of degree 1 in $G'[E(\Gamma') \cap E(C'_e)]$ is a member of X'.

Proof: Suppose that u has degree 1 in $G'[E(\Gamma') \cap E(C'_e)]$. Hence, u is incident with an edge of $E(C'_e) - E(\Gamma')$. Since Γ' is eulerian, u is also incident with at least two edges of $E(\Gamma')$, and so u has degree at least 3 in G'. Since $G \in \mathcal{F}$ and by the definition of X', $u \in X'$. □

Proof of Theorem 9, continued: By (23) and (17), at least 2 vertices have degree 1 in $G'[E(\Gamma') \cap E(C'_e)]$. By Lemma 5, they are in X', and so

$$|X'| \geq 2. \tag{24}$$

Case 1 Suppose that $|X'| = 3$. Since equality holds in (22),

$$|E(C'_e)| = 4,$$

and so we can denote the 4-cycle C'_e in G' by $vwxyv$. By (17) and (23),

$$|E(\Gamma') \cap E(C'_e)| = \{vy, xy\}, \tag{25}$$

and so v and x must be in X', by Lemma 5. By (25), (19) is false, and so (18) holds. Therefore, in Case 1,

$$X' = \{v, w, x\}$$

and y is a trivial vertex, and so $G'[E(\Gamma')\Delta E(C'_e)]$ is an eulerian subgraph of G' that satisfies (15) and violates (16). This contradiction precludes Case 1.

Case 2 Suppose $|X'| = 2$. By (21),

$$4 \leq |E(C'_e)| \leq 5.$$

Subcase 2A Suppose $|E(C'_e)| = 4$. Denote the 4-cycle C'_e by $vwxyv$. Then (17) and (23) imply $E(\Gamma') \cap E(C'_e) = \{vy, yx\}$ and hence $X' = \{v, x\}$, by Lemma 5. But then both (18) and (19) are false, an impossibility.

Subcase 2B Suppose $|E(C'_e)| = 5$. We denote the 5-cycle C'_e by $vwxyzv$. By the hypothesis of Subcase 2B, by (21) and by (24), $|X'| = 2$. This, Lemma 5, (17), and (23) imply that $\Gamma' \cap C'_e$ is a path in $C'_e - w$ of length at least 2 in G', where X' denotes the set of ends of this path. Therefore by Lemma 5 and $|X'| = 2$, $w \notin X'$, whence (18) fails, and so (19) holds and we thus have $x \notin X'$ also. Hence, by (23) we must have

$$\{vz, zy\} = E(\Gamma') \cap E(C'_e),$$

and $X' = \{v, y\}$, and hence w, x, and z are trivial vertices. Define

$$\Gamma'' = G'[E(\Gamma')\Delta E(C'_e)].$$

Then Γ'' is an eulerian subgraph of G' that violates the maximality of Γ' in (15). Therefore, Subcase 2B is impossible, too, and Case 2 is complete.

By (22) and (24), Cases 1 and 2 cover all possibilities.

Hence, by contradiction, we have shown that

(26) $\qquad V(\Gamma')$ contains all nontrivial vertices of G',

and

(27) $\qquad G' - V(\Gamma')$ is edgeless.

We finish the proof of Theorem 9 by lifting the eulerian subgraph $\Gamma' \subset G'$ to an eulerian subgraph $\Gamma \subset G$ that satisfies the conclusions of Theorem 9.

Since Γ' is eulerian,

(28) $\qquad G'[V(\Gamma')] \in \mathcal{SL}.$

Define $E' = E'(\mathcal{SL}^O) \subseteq E(G)$ by (6). By Theorem 5, the components $H_1, H_2, \ldots, H_c$ (say) of $G - E'$ are the maximal subgraphs of G in $\mathcal{SL}^O$. The graph G' is obtained from G by contracting each H_i to a distinct vertex $x_i \in V(G')$ ($1 \leq i \leq c$). Without loss of generality, suppose

$$V(\Gamma') = \{x_1, x_2, ..., x_t\},$$

where $t \leq c$. Define

$$V = \bigcup_{i=1}^{t} V(H_i).$$

Thus, $G'[V(\Gamma')]$ is the $\mathcal{SL}^O$-reduction of $G[V]$. By (8) with $S = \mathcal{SL}$ and with $G[V]$ in place of G, we have

$$(29) \qquad G[V] \in \mathcal{SL} \iff G'[V(\Gamma')] \in \mathcal{SL}.$$

By (28) and (29), $G[V]$ has a spanning eulerian subgraph, say Γ. By (26) and the definition of trivial vertices,

$$(30) \qquad V(G') - V(\Gamma') = V(G) - V(\Gamma),$$

and so by Lemma 3, all vertices of $V(G) - V(\Gamma)$ have degree at most 2 in G. Hence, (ii) of Theorem 9 holds. If $G - V(\Gamma)$ has an edge, then by (30) and the definition of G', so does $G' - V(\Gamma')$. But this contradicts (27), and so (i) of Theorem 9 holds. □

Theorem 9 holds (and is weaker) if the hypothesis $G \in \mathcal{F}(\mathcal{SL}^O)$ is replaced either by $G \in \mathcal{F}(\mathcal{CL})$ or by the assumption that G contains no induced $K_{1,3}$ as a subgraph. A similar remark applies to the corollaries below.

Let G be a graph. The line graph of G is the graph with vertex set $E(G)$, such that two vertices of the line graph are adjacent whenever the corresponding edges in $E(G)$ are adjacent in G.

Corollary 9A Let $G \in \mathcal{F}(\mathcal{SL}^O)$, define E' by (6), and suppose that every edge of E' is in a cycle of G having length at most 7. Then the line graph of G is hamiltonian, unless $|E(G)| < 3$.

Proof: Harary and Nash-Williams [6] proved that a graph G with at least three edges has a hamiltonian line graph if and only if G has an eulerian subgraph Γ such that each edge of G has at least one end in $V(\Gamma)$. Any graph G satisfying the conclusion of Theorem 9 has such an eulerian subgraph Γ. □

Corollary 9B Let G be a graph satisfying the hypothesis of Theorem 9. If $\delta(G) \geq 3$, then $G \in \mathcal{SL}$.

Proof: This follows immediately from Theorem 9. □

There are some examples to show that the conditions of the hypothesis of Corollary 9B cannot be omitted. Let G_5 be the graph obtained from G_2 of Figure 2 by joining to each 3-cycle C of G_2 an extra vertex and three new edges, to form a K_4 containing C. Then G_5 has two K_4's that are joined by three disjoint length 2 paths. Since $\delta(G_5) = 2$, G_5 violates the hypothesis of Corollary 9B. Let G_6 be the graph obtained from $K_{2,3}$ by adding three copies of K_4 and attaching each to a distinct divalent vertex of the $K_{2,3}$. Note that $G_6 \notin \mathcal{F}(\mathcal{SL}^O)$, contrary to the hypothesis of Corollary 9B. Since $G_5, G_6 \notin \mathcal{SL}$, neither G_5 nor G_6 satisfies the conclusion of Corollary 9B.

Let $D_1(G)$ be the set of vertices having degree 1 in G. The second author [9] proved:

Theorem 10 Let G be a simple graph of order $n \geq 46$. If
(i) $\kappa'(G - D_1(G)) \geq 2$;
(ii) $G \in \mathcal{F}(\mathcal{SL}^O)$; and
(iii) for any edge $xy \in E(G)$,

$$d(x) + d(y) > \frac{2n}{5} - 2,$$

then the line graph of G is hamiltonian. □

Benhocine, Clark, Köhler, and Veldman [1] conjectured that Theorem 10 holds even if the hypothesis (ii) is omitted.

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