The Flower Intersection Problem for Steiner Systems $S(3,4,v)$, $v = 4\cdot 2^n$, $5\cdot 2^n$ [1]

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Abstract. We determine those pairs (k,v), $v = 4\cdot 2^n$, $5\cdot 2^n$, for which there exists a pair of Steiner quadruple systems on the same v-set, such that the quadruples in one system containing a particular point are the same as those in the other system and moreover the two systems have exactly k other quadruples in common.

1. Introduction.

A Steiner system $S(t-1,t,v)$ of order v is a pair (S,a) where S is a v-set and a is a collection of t-subsets of S, usually called blocks, such that every $(t-1)$-subset of A occurs in exactly one block of a.

A Steiner triple system is an $S(2,3,v)$ and a Steiner quadruple system is an $S(3,4,v)$.

A partial Steiner system of order n is a pair (P,b) where P is a n-set and b is a collection of t-subsets of P such that every $(t-1)$-subset of P occurs in at most one block of b.

For $t = 3$ or 4 (P,b) is called a partial Steiner triple system (PTS) or a partial Steiner quadruple system (PQS) respectively.

Two partial Steiner systems (P,a) and (P,b) are said to be disjoint and mutually balanced (DMB) if $|a \cap b| = 0$ and any $(t-1)$-subset of P is contained in a block of a if and only if it is contained in a block of b.

H. Hanani [6] proved that an $S(3,4,v)$ (S,a) exists if and only if $v \equiv 2$ or $4 \pmod 6$. It is easy to see that $|a| = q_v = v(v-1)(v-2)/24$.

Let $J(v)$ ([4], [9]) be the set of all integers k such that there exists a pair of Steiner quadruple systems (S,a) and (S,b) of order v having exactly k blocks in common (that is, $|a \cap b| = k$).

Let $I(v) = \{0,1,2,\dots,q_v-14,q_v-12,q_v-8,q_v\}$ for every admissible $v \geq 8$.

In [3], [4], [8] and [12] the following results are proved:

(i) $J(v) \subseteq I(v)$ for all $v \equiv 2$ or $4 \pmod 6$ $v \geq 8$ [4].
(ii) $J(4) = \{1\}$. $J(8) = \{0,2,6,14\}$. $J(10) = \{0,2,4,6,8,12,14,30\}$ [8].
(iii) $J(v) = I(v)$ for all $v = 2^{n+2}$, $5\cdot 2^n$ $n \geq 2$ ([4], [3] and [12]).

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The flower at a point x of a Steiner quadruple system is the set of all quadruples containing x. The flower intersection problem for $S(3,4,v)$ is the determination for each $v \equiv 2$ or $4 \pmod 6$ of the set $J^f(v)$ of all k such that there exists a pair of Steiner quadruple systems (S,a) and (S,b) of order v having $k+(v-1)(v-2)/6$ quadruples in common, $(v-1)(v-2)/6$ of them being the quadruples of a common flower.

The similar problem for Steiner triple systems has been completely settled by Hoffman and Lindner [7].

For any $v \geq 8$ let $I^f(v) = \{0,1,\ldots,f_v-14,\ f_v-12, f_v-8, f_v\}$, $f_v = (v-1)(v-2)(v-4)/24$. From (i) it follows easily that $J^f(v) \subseteq I^f(v)$ for $v \geq 8$. It can be checked that $J^f(4) = \{0\}$ and $J^f(8) = \{7\}$. We prove here that $J^f(10) = \{0,18\}$, $I^f(16) - \{16\} \subseteq J^f(16)$ and $J^f(v) = I^f(v)$ for every $v = 4\cdot 2^{n+1}, 5\cdot 2^n \quad n \geq 2$.

Let $F = \{F_1, F_2, \ldots, F_{2m-1}\}$ and $G = \{G_1, G_2, \ldots, G_{2m-1}\}$ be any two 1-factorizations of the complete graph on $2m$ vertices. We will say that F and G have k edges in common if and only if $k = \sum_{i=1}^{2m-1} |F_i \cap G_i|$.

In [13] Webb has shown that for every $2m \geq 8$ there exist two 1-factorizations with $h + 2m - 1$ edges in common, $2m-1$ of them being the edges containing the same point x, and $h \in W^f(2m) = \{0,1,\ldots, N_f = (2m-1)(2m-2)/2) - \{N_f-1, N_f-2, N_f-3, N_f-5\}$.

Starting from Webb's result, Lindner and Wallis proved in [10] that for any $2m \geq 8$ there exist two 1-factorizations with k edges in common for every $k \in W(2m) = \{0,1,\ldots,N = 2m(2m-1)/2\} - \{N-1, N-2, N-3, N-5\}$.

Now we describe two well-known constructions for quadruple systems of order $2v$ which are the main tools used in what follows.

Construction A. (For example, see [9]): Let (X,a) and (Y,b) be two $S(3,4,v)$ with $|X \cap Y| = 0$. Let $F = \{F_1, F_2, \ldots, F_{v-1}\}$ and $G = \{G_1, G_2, \ldots, G_{v-1}\}$ be any two 1-factorizations on X and Y respectively. Define a collection s of blocks of $S = X \cup Y$, as follows:

(a1) any block belonging to a or b belongs to s;

(a2) if $x_1, x_2 \in X (x_1 \neq x_2)$ and $y_1, y_2 \in Y (y_1 \neq y_2)$ then $\{x_1, x_2, y_1, y_2\} \in s$ if and only if $\{x_1, x_2\} \in F_i$ and $\{y_1, y_2\} \in G_i$.

It is a routine matter to check that (S,s) is an $S(3,4,2v)$. We will denote (S,s) by $[X \cup Y]\ [a,b,F,G]$.

Construction B. (See [2]): Let (Q,q) be an $S(3,4,v)$, Q' be a finite set such that $|Q| = |Q'|$, $|Q \cap Q'| = 0$ and let φ be a bijection from Q onto Q' with $x' = \varphi(x)$, for every $x \in Q$. Obviously (Q', q') is an $S(3,4,v)$ where $q' = \varphi(q) = \{\{\varphi(x), \varphi(y), \varphi(z), \varphi(u)\} / \{x,y,z,u\} \in q\}$.

If $q_1 \subseteq q$, we define a collection $p(q_1)$ of blocks of $P = Q \cup Q'$ as follows:

(b1) any block belonging to q_1 or q_1' $(= \varphi(q_1))$ belongs to $p(q_1)$;

(b2) $\{\{x_1,x_2,x_3',x_4'\},\{x_1,x_2',x_3,x_4'\},\{x_1,x_2',x_3',x_4\},\{x_1',x_2,x_3,x_4'\},$
$\{x_1',x_2,x_3',x_4\},\{x_1',x_2',x_3,x_4\}\} \subset p(q_1)$ if and only if
$\{x_1,x_2,x_3,x_4\} \in q_1$;

(b3) $\{\{x_1,x_2,x_3,x_4'\},\{x_1,x_2,x_3',x_4\},\{x_1,x_2',x_3,x_4\},\{x_1',x_2,x_3,x_4\},$
$\{x_1',x_2',x_3',x_4\},\{x_1',x_2',x_3,x_4'\},\{x_1',x_2,x_3',x_4'\},\{x_1,x_2',x_3',x_4'\}\}$
$\subset p(q_1)$ if and only if $\{x_1,x_2,x_3,x_4\} \in q - q_1$;

(b4) $\{x_1,x_2,x_1',x_2'\} \in p(q_1)$ for every $x_1,x_2 \in Q, x_1 \neq x_2$.

It is a routine matter to check that $(P,p(q_1))$ is an $S(3,4,2v)$. We will denote $(P,p(q_1))$ by $((Q \cup Q'),(q,q_1))$.

2. Theorems.

Let (X,t) and (X,t') be two disjoint and mutually balanced partial quadruple systems (DMB PQS). Let $d(x) = |\{b \in t / x \in b\}|$.

Theorem 2.1. *$k \in J^f(v)$ for every integer k such that $k \geq (v-1)\,[(v-4)(v-2)-24]/24$ and $(k+(v-1)(v-2)/6) \in J(v)$.*

Proof: Let (S,a) and (S,b) be two $S(3,4,v)$ having $k+(v-1)(v-2)/6$ blocks in common. Let $t = a-(a\cap b)$, $t' = b-(a\cap b)$ and $X = \{x \in S / x \in c$ for some $c \in t\}$. Clearly (X,t) and (X,t') are two DMB PQSs such that $|t| = |t'| = m$ and $m \leq v-1$. If there is not a flower contained in $a\cap b$, it is $|X| = v$. However it is [4] $\sum_{x\in X} d(x) = 4m$ and $d(x) \geq 4$, hence $v = |X| \leq m$. ∎

Theorem 2.2. *$k + \sigma + hv/2 \in J^f(2v)$ for any $\sigma \in J(v)$, $k \in J^f(v)$ and $h \in W^f(v)$.*

Proof: Let $(X,a),(X,b)$ be two $S(3,4,v)$ having $k+(v-1)(v-2)/6$ quadruples in common, $(v-1)(v-2)/6$ of them being the blocks of the common flower at a point $x \in X$. Let $F^{(i)} = \{F_1^{(i)},F_2^{(i)},\dots,F_{v-1}^{(i)}\}$ $(i = 1,2)$ be two 1-factorizations on X having $h+v-1$ edges in common, $v-1$ of them being the edges containing the same point x. Let (Y,a') and (Y,b') be two $S(3,4,v)$ such that $|X\cap Y|=0$ and let $G=\{G_1,G_2,\dots,G_{v-1}\}$ be a 1-factorization on Y.

It is easy to check that $[X\cup Y]\,[a,a',F^{(1)},G]$ and $[X\cup Y]\,[b,b',F^{(2)},G]$ are two Steiner quadruple systems having $|a'\cap b'|+hv/2+k+(2v-1)(2v-2)/6$ quadruples in common, $(2v-1)(2v-2)/6$ of them being the quadruples of the common flower at the point x. ∎

From Theorem 2.2 it follows, by a simple calculation,

Theorem 2.3. *If $J^f(v) = I^f(v)$ and $J(v) = I(v)$ for every admissible $v \geq 16$ then $J^f(2v) = I^f(2v)$.*

Let (Q,q) and (Q,c) be two $S(3,4,v)$ $(Q = \{1,2,\dots,v\})$ such that $c_1 \subset q\cap c$, c_1 being the flower at the point 1. Let Q' be a v-set such that $|Q\cap Q'| = 0$

and let φ be a bijection from Q onto Q' with $x' = \varphi(x)$ for every $x \in Q$. Obviously (Q', c') is an $S(3,4,v)$ where $c' = \varphi(c) = \{\{\varphi(x), \varphi(y), \varphi(z), \varphi(u)\} / \{x, y, z, u\} \in c\}$. Let (Q', t) be an $S(3,4,v)$ such that $|t \cap c'_1| = h(c'_1 = \varphi(c_1))$. Let $F = \{F_1, F_2, \dots, F_{v-1}\}$ be the 1-factorization on Q' such that, for every $i = 1, 2, \dots, v-1$:

1) $\{1', (i+1)'\} \in F_i$;
2) $\{x', y'\} \in F_i$ if and only if $\{1, i+1, x, y\} \in c_1$.

Let $G = \{G_1, G_2, \dots, G_{v-1}\}$ be a 1-factorization on Q such that $\{1, i+1\} \in G_i$ for every $i = 1, 2, \dots, v-1$ and $\sigma = \sum_{i=1}^{v-1} |\varphi(G_i - \{\{1, i+1\}\}) \cap (F_i - \{\{1', (i+1)'\}\})|$.

From Construction A and B, we obtain respectively that $(P, s) = [Q \cup Q'][q, t, G, F]$ and $(P, p) = ((Q \cup Q'), (c, c_1))$ are two $S(3,4,2v)$.

Theorem 2.4. *Let (P, s) and (P, p) be the above $S(3,4,2v)$. (P, s) and (P, p) intersect in $p_1 \cup g$, p_1 being the flower at the point 1 and $|g| = h + 2\sigma$.*

Proof: It is easy to see that the common blocks of p and s are the common blocks either in (b1), (b2) or (b4) of Construction B.

If $b \in p \cap s$ is in (b1) it follows either $b \in c_1$ or $b \in c'_1 \cap t$.

Let $U_i = \{\varphi(G_i - \{\{1, i+1\}\}) \cap (F_i - \{\{1', (i+1)'\}\})\}$ and let $\{x, y, w', z'\} \in p \cap s$ be a block in (b2). It follows that $\{x, y, w, z\} \in c_1$ hence either $\{1, i+1\} = \{x, y\}$ or $\{1, i+1\} = \{w, z\}$ for some $i \in \{1, 2, \dots, v-1\}$. If $\{1, i+1\} = \{x, y\}$ then $\{w', z'\} \in F_i$ and $\{x, y, w', z'\} = \{1, i+1, w', z'\} \in p_1$. If $\{1, i+1\} = \{w, z\}$ it follows $\{x, y, w', z'\} = \{1', (i+1)', x, y\}$, $\{1, i+1, x, y\} \in c_1$, hence $\{x', y'\} \in F_i$. Moreover $\{1', (i+1)'\} \in F_i$ therefore $\{x, y\} \in G_i$, hence $\{x', y'\} \in U_i$.

At last let $\{x, y, x', y'\} \in p \cap s$ be a block in (b4). If $1 \in \{x, y\}$, we obtain $\{x, y, x', y'\} \in p_1$. If $1 \notin \{x, y\}$ then $\{x', y'\} \in U_i$ for some i. This completes the proof. ∎

Theorem 2.5. *$k + h + v(v-1)(v/2-1)/2 - (v-4)^2(\sigma + 2\tau)/4 \in J^f(2v)$ for every $k \in J^f(v)$, $h \in J(v)$, $\sigma \in \{0, 2, 3\}$ and $\tau \in \{0, 1, \dots, (v-4)/2\}$.*

Proof: Let (X, a) and (X, b) be two $S(3,4,v)$ ($X = \{1, 2, \dots, v\}$) intersecting in $a_1 \cup g$, a_1 being the flower at point 1 and $|g| = k$. Let (Y, c) and (Y, d) be two $S(3,4,v)$ such that $|Y \cap X| = 0$ and $|c \cap d| = h$. If $(X - \{1\}, t)$ is an $S(2,3,v-1)$ we define the following 1-factorization $F = \{F_1, F_2, \dots, F_{v-1}\}$ on X:

1) $\{1, i+1\} \in F_i$, $i = 1, 2, \dots, v-1$;
2) $\{x, y\} \in F_i$ if and only if $\{i+1, x, y\} \in t$.

Let φ be a bijection from X onto Y and let $G = \{G_1, G_2, \dots, G_{v-1}\}$ be the 1-factorization on Y such that $\{x, y\} \in G_i$ if and only if $\{\varphi^{-1}(x), \varphi^{-1}(y)\} \in F_i$. Let $(S, s) = [X \cup Y][a, c, F, G]$ and $(S, s') = [X \cup Y][b, d, F, G]$. Clearly if

$\{2,x,y\} \in t$ then $H_1 = F_1 - \{\{1,2\},\{x,y\}\}$, $H_{x-1} = F_{x-1} - \{\{1,x\},\{2,y\}\}$, and $H_{y-1} = F_{y-1} - \{\{1,y\},\{2,x\}\}$ are three 1-factors on $X - \{1,2,x,y\}$. If β is a permutation on $\{1, x-1, y-1\}$, let Φ_β be the set of blocks $\{x_1,x_2,y_1,y_2\}$ such that $\{x_1,x_2\} \in H_j$ and $\{\varphi^{-1}(y_1),\varphi^{-1}(y_2)\} \in H_{\beta(j)}$. Obviously (S,s) and $(S,(s' - \Phi_{\text{identity}}) \cup \Phi_\beta)$ are two $S(3,4,2v)$ intersecting in $s_1 \cup \nu$ blocks, s_1 being the flower at the point 1 and $|\nu| = k + h + v(v-1)(v/2-1)/2 - \sigma(v-4)^2/4$, $\sigma \in \{0,2,3\}$. By repeating this argument we obtain the proof. ∎

It is well-known [1] that for any even positive integer $n \leq v/2$ there exists a 1-factorization of K_v containing a sub 1-factorization of K_n. Hence, similarly to the above theorem, it is possible to prove the following

Theorem 2.6. *$k + h + v(v-1)(v/2-1)/2 - \nu^2\varepsilon/4 \in J^f(2v)$ for every $k \in J^f(v)$, $h \in J(v)$, $\varepsilon \in \{0,2,3,\ldots,\nu-1\}$ and even positive integer $\nu \leq v/2$.*

3. $J^f(4\cdot 2^n)$.

Theorem 3.1. $\{0,1,2,3,4,5,6,8,10,12,14,18,22,26,30,34,42\} \subset J^f(16)$.

Proof: Let $Q = \{1,2,\ldots,8\}$ and let

$$q = \{\{1,2,3,4\},\{1,2,5,6\},\{1,2,7,8\},\{1,3,5,7\},\{1,3,6,8\},\{1,4,5,8\},\{1,4,6,7\},\{2,3,5,8\},\{2,3,6,7\},\{2,4,5,7\},\{2,4,6,8\},\{3,4,5,6\},\{3,4,7,8\},\{5,6,7,8\}\}.$$

Obviously (Q,q) is an $S(3,4,8)$. Let $c = q$ and c_1 be the flower at the point 1. Let Q' be the set $\{1',2',\ldots,8'\}$ and let φ be the bijection from Q onto Q' with $x' = \varphi(x)$ for every $x \in Q$.

Let $H_1 = \{\{1,2\},\{3,4\}\}$, $H_2 = \{\{1,3\},\{2,4\}\}$, $H_3 = \{\{1,4\},\{2,3\}\}$, $H_4 = \{\{1,5\},\{2,6\}\}$, $H_5 = \{\{1,6\},\{2,5\}\}$, $H_6 = \{\{1,7\},\{2,8\}\}$, $H_7 = \{\{1,8\},\{2,7\}\}$, $H_1^\star = \{\{5,6\},\{7,8\}\}$, $H_2^\star = \{\{5,7\},\{6,8\}\}$, $H_3^\star = \{\{5,8\},\{6,7\}\}$, $H_4^\star = \{\{3,7\},\{4,8\}\}$, $H_5^\star = \{\{3,8\},\{4,7\}\}$, $H_6^\star = \{\{3,5\},\{4,6\}\}$, $H_7^\star = \{\{3,6\},\{4,5\}\}$.

Clearly, $F_i = \{\{\varphi(x),\varphi(y)\}/\{x,y\} \in H_i \cup H_i^\star\}$ $i = 1,2,\ldots,7$ is a 1-factorization on Q'. Let α be a permutation on $\{1,2,3\}$ and let β and γ be two permutations on $\{4,5\}$ and $\{6,7\}$, respectively. Let $G_i = H_i \cup H^\star_{\alpha(i)}$ $i = 1,2,3$, $G_i = H_i \cup H^\star_{\beta(i)}$ $i = 4,5$ and $G_i = H_i \cup H^\star_{\gamma(i)}$ $i = 6,7$. $G = \{G_1,G_2,\ldots,G_7\}$ is a 1-factorization on Q such that

$$\sigma = \sum_{i=1}^{7} |\varphi(G_i - \{\{1,i+1\}\}) \cap (F_i - \{\{1',(i+1)'\}\})| \in \{7,9,11,13,15,21\}.$$

If G is one of the following 1-factorizations, we obtain $\sigma \in \{0,1,2,3,4,5,6\}$.

12	13	14	15	16	17	18
74	27	25	24	28	23	26
35	46	36	67	54	84	34
68	58	78	38	37	56	57

12	13	14	15	16	17	18
74	25	27	24	28	23	26
35	46	36	67	54	84	34
68	78	58	38	37	56	75

12	13	14	15	16	17	18
34	25	27	24	28	23	26
57	46	36	67	54	48	74
68	78	58	38	73	56	35

12	13	14	15	16	17	18
34	25	27	26	28	23	24
57	46	36	47	54	84	67
68	78	58	38	73	56	35

12	13	14	15	16	17	18
56	24	28	23	27	26	25
78	58	57	68	35	38	37
34	67	36	47	48	45	46

12	13	14	15	16	17	18
56	24	28	26	27	23	25
78	58	57	38	35	68	37
34	67	36	47	48	45	46

12	13	14	15	16	17	18
56	24	28	23	27	26	25
78	58	57	68	38	35	37
34	67	36	47	45	48	46

Clearly, for $h \in \{0,1,3,7\}$ there is an $S(2,3,7)$ (T,t'), $T = \{2',3',\dots,8'\}$, such that $|t' \cap \{\{x',y',z'\}/\{1,x,y,z\} \in q\}| = h$. Since there exists, up to isomorphism, only one triple system of order 7 it is of course derived. Hence the proof follows from Theorem 2.4. ∎

Theorem 3.2. $\{28,34,36,38,44,46,48,50,52,54,56,58,60,62,64,66,70,72,74,78,86\} \subset J^f(16)$.

Proof: Let F and $F^{(j)}$ $(j = 1,2,\dots,10)$ be the following 1-factorizations on $X' = \{0',1',\dots,7'\}$ and $X = \{0,1,\dots,7\}$

F'_1	F'_2	F'_3	F'_4	F'_5	F'_6	F'_7
$0'\ 1'$	$0'\ 2'$	$0'\ 3'$	$0'\ 4'$	$0'\ 5'$	$0'\ 6'$	$0'\ 7'$
$2'\ 3'$	$1'\ 3'$	$1'\ 2'$	$1'\ 5'$	$1'\ 6'$	$1'\ 7'$	$1'\ 4'$
$4'\ 7'$	$4'\ 5'$	$4'\ 6'$	$2'\ 6'$	$2'\ 7'$	$2'\ 4'$	$2'\ 5'$
$5'\ 6'$	$6'\ 7'$	$5'\ 7'$	$3'\ 7'$	$3'\ 4'$	$3'\ 5'$	$3'\ 6'$

$F_1^{(1)}$	$F_2^{(1)}$	$F_3^{(1)}$	$F_4^{(1)}$	$F_5^{(1)}$	$F_6^{(1)}$	$F_7^{(1)}$
0 2	0 1	0 4	0 6	0 5	0 3	0 7
1 4	2 4	1 2	1 5	1 6	1 7	1 3
3 5	3 7	3 6	2 7	2 3	2 5	2 6
6 7	5 6	5 7	3 4	4 7	4 6	4 5

$F_4^{(2)}$	$F_5^{(2)}$	$F_6^{(2)}$	$F_7^{(2)}$	$F_6^{(3)}$	$F_7^{(3)}$	$F_2^{(6)}$	$F_3^{(6)}$
0 6	0 5	0 3	0 7	0 3	0 7	0 1	0 4
1 3	1 7	1 5	1 6	1 6	1 5	2 4	1 2
2 7	2 3	2 6	2 5	2 5	2 6	3 6	3 7
4 5	4 6	4 7	3 4	4 7	4 3	7 5	5 6

$$F_1^{(j)} = F_1^{(1)} \text{ for every } j = 2, 3, \ldots, 10;$$
$$F_2^{(j)} = F_2^{(1)}, F_3^{(j)} = F_3^{(1)} \text{ for every } j = 2, 3, 4, 5;$$
$$F_2^{(j)} = F_2^{(6)}, F_3^{(j)} = F_3^{(6)} \text{ for every } j = 7, 8, 9, 10;$$
$$F_4^{(3)} = F_4^{(2)}, F_5^{(3)} = F_5^{(2)}, F_4^{(4)} = F_4^{(2)}, F_5^{(4)} = F_5^{(1)}, F_6^{(4)} = F_6^{(1)},$$
$$F_7^{(4)} = F_7^{(3)}, F_4^{(5)} = F_4^{(1)}, F_5^{(5)} = F_5^{(2)}, F_6^{(5)} = F_6^{(3)}, F_7^{(5)} = F_7^{(1)},$$
$$F_k^{(j)} = F_k^{(j-5)} \text{ for every } j = 6, 7, 8, 9, 10 \text{ and } k = 4, 5, 6, 7.$$

Let m_1, m_2, d_1, d_2 be the following block-sets

m_1	m_2
1 4 4′ 7′	1 4 4′ 5′
1 4 5′ 6′	1 4 6′ 7′
0 2 4′ 7′	0 2 4′ 5′
0 2 5′ 6′	0 2 6′ 7′
1 0 4′ 5′	1 0 4′ 7′
1 0 6′ 7′	1 0 5′ 6′
4 2 4′ 5′	4 2 4′ 7′
4 2 6′ 7′	2 4 5′ 6′

d_1	d_2
5 6 4′ 5′	5 6 4′ 6′
5 6 6′ 7′	5 6 5′ 7′
3 7 4′ 5′	3 7 4′ 6′
3 7 6′ 7′	3 7 5′ 7′
5 7 4′ 6′	5 7 4′ 7′
5 7 5′ 7′	5 7 5′ 6′
3 6 4′ 6′	3 6 4′ 7′
3 6 5′ 7′	3 6 5′ 6′
3 5 4′ 7′	6 7 4′ 5′
3 5 5′ 6′	6 7 6′ 7′
6 7 4′ 7′	3 5 6′ 7′
6 7 5′ 6′	1 4 0′ 4′
3 5 0′ 1′	1 4 1′ 5′
1 4 0′ 1′	3 5 0′ 4′
1 4 4′ 5′	3 5 1′ 5′
1 5 0′ 4′	1 5 0′ 1′
1 5 1′ 5′	1 5 4′ 5′
3 4 0′ 4′	3 4 0′ 1′
3 4 1′ 5′	3 4 4′ 5′

Let (X,a), (X',b_1) and (X',b_2) be three $S(3,4,8)$. Let $(Q,q) = [X' \cup X][b_1,a,F,F^{(1)}]$ and $(Q,q^{(j)}) = [X' \cup X][b_2,a,F,F^{(j)}]$. Clearly, $(Q,(q-m_1)\cup m_2)$ and $(Q,(q^{(j)}-m_1)\cup m_2)$ are two $S(3,4,16)$. Let $c^{(j)} = (q^{(j)}-m_1)\cup m_2)$. Since $d_1 \subseteq (q-m_1)\cup m_2$ and $d_1 \subseteq c^{(j)}$, $(Q,(q-m_1)\cup m_2)$ and $(Q,(c^{(j)}-d_1)\cup d_2)$ are two $S(3,4,16)$ agreeing on the blocks of the flower at the point 0 and exactly $7+|b_1 \cap b_2|+k$ ($k \in \{21,29,37,41,45,49,53,57,65\}$) others. ∎

Theorem 3.3. $76, 84 \in J^f(16)$.

Proof: Let $X = \{1,2,\ldots,8\}$ and $Y = \{1',2',\ldots,8'\}$. Let φ be the bijection of X onto Y such that $x' = \varphi(x)$ for every $x \in X$. Let F and G be the following 1-factorizations on X and Y respectively:

F_1	F_2	F_3	F_4	F_5	F_6	F_7
1 2	1 3	1 4	1 5	1 6	1 7	1 8
3 4	2 4	2 3	2 8	2 5	2 6	2 7
5 6	5 7	5 8	3 7	3 8	3 5	3 6
7 8	6 8	6 7	4 6	4 7	4 8	4 5

G_1	G_2	G_3	G_4	G_5	G_6	G_7
1′ 2′	1′ 3′	1′ 5′	1′ 4′	1′ 6′	1′ 7′	1′ 8′
3′ 5′	2′ 5′	2′ 3′	2′ 8′	2′ 4′	2′ 6′	2′ 7′
4′ 6′	4′ 7′	4′ 8′	3′ 7′	3′ 8′	3′ 4′	3′ 6′
7′ 8′	6′ 8′	6′ 7′	5′ 6′	5′ 7′	5′ 8′	4′ 5′

Let

a
1 2 3 5
1 2 4 7
1 2 6 8
1 3 4 8
1 4 5 6
1 5 7 8
1 3 6 7
2 4 5 8
2 3 4 6
2 5 6 7
2 3 7 8
3 4 5 7
4 6 7 8
3 5 6 8

v
1 2 1′ 2′
1 2 3′ 5′
3 4 1′ 2′
3 4 3′ 5′
1 3 1′ 3′
1 3 2′ 5′
2 4 2′ 3′
2 4 2′ 5′

q_1
1′ 2′ 3′ 5′
1′ 2′ 5 6
1′ 3′ 5 7
1′ 5′ 6 7
2′ 3′ 5 8
2′ 5′ 6 8
3′ 5′ 7 8

w
1 2 1′ 3′
1 2 2′ 5′
3 4 1′ 3′
3 4 2′ 5′
1 3 1′ 2′
1 3 3′ 5′
2 4 1′ 2′
2 4 3′ 5′

q_2
1 2 3 4
1 2 5 6
1 2 7 8
1 3 5 7
1 4 6 7
1 3 6 8
1 4 5 8
2 3 5 8
2 4 5 7
2 4 6 8
2 3 6 7
3 4 5 6
3 4 7 8
1′ 5 6 7
2′ 5 6 8
3′ 5 7 8
5′ 6 7 8
1′ 2′ 3′ 5
1′ 2′ 5′ 6
1′ 3′ 5′ 7
2′ 3′ 5′ 8

Let $b = \{\varphi(c)/c \in a\}$. Clearly, (X,a) and (Y,b) are two $S(3,4,8)$. Let $(Q,q) = [X \cup Y][a,b,F,G]$ and $q^\star = a \cup q_1$. It is easy to check that (Q,q) and $(Q,(q-q^\star) \cup q_2)$ are two $S(3,4,16)$ agreeing on the blocks of the flower at the point $7'$ and exactly 84 others.

Similarly, from (Q,q) and $(Q,(q-(q^\star \cup v)) \cup (q_2 \cup w))$ it follows $76 \in J^f(16)$. ∎

Theorem 3.4. $\{20,24,32,40,68,80\} \subset J^f(16)$.

Proof: Consider the $S(3,4,8)$ (X,a), (Y,b), the two 1-factorizations F, G and the bijection φ, all defined in the Theorem 3.3.

Let $F^{(j)}$ $(j = 1,2,\ldots,6)$ be the following 1-factorizations on Y:

$$F_1^{(j)} = G_1 \text{ for } j = 1,2,3,5,6;\ F_2^{(j)} = G_3 \text{ for } j = 1,2,3;$$
$$F_3^{(j)} = G_2 \text{ for } j = 1,2,3;\ F_i^{(1)} = G_i \text{ for } i = 4,5,6,7;$$
$$F_i^{(4)} = F_i^{(2)}, F_i^{(5)} = F_i^{(2)} \text{ and } F_i^{(6)} = F_i^{(3)} \text{ for } i = 4,5,6,7;$$
$$F_2^{(6)} = F_2^{(5)}, F_3^{(6)} = F_3^{(2)}.$$

$F_4^{(2)}$	$F_5^{(2)}$	$F_6^{(2)}$	$F_7^{(2)}$	$F_4^{(3)}$	$F_5^{(3)}$	$F_6^{(3)}$
3′ 7′	5′ 7′	1′ 7′	2′ 7′	3′ 7′	5′ 7′	1′ 7′
1′ 8′	1′ 4′	2′ 4′	1′ 6′	1′ 4′	1′ 8′	2′ 8′
2′ 6′	2′ 8′	3′ 8′	3′ 4′	2′ 6′	2′ 4′	3′ 4′
4′ 5′	3′ 6′	5′ 6′	5′ 8′	5′ 8′	3′ 6′	5′ 6′

$F_7^{(3)}$	$F_1^{(4)}$	$F_2^{(4)}$	$F_3^{(4)}$	$F_2^{(5)}$	$F_3^{(5)}$
2′ 7′	7′ 8′	7′ 4′	7′ 6′	1′ 3′	1′ 5′
3′ 8′	1′ 3′	1′ 2′	1′ 5′	2′ 5′	2′ 3′
1′ 6′	2′ 5′	3′ 5′	2′ 3′	4′ 8′	6′ 8′
4′ 5′	4′ 6′	4′ 6′	6′ 8′	6′ 7′	7′ 4′

Let

s_1	
1′ 2′ 3′ 5′	
1 2 3 5	1 2 4 7
1 2 6 8	1 3 4 8
1 4 5 6	1 5 7 8
1 3 6 7	2 4 5 8
2 3 4 6	2 5 6 7
2 3 7 8	3 4 5 7
4 6 7 8	3 5 6 8
5 6 1′ 2′	7 8 3′ 5′
5 8 1′ 3′	5 7 2′ 3′
6 8 1′ 5′	6 7 2′ 5′
5 7 1′ 5′	5 8 2′ 5′
6 7 1′ 3′	6 8 2′ 3′

s_2	
8 2′ 3′ 5′	
1 2 3 4	1 2 5 6
1 2 7 8	1 3 5 7
1 4 6 7	1 3 6 8
1 4 5 8	2 3 5 8
2 4 5 7	2 4 6 8
2 3 6 7	3 4 5 6
3 4 7 8	5 8 1′ 5′
5 7 2′ 5′	6 8 1′ 3′
6 7 2′ 3′	5 6 7 1′
5 6 8 2′	5 7 8 3′
6 7 8 5′	5 1′ 2′ 3′
6 1′ 2′ 5′	7 1′ 3′ 5′

Let $(S,s) = [X \cup Y][a,b,F,F^{(1)}]$ and $(S,s^{(j)}) = [X \cup Y][a,b,F,F^{(j)}]$ $(j = 1,2,\dots,6)$. Since $s_1 \subseteq s^{(j)}$, (S,s) and $(S,(s^{(j)} - s_1) \cup s_2)$ agreeing on the blocks of the flower at the point $7'$ and exactly $k \in \{20,24,32,40,68,80\}$ others. ∎

Theorem 3.5. $I^f(16) - \{16\} \subseteq J^f(16)$.

Proof: From [4] (Theorem 3.2) it follows that $82 \in J^f(16)$. From [12] (Theorem 2.1) it follows that $87, 88 \in J^f(v)$. The theorems of Section 2 and the above theorems complete the proof. ∎

Remarks: The authors have not been able to handle the case $16 \in J^f(16)$. Therefore it is an open problem.

Corollary 3.1. $J^f(4\cdot 2^n) = I^f(4\cdot 2^n)$ *for every integer* $n \geq 3$.

Proof: The (i) of Section 1 and Theorem 2.3 imply $I^f(32) - \{996\} \subseteq J^f(32)$. The Theorem 2.5 gives $996 \in J^f(32)$. At last the (iii) of Section 1 and Theorem 2.3 complete the proof. ∎

4. $J^f(5\cdot 2^n)$.

Theorem 4.1. $J^f(10) = \{0, 18\}$.

Proof: Since $J(10) = \{0,2,4,6,8,12,14,30\}$ [8] it is $J^f(10) \subseteq \{0,2,18\}$. Let

p_1	p_2	t
2 4 3 8	2 4 0 9	1 2 4 5
2 4 6 9	2 4 6 3	1 2 8 9
2 4 7 0	2 4 8 7	1 2 6 0
2 5 8 0	2 5 3 9	1 2 3 7
2 5 7 9	2 5 6 8	1 4 6 8
2 5 3 6	2 5 7 0	1 4 7 9
4 5 7 8	4 5 7 3	1 4 3 0
4 5 6 0	4 5 6 9	1 5 9 0
4 5 3 9	4 5 8 0	1 5 6 7
2 6 7 8	2 3 8 0	1 5 3 8
2 3 9 0	2 6 7 9	1 7 8 0
4 3 6 7	4 3 8 9	1 3 6 9
4 8 9 0	4 6 0 7	
5 6 8 9	5 3 6 0	
5 3 7 0	5 7 8 9	
3 7 8 9	3 7 6 8	
3 6 8 0	3 7 9 0	
6 7 9 0	6 8 9 0	

Clearly, $(X, p_1 \cup t)$ and $(X, p_2 \cup t)$ $(X = \{0,1,\dots,9\})$ are two $S(3,4,10)$ intersecting in the flower t at the point 1. Hence, $0 \in J^f(10)$.

Now we will show that $2 \notin J^f(10)$. Assume $2 \in J^f(10)$, there are two DMB PQSs (A, a_1) and (A, a_2) such that $|A| = 9$ and $|a_1| = |a_2| = 16$. Obviously for every $x \in A$ $|\{c \in a_i / x \in c\}| \leq 8$ and there is at least one element say x' in A such that $|s_i = \{c \in a_i / x' \in c\}| = 8$ for $i = 1,2$. Let $r_i = \{c - \{x'\} / c \in s_i\}$ and $D = A - \{x'\}$. Obviously (D, r_i) are two DMB PTSs. It is proved in [11] that there exist, to within isomorphism, only two DMB PTSs with $|r_i| = 8$ blocks and $|D| = 8$ elements. Moreover every x of D appears in precisely three blocks of r_i. It is easy to see that $a_i - s_i = \{\{x\} \cup b / b \in r_j$, for some $x \in D\}$, $i \neq j$ and $\{i,j\} = \{1,2\}$. Let $\{x, x_1, x_2, x_3\} \in a_1 - s_1$ with $\{x_1, x_2, x_3\} \in r_2$. Obviously it is $\Gamma = \{\{x, x_1, x_2, y_1\}, \{x, x_1, x_3, y_2\}, \{x, x_2, x_3, y_3\}\} \subset a_2 - s_2$. Moreover x occurs in exactly three blocks of s_2 and in at least three blocks of $a_2 - (s_2 \cup \Gamma)$. Hence, there exist at least 9 blocks of a_2 containing x. ∎

Theorem 4.2. $J^f(20) = I^f(20)$.

Proof: Let (Q,q) be an $S(3,4,10)$ $(Q=\{0,1,\ldots,9\})$ containing the flower

$$c_1=\{\{1,2,3,4\},\{1,2,5,6\},\{1,2,7,8\},\{1,2,0,9\},\{1,3,5,8\},\\ \{1,3,7,9\},\{1,3,0,6\},\{1,4,5,9\},\{1,4,7,6\},\{1,4,0,8\},\\ \{1,5,7,0\},\{1,6,8,9\}\}.$$

Let $c=q$ and let φ be the bijection from Q onto $Q'=\{0',1',\ldots,9'\}$ with $x'=\varphi(x)$ for $x\in Q$. Let F and G be the two following 1-factorizations on Q' and Q respectively:

F_1'	F_2'	F_3'	F_4'	F_5'	F_6'	F_7'	F_8'	F_9'
1' 2'	1' 3'	1' 4'	1' 5'	1' 6'	1' 7'	1' 8'	1' 9'	1' 0'
3' 4'	2' 4'	2' 3'	2' 6'	2' 5'	2' 8'	2' 7'	2' 0'	2' 9'
5' 6'	5' 8'	5' 9'	3' 8'	3' 0'	3' 9'	3' 5'	4' 5'	3' 6'
7' 8'	7' 9'	7' 6'	4' 9'	4' 7'	4' 6'	4' 0'	6' 8'	4' 8'
9' 0'	0' 6'	8' 0'	7' 0'	8' 9'	5' 0'	6' 9'	3' 7'	5' 7'

G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9
1 2	1 3	1 4	1 5	1 6	1 7	1 8	1 9	1 0
3 5	2 5	2 6	2 3	2 4	2 0	2 9	2 8	2 7
4 6	4 9	3 9	4 0	3 7	3 6	3 0	3 4	3 8
7 0	6 7	5 0	7 9	5 8	5 9	5 6	5 7	5 4
9 8	8 0	7 8	6 8	9 0	4 8	4 7	6 0	6 9

For $h=1,3$ there is an $S(2,3,9)$ (T,t') $(T=\{2',3',\ldots,9',0'\})$ such that $|t'\cap\{\{x',y',z'\}/\{1,x,y,z\}\in q\}|=h$. Since there exists, up to isomorphism, only one triple system of order 9 it is of course derived. Hence from Theorem 2.4 it follows that $1,3\in J^f(20)$.

Let (X,a_i) $i=1,2$ be two $S(3,4,10)$ $(X=\{0,1,\ldots,9\})$ such that $|a_1\cap a_2|\in\{0,2,14,30\}$. Let (Y,b_i) be two $S(3,4,10)$ $(Y=\{0',1',\ldots,9'\})$ agreeing on the blocks of the flower at the point $9'$ and exactly 18 others. Let $F=\{F_1,F_2,\ldots,F_9\}, G=\{G_1,G_2,\ldots,G_9\}$, $G'=\{G_1',G_2',\ldots,G_9'\}$ be the following 1-factorizations:

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9
1 2	1 3	1 4	1 5	4 5	4 6	1 6	1 7	1 9
5 6	7 5	5 8	4 8	6 9	5 9	2 8	2 6	2 5
7 8	4 2	3 9	2 9	1 8	2 7	3 5	3 4	3 6
3 0	6 8	2 0	3 7	2 3	3 8	7 9	8 9	4 7
4 9	9 0	7 6	6 0	7 0	1 0	4 0	5 0	8 0

G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9
4′ 6′	4′ 7′	4′ 5′	0′ 4′	0′ 6′	0′ 9′	0′ 5′	0′ 7′	0′ 8′
5′ 7′	5′ 6′	6′ 7′	1′ 5′	1′ 2′	1′ 6′	3′ 7′	1′ 4′	1′ 7′
0′ 2′	0′ 3′	0′ 1′	2′ 3′	5′ 8′	2′ 7′	2′ 4′	2′ 6′	2′ 5′
1′ 3′	1′ 8′	2′ 8′	6′ 9′	3′ 4′	3′ 5′	1′ 9′	5′ 9′	3′ 6′
8′ 9′	2′ 9′	3′ 9′	7′ 8′	7′ 9′	4′ 8′	6′ 8′	3′ 8′	4′ 9′

G'_1	G'_2
4′ 6′	4′ 7′
5′ 7′	5′ 6′
0′ 2′	0′ 3′
1′ 3′	1′ 8′
8′ 9′	2′ 9′

$G'_i = G_i$ for $i = 2, 3, \dots, 9$.

$(Q, q_i) = [X \cup Y][a_i, b_i, F, G]$ and $(Q, q^*) = [X \cup Y][a_i, b_i, F, G']$ are four $S(3,4,20)$. Let

c_1	c_2
5 6 4′ 6′	5 6 4′ 5′
5 6 5′ 7′	5 6 6′ 7′
7 8 4′ 6′	7 8 4′ 5′
7 8 5′ 7′	7 8 6′ 7′
5 7 4′ 7′	5 7 4′ 6′
5 7 5′ 6′	5 7 5′ 7′
6 8 4′ 7′	6 8 4′ 6′
6 8 5′ 6′	6 8 5′ 7′
5 8 4′ 5′	5 8 4′ 7′
5 8 6′ 7′	5 8 5′ 6′
6 7 4′ 5′	6 7 4′ 7′
6 7 6′ 7′	6 7 5′ 6′

d_1	d_2
5 6 4′ 5′	5 6 4′ 6′
5 6 6′ 7′	5 6 5′ 7′
7 8 4′ 5′	7 8 4′ 6′
7 8 6′ 7′	7 8 5′ 7′
5 7 4′ 6′	5 7 4′ 7′
5 7 5′ 7′	5 7 5′ 6′
6 8 4′ 6′	6 8 4′ 7′
6 8 5′ 7′	6 8 5′ 6′
5 8 4′ 7′	6 7 4′ 5′
5 8 5′ 6′	6 7 6′ 7′
6 7 4′ 7′	5 8 6′ 7′
6 7 5′ 6′	1 4 0′ 4′
5 8 0′ 1′	1 4 1′ 5′
1 4 0′ 1′	5 8 0′ 4′
1 4 4′ 5′	5 8 1′ 5′
1 5 0′ 4′	1 5 0′ 1′
1 5 1′ 5′	1 5 4′ 5′
4 8 0′ 4′	4 8 0′ 1′
4 8 1′ 5′	4 8 4′ 5′

r_1	r_2
5 9 5′ 3′	5 9 3′ 4′
4 5 5′ 8′	4 5 4′ 8′
4 5 3′ 4′	4 5 5′ 3′
6 9 5′ 8′	6 9 4′ 8′
6 9 3′ 4′	6 9 5′ 3′
4 6 4′ 8′	4 6 5′ 8′
4 6 5′ 3′	4 6 3′ 4′
5 9 4′ 8′	5 9 5′ 8′

Let $f_i = (q_i^* - c_1) \cup c_2$ and $f_i' = (q_i - c_1) \cup c_2$. (Q, f_1) and $(Q, (f_2 - (d_1 \cup r_1)) \cup (d_2 \cup r_2))$, (Q, f_1') and $(Q, (f_2' - d_1) \cup d_2)$, (Q, f_1') and $(Q(f_2' - (r_1 \cup d_1)) \cup (r_2 \cup d_2))$ are three pairs of $S(3,4,20)$. Hence it is easy to see that $\{189, 179, 181, 185, 201\} \subset J^f(20)$.

From [3, Theorem 2.1 and Theorem 2.2] and [5, Theorem 2.1] it follows that $\{187, 195, 203, 205, 199, 207\} \subset J^f(20)$ and $191, 197 \in J^f(20)$.

Theorems 2.1, 2.2, 2.5 and 2.6 complete the proof. ∎

The (iii) of Section 1 and Theorem 2.1 and Theorem 4.2 give the following corollary.

Corollary 4.1. $J^f(5 \cdot 2^n) = I^f(5 \cdot 2^n)$ *for every integer* $n \geq 2$.

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