Upper and lower bounds for the number of monotone crossing free Hamiltonian cycles from a set of points

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Introduction

Let n points with straight line segments connecting every pair of points represent a drawing of the complete graph on n vertices, K_n. A crossing free Hamiltonian cycle (CFHC), in a drawing of K_n, is a tour of the n points beginning and ending at the same point, such that each point is visited exactly once, and no two non-adjacent edges intersect. Let $F(n)$ denote the maximum number of CFHCs obtainable from any drawing of K_n. Newborn and Moser [NM] showed that $(3/20) \cdot 10^{n/3} \leq F(n) \leq 2 \cdot 6^{n-2} \cdot \lfloor n/2 \rfloor!$, where $10^{1/3} \approx 2.15443$. A major improvement to the upper bound, namely $F(n) \leq 10^{13n}$, was later established by Ajtai, Chvátal, Newborn and Szemeredi [ACNS]. More recently the lower bound has been improved, first by Akl [A] to $F(n) > k \cdot 2.270719168^n$, for some constant k and later by Hayward [H] to $F(n) > c \cdot 3.268461786^n$ for some constant c.

In this note we examine a constrained version of the maximum CFHC problem and obtain upper and lower bounds to two versions of this constrained problem. A simple polygon P is said to be *monotone* with respect to a direction L, if every line normal to L intersects the boundary of P at most twice. When we use the term monotone polygon we will always assume that the polygon is monotone with respect to a horizontal line. We define a monotone crossing free Hamiltonian circuit (MCFHC) as a CFHC that is a monotone polygon.

Given a set of points $P = (p_1, p_2, \cdots, p_n)$, let x_i be the x-coordinate of point p_i. We will give the upper and lower bounds under two different assumptions. Let $\Psi_1(n)$ denote the maximum number of MCFHC's under the assumption that $x_1 < x_2 < \cdots < x_n$, that is, no two points have the same x-coordinate. We let $\Psi_2(n)$ denote the maximum number of MCFHC's with $x_1 \leq x_2 \leq \cdots \leq x_n$. The difference between these two assumptions is significant since we show the upper bound we obtain for $\Psi_1(n)$ is less than the lower bound for $\Psi_2(n)$.

Upper bounds

A monotone polygon can be partitioned into two polygonal chains beginning and ending at the same point such that a vertical line intersects non-vertical edges of each chain at most once. We will distinguish these two chains as the *upper* and *lower* chains for obvious reasons.

The upper bound for $\Psi_1(n) \leq 2^{n-2}$ is straightforward, since each point $p_2, p_3, \cdots, p_{n-1}$ can either appear on the upper or lower chain of an MCFHC, and the extremal points p_1 and p_n have their positions fixed in every MCFHC.

The upper bound for $\Psi_2(n)$ requires slightly more development. Our analysis will examine the choice of placement of a point in an MCFHC with respect to the number of other points sharing the same x-coordinate.

Case 0: If a point has a unique x-coordinate then there are at most two choices for it. It may appear on the upper or on the lower chain of the MCFHC.

Case 1: If a point shares its x-coordinate with exactly one other point then there are the following possibilities:

(1) The top point precedes the bottom point on the lower chain.
(2) The top point succeeds the bottom point on the lower chain.
(3) The top point precedes the bottom point on the upper chain.
(4) The top point succeeds the bottom point on the upper chain.
(5) The top point is on the upper and the bottom point is on the lower chain.

This results in at most $\sqrt{5}$ choices per point.

Case 2: If there are three points with the same x-coordinate then the following possibilities exist. (Label the points q_1, q_2, q_3, with q_1 above q_2 above q_3.)

(1) All three points on the upper chain in the order q_1, q_2, q_3.
(2) All three points on the upper chain in the order q_3, q_2, q_1. (Note the point q_2 can never appear first or last in the ordering. This would result in two edges being coincident, thus violating a property of MCFHC's.)
(3)&(4) Replace the word "upper" with the word "lower" in cases (1)&(2).
(5) Point q_1 is on the upper chain and q_2 precedes q_3 on the lower chain.
(6) Point q_1 precedes q_2 on the upper chain and point q_3 is on the lower chain.
(7)&(8) Replace the word "precedes" with "succeeds" in cases (5)&(6).

This results in at most $8^{1/3} = 2$ choices per point.

Case 3: If there are $k > 3$ points with the same x-coordinate then there are at most $(4(k-1))^{1/k}$ choices per point. We arrive at this figure by generalizing the argument given in case 2 above. Label the k points $q_1, q_2, \cdots, q_k$. The valid choices include the points q_1 to q_i for $i = 1, \cdots, k$ on the upper chain and the complementary subset of the k points on the lower chain with each set of points in either increasing or decreasing sequences.

It should be noted, that for $k > 3$ the quantity $(4(k-1))^{1/k}$ is less than 2. Therefore $\Psi_2(n)$ is maximized with $\lfloor (n-2)/2 \rfloor$ pairs of points on $\lfloor (n-2)/2 \rfloor$ vertical lines. Since we have choices for at most $n-2$ of the n points we derive that $\Psi_2(n) \leq (\sqrt{5})^{n-2}$.

Lower Bounds

We provide constructions of drawings of K_n, giving lower bounds for $\Psi_1(n)$ and $\Psi_2(n)$. Let $n = 2m + 3$ and consider the n points, P, placed as shown in figure 1. The polygonal chains $(v, a_0, \cdots, a_m)$ and $(v, b_0, \cdots, b_m)$ are convex and concave respectively. We assume the lines through the points a_i, b_i have very large slopes.

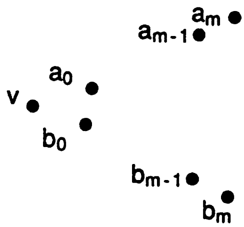

Figure 1

Let us examine a pair of points a_i, b_i. There are three ways in which a_i, b_i can appear in an MCFHC:

(0) a_i is on the upper chain, and b_i is on the lower chain,
(1) a_i, b_i are both on the lower chain,
(2) a_i, b_i are both on the upper chain.

In order to count the total number of MCFHC's using P we use the following rules for constructing a valid MCFHC. a_i, b_i can satisfy:

case (0) if a_{i-1}, b_{i-1} satisfy the cases (0), (1) or (2);
case (1) if a_{i-1}, b_{i-1} satisfy the cases (0), (1) or (2);
case (2) if a_{i-1}, b_{i-1} satisfy the cases (0) or (2);

We can now uniquely specify every MCFHC using the points P by giving a sequence of m 0's, 1's, and 2's obeying the above rules.

Let $T(j, t)$ be the number of choices for connecting the vertex v and the pairs $(a_0, b_0), (a_1, b_1), \cdots, (a_t, b_t)$, assuming that (a_t, b_t) has been picked according to case j. We conclude that:

$$\begin{aligned} T(0,t) &= T(0,t-1) + T(1,t-1) + T(2,t-1) \\ T(1,t) &= T(0,t-1) + T(1,t-1) + T(2,t-1) \\ T(2,t) &= T(0,t-1) \qquad\qquad\qquad + T(2,t-1). \end{aligned}$$

Since $T(0,t) = T(1,t)$ we can express the recurrence equations above as:

$$\begin{aligned} T(0,t) &= 2T(0,t-1) + T(2,t-1) \\ T(2,t) &= T(0,t-1) + T(2,t-1). \end{aligned}$$

Thus,

$$\begin{bmatrix} T(0,t) \\ T(2,t) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^t \begin{bmatrix} T(0,0) \\ T(2,0) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$> \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^t \begin{bmatrix} 1 \\ 2/(1+\sqrt{5}) \end{bmatrix} = \left(\frac{3+\sqrt{5}}{2}\right)^t \begin{bmatrix} 1 \\ 2/(1+\sqrt{5}) \end{bmatrix}$$

since $\frac{3+\sqrt{5}}{2}$ is an eigenvalue of $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2/(1+\sqrt{5}) \end{bmatrix}$ is a corresponding eigenvector. The total number of MCFHC's that can be constructed using the points P is:

$$T(0,m-1) + T(1,m-1) + T(2,m-1) = T(0,m).$$

Therefore,

$$\Psi_1(n) \geq T(0,m) \geq \left(\frac{3+\sqrt{5}}{2}\right)^m = \left(\frac{1+\sqrt{5}}{2}\right)^{n-3} > 1.618^{(n-3)}.$$

We now give a construction to provide a lower bound for $\Psi_2(n)$. Let P represent $n = 2m + 3$ points labelled as in figure 2. Place point v at the origin of the coordinate system and points a_i, b_i at coordinates $(i, 2^{2i})$ and $(i, -2^{2i})$ respectively. It can be shown that the line segments (a_i, b_{i+2}) and (b_i, a_{i+2}) do not intersect the line segment (a_{i+1}, b_{i+1}), for all $i = 0, \cdots, m-2$.

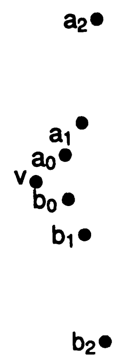

Figure 2

Let us establish the ways in which the pairs of points a_i, b_i can appear in an MCFHC. Adopting an anti-clockwise direction we have the following cases:

(0) a_i is on the upper chain and b_i is on the lower chain,

(1) a_i precedes b_i on the upper chain,
(2) a_i succeeds b_i on the upper chain,
(3) a_i precedes b_i on the lower chain,
(4) a_i succeeds b_i on the lower chain.

We can specify valid choices for a_i, b_i given the case for a_{i-1}, b_{i-1}. a_i and b_i may appear in the MCFHC according to:

case (0) if a_{i-1}, b_{i-1} satisfy the cases (0),(1),(2),(3),(4);
case (1) if a_{i-1}, b_{i-1} satisfy the cases (0),(1),(2);
case (2) if a_{i-1}, b_{i-1} satisfy the cases (0),(1),(2),(3),(4);
case (3) if a_{i-1}, b_{i-1} satisfy the cases (0),(3),(4);
case (4) if a_{i-1}, b_{i-1} satisfy the cases (0),(1),(2),(3),(4).

From this we see that $T(0,t) = T(2,t) = T(4,t)$ and that $T(1,t) = T(3,t)$. Therefore, we can write out the recurrence equations that represent these possibilities as follows:

$$\begin{bmatrix} T(0,t) \\ T(1,t) \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} T(0,t-1) \\ T(1,t-1) \end{bmatrix}.$$

Since $\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$ has eigenvalue $(2+\sqrt{5})$ with corresponding eigenvector $\begin{bmatrix} 1 \\ (\sqrt{5}-1)/2 \end{bmatrix}$ we derive that the number of MCFHC's that can be constructed is:

$$T(0,m) \geq (2+\sqrt{5})^m = (2+\sqrt{5})^{(n-3)/2}.$$

Therefore, $\Psi_2(n) \geq (2+\sqrt{5})^{(n-3)/2} > (2.058)^{n-3}$.

Conclusion

Let $\Psi_1(n)$ denote the maximum number of MCFHC's under the assumption that $x_1 < x_2 < \cdots < x_n$, and let $\Psi_2(n)$ denote the maximum number of MCFHC's with $x_1 \leq x_2 \leq \cdots \leq x_n$. We have shown that:

$$\left(\frac{1+\sqrt{5}}{2}\right)^{n-3} \leq \Psi_1(n) \leq 2^{(n-2)},$$

and

$$(2+\sqrt{5})^{(n-3)/2} \leq \Psi_2(n) \leq (\sqrt{5})^{(n-2)}.$$

Exact values for $\Psi_1(n)$ and $\Psi_2(n)$ remain unknown.

References

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