ON NEAR-OVALS AND NEAR-SYSTEMS IN SQS

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Abstract. The point set "oval" has been considered in Steiner triple systems (STS) and Steiner quadruple systems (SQS) [3], [2]. There are many papers about "subsystems" in STS and SQS. Generalizing and modifying the terms "oval" and "subsystem" we define the special point sets "near-oval" and "near-system" in Steiner quadruple systems. Considering some properties of these special point sets we specify how to construct SQS with near-ovals (S^{no}) and with near-systems (S^{ns}), respectively. For the same order of the starting system we obtain non-isomorphic systems S^{no} and S^{ns}.

1. Introduction.

A Steiner system $S(t, k, v)$ is a pair (P, B), where P is a v-set (called the set of points) and B is a collection of k-subsets of P (called the set of blocks) such that every t-subset of P is contained in exactly one member of B. A system $S(2, 3, v)$ is called a Steiner triple system (briefly STS). A system $S(3, 4, v)$ is called a Steiner quadruple system (briefly SQS). It is well-known that the necessary and sufficient condition for the existence of an SQS of order v is $v = 0, 1$ or $v \equiv 2$ or $4 \pmod 6$. If $v \in \text{SQS} := \{v \in N \mid v = 2 \text{ or } 4 \pmod 6\}$ we call v admissible. In the following we will denote an SQS of order v (point set P, block set B) by $\text{SQS}_v(P, B)$.

We call a block b with respect to a point set $M \subset P$ a

in-block		4
3-secant		3
2-secant	$\langle = \rangle$ $\lvert b \cap M \rvert =$	2
tangent		1
passant		0

An *oval* $Q \subset P$ is a point set with the following properties:

(1) There exists no in-block with respect to Q (that is, Q is an arc).

(2) If $p,\ q \in Q$ are two arbitrary points then there exists exactly one 2-secant through p and q.

Condition 2 in the definition of Q is equivalent to the property $|Q| = \frac{v}{2}$. For further information on ovals in SQS see [2].

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2. Near-ovals in SQS.

Let $k \in \mathbb{N}_0$ and let $S(P,B)$ be an SQS of order v.

A *near-k-oval* $N_k \subset P$ is a point set with the following properties:

(1) There exist exactly k in-blocks with respect to N_k.
(2) $|N_k| = \frac{v}{2}$.

Points of N_k are called in-points if they are elements of an in-block with respect to N_k. We denote by I the set of all in-points.

2.1 Lemma. *Two points $p, q \in N_k$ not both in-points are contained in exactly one 2-secant and $\frac{1}{2}v - 2$ 3-secants.*

Proof: Together with p and q each point $r \in N_k$ is contained in exactly one 3-secant. Hence there exist exactly $\frac{1}{2}v - 2$ different 3-secants through p and q and there remains only one block. This must be a 2-secant. ∎

From the definition of a near-k-oval we see that near-0-ovals are ovals in SQS. In this paper we will only consider near-1-ovals and call them briefly near-ovals. In the following let $N \subset P$ be a near-oval in S and let I be the in-block of N.

Now we tabulate some results concerning the number of blocks of different classes through points of N. One can prove them by using counting arguments similar to those of Lemma 2.1.

2.2 Lemma. *Two points $p, q \in I$ are contained in exactly one in-block, $\frac{1}{2}v - 4$ 3-secants and two 2-secants.*

Hence we can conclude: If we delete a point $x \in P$ in order to obtain a derived triple system of the quadruple system S we get an STS with an oval, if $x \in N \setminus I$, and an STS with a near-oval, if $x \in I$ (a near-oval in an STS of order u is defined as a point set with exactly one in-block and exactly $\frac{u-1}{2}$ points).

2.3 Lemma. *A point $p \in N \setminus I$ and a point $q \in I$, respectively, is contained in the following blocks of different classes:*

$\underline{p}$	$\underline{q}$	*block with respect to N*
0	1	*in-block*
$\frac{1}{8}(v-2)(v-4)$	$\frac{1}{8}(v-2)(v-4) - 3$	*3-secant*
$\frac{1}{2}v - 1$	$\frac{1}{2}v + 2$	*2-secant*
$\frac{1}{24}(v-2)(v-4)$	$\frac{1}{24}(v-2)(v-4) - 1$	*tangent.*

Now we are able to classify the blocks of S with respect to the near-oval N:

2.4 Theorem. *With respect to the near-oval N there exist exactly the following numbers of blocks in S:*

1	*in-block*
$\frac{1}{48}(v^3 - 6v^2 + 8v - 192)$	*3-secants*
$\frac{1}{8}(v^2 - 2v + 48)$	*2-secants*
$\frac{1}{48}(v^3 - 6v^2 + 8v - 192)$	*tangents*
1	*passant*

Hence we can conclude that the point set $\widehat{N} := P \backslash N$ *is a near-oval, too.*

We wish to prove the existence of SQS_v with a near-oval for admissible v. We first consider the 6 non-trivial quadruple systems of orders $v \leq 14$.

2.5 Proposition. *There exist exactly 14 near-ovals in* SQS_8 *and exactly 180 near-ovals in* SQS_{10}.

Proof: Each block j in SQS_8 is a (trivial) near-oval because there exists exactly one "parallel" k with respect to j and there exist exactly $b_8 = 14$ blocks in SQS_8. Each block j and each point $p \notin j$ determine together exactly one near-oval N in SQS_{10} because there cannot exist a further in-block with respect to N. There are exactly 6 points in SQS_{10} different to the points of j and there are exactly $b_{10} = 30$ blocks in SQS_{10} and thus exactly $6 \cdot 30 = 180$ near-ovals in SQS_{10}. ∎

A computer test of the 4 non-isomorphic SQS of order 14 indicates the following:

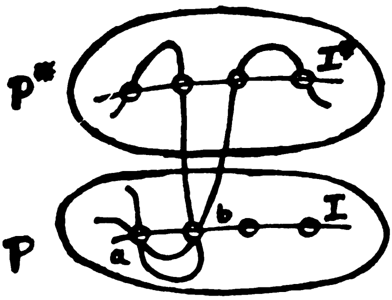
2.6 Proposition. *Two of the 4 non-isomorphic* SQS_{14} *do not contain any near-oval. The remaining two systems (each of which has automorphism-group of order 6) each contain exactly 24 near-ovals.*

For general $v \in$ SQS we need a construction method. We modify the oval construction method O in [2] (see also [1]) in the following way: Let $S(P, B)$ be the starting system and let $I \in B$ be an arbitrary block of S. We double the points of S by adding an isomorphic system $S^\star(P^\star, B^\star)$. The belonging isomorphism $\varphi: S \rightarrow S^\star$ may map the points of S to corresponding points with a "$\star$" of $S^\star$. With NO and S^{no} (P^{no}, B^{no}) we denote our construction method and the new constructed system, respectively. Then P^{no} and B^{no} are defined in the following way: $P^{no} := P \cup P^\star$. $B^{no} := B_1 \cup \ldots \cup B_5$, where

$B_1 := I \cup I^\star$ (in-block and passant with respect to P). $|B_1| = 2$.

$B_2 := \{\{x, y, x^\star, y^\star\} \mid x, y \in P, x \neq y\}$ (2-secants of the first kind). $|B_2| = \frac{1}{2}v(v-1)$.

B_3: = $\{\{x, y, z^\star, u^\star\} \mid \{x, y, z, u\} = I\}$ (2-secants of the second kind). $|B_3| = 6$.



2-secants of the first and second kind, resp. (through $a,\ b \in I$)

Figure 1

B_4: $= \{\{x, y, z, u^\star\} \mid \{x, y, z, u\} \in B \backslash I\}$ (3-secants). $|B_4| = 4 \cdot |B| - 4$.
B_5: $= \{\{x, y^\star, z^\star, u^\star\} \mid \{x, y, z, u\} \in B \backslash I\}$ (tangents). $|B_5| = 4 \cdot |B| - 4$.
By the construction NO we see that 3 points of P^{no} are contained in at least one block of B^{no} and $|B^{no}| = 2 + \frac{1}{2}v(v-1) + 6 + 2 \cdot (4 \cdot |B| - 4) = \frac{1}{24} \cdot 2v(2v-1)(2v-2) = b_{2v}$. Hence there is exactly one block through 3 arbitrary points of P^{no}, and S^{no} is a SQS of order $2v$. P and $P^\star$ are near-ovals in the new system S^{no} and we can conclude:

2.7 Theorem. *Let $v \geq 8$. There exist SQS_v with near-ovals for orders $v \equiv 4$ or $8 \pmod{12}$.*

3. Near-systems in SQS.

Let $S(P, B)$ be an SQS of order v and with respect to a point set $M \subset P$ let us denote the set of all in-blocks by J.

We call M a *near-system* of S if M has the following properties:

(1) There exists exactly one 4-element subset $i \subset M$ such that $(M, J \cup \{i\})$ is an SQS.

(2) $|M| = \frac{1}{2}v$.

One can imagine M as an SQS from which one block has been taken away. M is indeed a special partial SQS. Condition 2 in the definition of M implies that a necessary condition for the existence of SQS with a near-system is $v \equiv 4$ or $8 \pmod{12}$.

Now we tabulate some properties concerning how many blocks of different classes through one or two points of M exist. We use in these proofs similar counting arguments as in the proof of 3.1. In the following let $M \subset P$ be a near-system and i be the "missing block" as in the definition of M.

3.1 Lemma. *Two points $p, q \in M$ not both points of i are contained in exactly $\frac{1}{4}v - 1$ in-blocks and $\frac{1}{4}v$ 2-secants.*

Proof: $p \notin i$ implies: p and q are not contained in a 3-secant with respect to M. Hence, p, q and each other point of $M \setminus \{p, q\}$ determine exactly one in-block with respect to M. Thus we have exactly $\frac{1}{2}(\frac{1}{2}v - 2) = \frac{1}{4}v - 1$ in-blocks. The remaining blocks through p and q are 2-secants: $\frac{1}{2}v - 1 - (\frac{1}{4}v - 1) = \frac{1}{4}v$. ∎

3.2 Lemma. *Two points $p, q \in i$ are contained in exactly two 3-secants, $\frac{1}{4}v - 2$ in-blocks and $\frac{1}{4}v - 1$ 2-secants.*

3.3 Lemma. *A point $p \in M \setminus i$ and a point $q \in i$, respectively, is contained in the following blocks of different classes:*

$\underline{p}$	$\underline{q}$	*block with respect to M*
0	3	*3-secant*
$\frac{1}{24}(v-2)(v-4)$	$\frac{1}{24}(v-2)(v-4) - 1$	*in-block*
$\frac{1}{8}v(v-2)$	$\frac{1}{8}v(v-2) - 3$	*2-secant*
0	1	*tangent.*

Now we can classify the blocks of S with respect to M.

3.4 Proposition. *With respect to M there exist exactly the following numbers of blocks:*

$\frac{1}{192}v(v-2)(v-4) - 1$	*in-blocks*
4	*3-secants*
$\frac{1}{32}v^2(v-2) - 6$	*2-secants*
4	*tangents*
$\frac{1}{192}v(v-2)(v-4) - 1$	*passants*

3.5 Corollary. *If M is a near-system in S, then $\widehat{M} := P \setminus M$ is a near-system in S, too.*

Proof: There are exactly $b_{\frac{1}{2}v} - 1$ passants with respect to M and therefore the same number of in-blocks with respect to $\widehat{M}$ (let us denote the set of all in-blocks with respect to $\widehat{M}$ by $\widehat{J}$). There are exactly 4 points $x, y, z, t \in \widehat{M}$ such that no triple of them lies on an in-block with respect to $\widehat{M}$ because there are exactly 4 3-secants with respect to $\widehat{M}$. If we add $\{x, y, z, t\}$ to the set of all in-blocks (and let $H := \widehat{J} \cup \{x, y, z, t\})$, then 3 points of $\widehat{M}$ are contained in exactly one in-block and $(\widehat{M}, H)$ is an SQS. ∎

The term near-system is analogously defined in STS:

Let $S^\star(P^\star, B^\star)$ be a STS of order u, $M^\star \subset P^\star$ and $J^\star \subset B^\star$ the set of all in-blocks with respect to $M^\star$. Then $M^\star$ is called a near-system of $S^\star$ if

(1) There exists exactly one 3-element subset $i^\star \subset M^\star$ such that $(M^\star, J^\star \cup \{i^\star\})$ is an STS.

(2) $|M^\star| = \frac{u-1}{2}$.

Now we can delete a point from a quadruple system with a near-system and obtain the following:

3.6 Theorem. *Let $u = v - 1$. The derived STS of an SQS_v with a near-system is either an STS_u with subsystem of order $\frac{1}{2}(u-1)$ and hence an STS with a hyperoval (see [4]) or an STS with a near-system.*

Proof: Let $S(P, B)$ be an SQS_v, $M \subset P$ a near-system of S and i the "missing block" of M. We delete a point $p \in M$ from S and get the derived STS $S^\star(P^\star, B^\star)$.

α) Let $p \notin i$. There are no 3-secants with respect to M containing p. All blocks containing p and two other points of M are therefore in-blocks. Hence in $S^\star$ there is exactly one in-block through 2 point of $M^\star$ and that means that $M^\star$ is a subsystem of $S^\star$.

β) Let $p \in i$. $|M| = \frac{1}{2}(v-2) = \frac{1}{2}(u-1)$. Two points of M not both elements of i together with p are contained in exactly one in-block with respect to M in S. Hence two points of $M^\star$ not both elements of $i^\star := i \backslash p$ are contained in exactly one in-block with respect to $M^\star$ in $S^\star$. If we add $i^\star$ to the set $J^\star$ of all in-blocks of $M^\star$ in $S^\star$, then $(M^\star, J^\star \cup \{i^\star\})$ is an STS and thus $M^\star$ is a near-system in $S^\star$.

∎

Now we want to prove the existence of SQS_v with a near-system for all $v \in$ SQS fulfilling the necessary condition.

3.7 Proposition. *In SQS_8 there exist exactly 56 near-systems.*

Proof: Each 4-element subset of P which is not a block in SQS_8 is a (trivial) near-system M (regard that $(M, \{M\})$ is a trivial SQS). There are exactly $\binom{8}{4} = 70$ 4-element subsets in SQS_8 thereof $b_8 = 14$ blocks. ∎

We modify the subsystem construction method U from [2] (see also [1]). Let $S(P, B)$ be the starting system and $i \in B$ be an arbitrary block. We double the points by adding an isomorphic system $S^\star(P^\star, B^\star)$. The belonging isomorphism $\varphi: S \longrightarrow S^\star$ may map the points of S to corresponding points with a "$\star$" of $S^\star$. With NS and S^{ns} we denote our construction method and the new constructed system, respectively. P^{ns} and B^{ns} are defined in the following way: $P^{ns} := P \cup P^\star$. $B^{ns} := B_I \cup \ldots \cup B_V$ where

B_I: = $(B \backslash i) \cup (B^\star \backslash i^\star)$ (in-blocks and passants with respect to P). $|B_I| = 2 \cdot |B| - 2$.

B_{II}: = $\{\{x, y, x^\star, y^\star\} \mid x, y \in P, x \neq y\}$ (2-secants of the first kind). $|B_{II}| = \frac{1}{2}v(v-1)$.

B_{III}: = $\{\{x, y, z^\star, t^\star\} \mid \{x, y, z, t\} \in B \backslash i\}$ (2-secants of the second kind). $|B_{III}| = 6 \cdot |B| - 6$.

B_{IV}: = $\{\{x, y, z, t^\star\} \mid \{x, y, z, t\} = i\}$ (3-secants). $|B_{IV}| = 4$.

B_V: = $\{\{x^\star, y^\star, z^\star, t\} \mid \{x, y, z, t\} = i\}$ (tangents). $|B_V| = 4$.

3 points of P^{ns} are contained in at least one block of B^{ns} and $|B^{ns}| = \frac{1}{24} \cdot 2v \cdot (2v-1)(2v-2) = b_{2v}$. Hence 3 points are contained in exactly one block and S^{ns} is an SQS_{2v}. P and $P^\star$, respectively, are near-systems in S^{ns}.

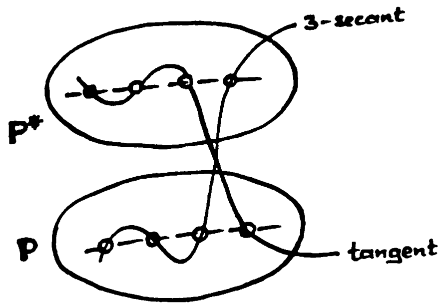


Figure 2

Starting with $v \in SQS$ we obtain an SQS_{2v}. Thus we can conclude:

3.7 Theorem. *There exist SQS_v with near-systems iff $v \equiv 4$ or $8 \pmod{12}$, $v \geq 8$.*

4. Comparing the systems S^{no} and S^{ns}.

Let $S(P, B)$ be an SQS of order v (later we will use S as a starting system).

4.1 Lemma. *Let $v \geq 10$. There does not exist any quadruple system S containing a point set M, where $|M| > \frac{1}{2}v$, and having exactly one in-block with respect to M (that is, a near-oval is a maximal point set with exactly one in-block).*

Proof: Let M be a point set with exactly one in-block I, $|M| = \frac{1}{2}v + 1$, and let $r \in I$. $Q := M \backslash r$ is an oval. This implies: There are exactly $\frac{1}{24}(v-2)(v-4)$ 3-secants through r (with respect to Q). $v \geq 10$ implies that there are at least two 3-secants through r (with respect to Q) and thus at least one further in-block which is a contradiction. We have a similar proof if $|M| > \frac{1}{2}v + 1$. ∎

4.2 Lemma. *Let $v \geq 14$. There does not exist any point set R with more than $\frac{1}{2}v$ points containing exactly two in-blocks.*

Proof: Let $|R| = \frac{1}{2}v + 1$ and p be a point of an in-block of R. $Q := R \backslash p$ is an oval or a near-oval in S. In both cases there are exactly $\frac{1}{24}(v-2)(v-4)$ 3-secants through p with respect to Q (that is, tangents through p in 2.3). $v \geq 14$ implies that there are at least five 3-secants through p and thus further in-blocks with respect to Q which is a contradiction. ∎

In the following let S^{ns} be an SQS with a near-system constructed with the help of NS from a starting system $S(P, B)$ of order v. Let J be the set of all in-blocks of M and i be the "missing block".

4.3 Lemma. *Let $v \geq 14$. If there exists a near-oval Q in S^{ns}, then $|Q \cap P| = |Q \cap P^\star| = \frac{1}{2}v$.*

Proof: Let $|Q \cap P| \geq \frac{1}{2}v$.

α) $|i \cap P| < 4$. After adding i to the near-system P, $Q \cap P$ is an oval or a near-oval in the system $(P, J \cup \{i\})$. Hence $|Q \cap P| = \frac{1}{2}v$ (4.1).

β) $|i \cap P| = 4$. After adding i $Q \cap P$ is a near-oval or a point set with exactly two in-blocks. Hence $|Q \cap P| = \frac{1}{2}v$ (4.2). ∎

4.4 Lemma. *Let $v \geq 14$. If Q is a near-oval in S^{ns} then $|Q_1 \cap Q_2| \leq 1$ where $Q_1 := Q \cap P$, $Q_2^\star := Q \cap P^\star$ and $Q_2 := \varphi^{-1}(Q_2^\star)$ (for φ see the definition of NS).*

Proof: $|Q_1 \cap Q_2| > 2$ leads to a contradiction. Let $Q_1 \cap Q_2 =: \{p, q\}$, thus $|Q_1 \cap Q_2| = 2$. There are exactly two points $x, y \in P$ such that $x, y \notin Q_1 \cup Q_2$. $I := \{p, q, p^\star, q^\star\}$ is the in-block of Q (2-secant of the first kind with respect to P). After adding i to the near-system P, Q_1 is an oval or a near-system in $\widehat{S}(P, J \cup \{i\})$. Together with p and q each point $r_i \in Q_1 \backslash \{p, q\}$ is contained in a 3-secant or in an in-block with respect to Q_1 in $\widehat{S}$. $v \geq 14$ implies that there exists at least one 3-secant $\{p, q, r, z\}$ with $r \in Q_1 \backslash \{p, q\}$, $z \in Q_2 \backslash \{p, q\}$. Because $\{p, q, r, z\} \neq i$ there exists the 2-secant $s = \{p, r, q^\star, z^\star\}$ with respect to P in S^{ns} (a 2-secant of the second kind) which is a further in-block with respect to Q in S^{ns} and this is a contradiction. ∎

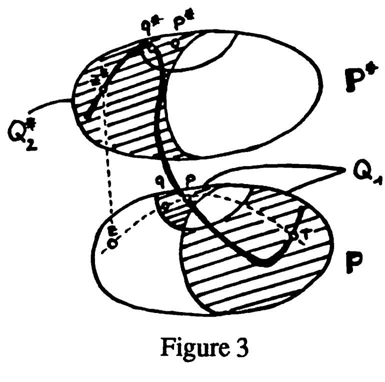


Figure 3

4.5 Proposition. *Let $v \geq 14$. There does not exist any near-oval in S^{ns}.*

Proof: Let $Q \subset P^{ns}$ be a near-oval in S^{ns} and Q_1 and $Q_2^\star$ are defined as in 4.4. We assume $Q_1 \cap Q_2 = p \in P$ (the proof is similar if $Q_1 \cap Q_2 = \{\ \}$). There exists exactly one point $q \in P \backslash (Q_1 \cup Q_2)$. $v \geq 14$ implies the existence of 4 further points $r, s, t, u \in Q_2 \backslash p$. After adding i to the near-system P, Q_1 is either an oval or a near-oval or a point set with exactly two in-blocks in $\widehat{S}(P, J \cup \{i\})$.

Case 1: Q_1 is an oval in $\widehat{S}$. There exists exactly one 2-secant through two points of Q_1 and we have 6 pairs of points of $\{r, s, t, u\}$. Hence there exist at least two 2-secants s_1 and s_2 with respect to Q_1 both different to i. Looking at the construction NS we have at least two in-blocks with respect to Q (2-secants of the second kind) which is a contradiction.

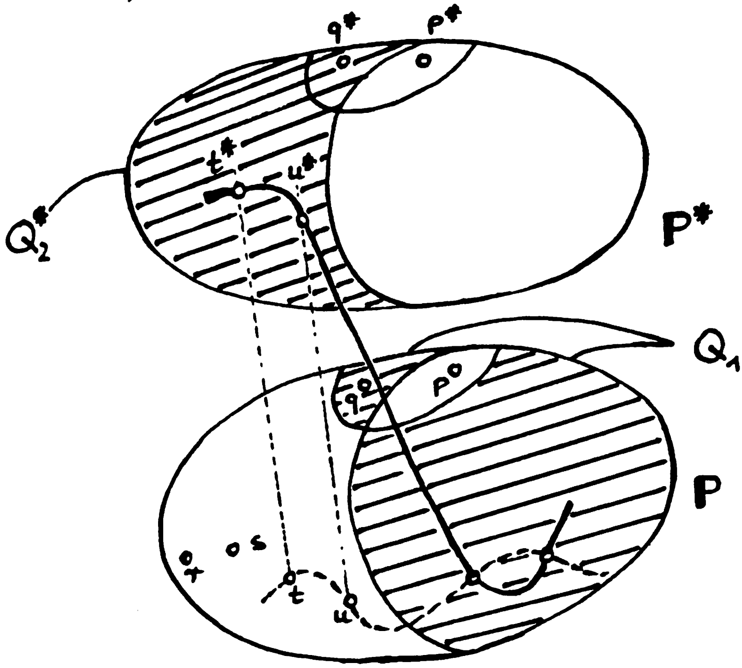


Figure 4

Case 2: Q_1 is a near-oval in $\widehat{S}$. Hence $P \backslash Q_1$ is a near-oval, too. The proof is similar to Case 1 because two points of a near-oval are contained in at least one 2-secant (2.1/2.2).

Case 3: Q_1 is a point set with exactly two in-blocks. After adding $i^\star$ to the near-system $P^\star$ the point set $Q_2^\star$ then must be an oval or a near-oval in $\widehat{S}^\star(P^\star, J^\star \cup \{i^\star\})$ and the proof is similar to Case 1 or Case 2. ∎

Hence we can conclude:

4.6 Theorem. *Let* $v \geq 14$. *Starting with quadruple systems* S_1 *and* S_2 *of the same order* v *we obtain the non-isomorphic systems* S_1^{no} *and* S_2^{ns}.

5. Concluding remarks.

Many problems and questions arise if we are concerned with these special point sets. Here we list some of them:

- Are there SQS_v with near-ovals for orders $v \equiv 2$ or $10 \pmod{12}$, $v \geq 16$? Moreover the spectrum of SQS_v with near-ovals should be completely determined (the same problem is even unsolved for ovals in SQS).
- Near-k-ovals with $k > 1$ can be constructed in a similar way as in Chapter 2 for orders $v \equiv 4$ or $8 \pmod{12}$ (this will be the subject of a further paper of the author). Are there construction methods for the remaining admissible orders? There are many possible arrangements of the blocks in a near-k-oval with fixed k and such near-k-ovals will have different properties.
- How many near-ovals (near-k-ovals) exist in a given quadruple system. Is there an upper bound on k for SQS_v of orders $v \equiv 2$ or $10 \pmod{12}$? Are there any constructions for these orders?
- Similar problems arise if we regard SQS with near-k-systems — a generalization of near-systems (now with k "missing blocks") and there are, too, many possible arrangements of the "missing blocks" in such a near-k-oval.

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