Concerning 3-factorizations of $3K_{n,n}$

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ABSTRACT

Standard doubling and tripling constructions for block designs with block size three (triple systems) employ factorizations of complete graphs and of complete bipartite graphs. In these constructions, repeated edges in a factor lead to repeated blocks in the design. Hence the construction of triple systems with a prescribed number of repeated blocks is facilitated by determining the possible structure of repeated edges in the factors of a λ-factorization of λK_n and $\lambda K_{n,n}$. For $\lambda=3$, a complete determination of the possible combinations of numbers of doubly and triply repeated edges in 3-factorizations of λK_n has been completed for $n \geq 12$. In this paper, we solve the analogous problem for the complete bipartite graphs in the case $\lambda=3$. The case $\lambda=1$ is trivial, and the case $\lambda=2$ has been previously solved by Fu.

1. The Background

Let $G=(V,E)$ be a multigraph. A λ-*factor* of G is a λ-regular spanning submultigraph of G, and a λ-*factorization* is a partition of the edges into λ-factors. When G has multiple edges, the λ-factors may also contain multiple edges. We are interested in classifying certain factorizations according to the number of repeated edges of each multiplicity. Our particular interest is in 3-factorizations, and hence we define the *type* of such a factorization to be the pair (t,s), where t is the total number of doubly repeated edges in factors, and $|E|/3-s$ is the total number of triply repeated edges.

Given a simple graph G, the multigraph λG is obtained by repeating each edge of G λ times. In a previous paper, we determined the possible types of 3-factorizations of $3K_{2n}$ for all n, leaving some possible exceptions for $n=4$ and 5. In this paper, we consider the analogous problem for $3K_{n,n}$.

Before stating the main result, it is important to address the motivation for this research. Our main motivation arises in the construction of triple systems, i.e. balanced incomplete block designs with block size three. In the standard $v \rightarrow 2v+1$ construction for triple systems with index λ, a primary ingredient is a λ-factorization of λK_{v+1}. Repeated edges in the factorization lead to repeated blocks in the triple system. Hence a characterization of types of repeated edges assists in the determination of types of repeated blocks.

The $2v+1$ construction alone is not sufficient, and is typically supplemented by a

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$v \rightarrow 2v+7$ construction (see, for example, [3]). In this case, however, the factorization employed is of some subgraph of λK_{v+7}. A particularly useful $2v+7$ construction employs, as a portion of the required λ-factorization, a λ-factorization of $\lambda K_{½(v+7),½(v+7)}$. In this context, the possible types of repeated edges are of some interest; hence we address the smallest open case, $\lambda=3$. The case $\lambda=2$ has been settled by Fu [2], and the analogous problem for twofold triple systems settled by Rosa and Hoffman [4]. The determination of the types of threefold triple systems is the subject of a paper under preparation by Mathon, Rosa, Shalaby and the present author.

Let G be a regular bipartite graph on $2n$ vertices. We let $\mathcal{Y}(G)$ denote the set of types of 3-factorizations of $3G$. Then $\mathcal{Y}(n,d)$ denotes the union of $\mathcal{Y}(G)$ over all d-regular bipartite graphs G on $2n$ vertices. In this terminology, our goal is to determine $\mathcal{Y}(n,n)$.

Our task breaks naturally into two parts. First, we must determine necessary conditions for a pair (t,s) to be in $\mathcal{Y}(n,d)$. Second, we must provide constructions for factorizations of all types which can be realized. It will become apparent that both problems are complicated by a very large number of cases. Hence we will in general just present the method by which the factorizations are obtained, and avoid an explicit presentation of the thousands of factorizations that are needed.

Now we can state the main theorem of [1], followed by the main theorem of this paper.

Let $\mathcal{A}(n,d) = \{(t,s) : 0 \leq t \leq s \leq nd\} \setminus \{(0,1), (0,2), (0,3), (0,4), (0,5), (0,7), (0,8), (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (1,9), (1,10), (1,11), (2,2), (2,3), (2,4), (2,5), (2,6), (2,7), (2,8), (2,9), (2,10), (3,3), (3,4), (3,5), (3,6), (3,7), (3,8), (3,10), (4,5), (4,7), (4,8), (5,5), (5,6), (5,7), (5,8), (5,9), (5,10)\}$.

Theorem A [1]: For $n \geq 6$, there is a 3-factorization of $3K_{2n}$ of type (t,s) if and only if $(t,s) \in \mathcal{A}(n,2n-1)$. □

In order to state the theorem for $3K_{n,n}$, we define $\mathcal{B}(n,d) = \mathcal{A}(n,d) \setminus \{(0,6), (0,10), (0,11), (0,13), (1,12), (1,13), (1,14), (1,17), (2,11), (2,12), (2,13), (3,9), (3,11), (3,13), (4,6), (4,10), (4,11), (5,11), (5,12), (5,13), (6,10), (7,10), (7,11)\}$. Then we have

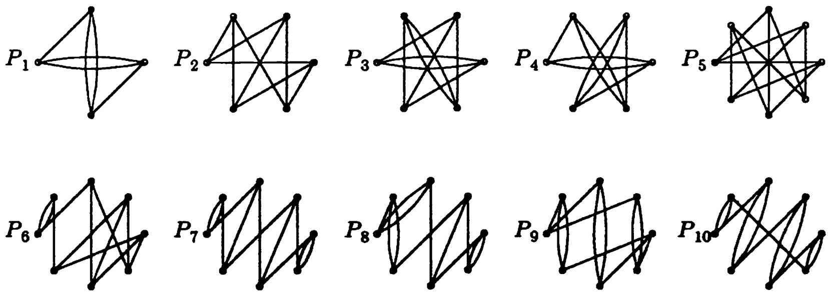
Main Theorem: For $n \geq 10$, $\mathcal{Y}(n,n) = \mathcal{B}(n,n)$.

In section 2, we establish the necessary conditions. Then in section 3, we recall the main recursive constructions used in establishing sufficiency from [1]. We develop additional constructions for the bipartite case as well. Subsequent sections then *outline* the construction of the small examples required in the recursions.

2. Necessary Conditions

We follow the same strategy as [1] in determining the necessary conditions. Suppose that a 3-factorization $\mathcal{F}$ of $3K_{n,n}$ of type (t,s) exists. Let $R(\mathcal{F})$ be the graph whose edges are those edges *not* appearing in three-times repeated edges in a factor of $\mathcal{F}$; we call this the *remainder* of $\mathcal{F}$. Each factor of $\mathcal{F}$ induces a cubic submultigraph of $R(\mathcal{F})$ having no three-times repeated edges; these submultigraphs are termed *portions*. In [1], it is observed that if a vertex of $R(\mathcal{F})$ has degree d, it belongs to precisely d of the portions. Hence the minimum degree in $R(\mathcal{F})$ is at least two. Moreover, the number of two-times repeated edges must be at least as large as the number of edges in $R(\mathcal{F})$ incident with a vertex of degree two. We make a further observation along these lines: if a two-times repeated edge of a factor is incident with a vertex of degree three in $R(\mathcal{F})$, at least two doubly repeated edges in factors are incident with this vertex. From this, it is easy to deduce that if $R(\mathcal{F})$ has maximum degree 3, $t \notin \{1,2,3,5\}$.

Our basic strategy is to consider candidates for $R(\mathcal{F})$, and to establish in the required cases that no suitable set of portions exist which partition $3G$. We first determine all portions on fewer than ten vertices, depicted next:



In total, there are sixty-six pairs (t,s) which we must eliminate. However, forty-three of these are eliminated by observing that every 3-factorization of $3G$ can be completed to a 3-factorization of $3K_n$ of the same type, provided that n is large enough (this follows from standard embedding theorems for latin squares). Hence we need only treat the remaining twenty-three pairs.

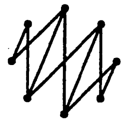
Lemma 2.1: If $(t,s) \notin \mathcal{B}(n,n)$, there is no 3-factorization of $3K_{n,n}$ of type (t,s).

Proof:

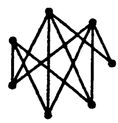
We consider the remaining values, by treating each value of s in turn. First, if $s=6$, $R(\mathcal{F})$ must be a bipartite graph with six edges, and minimum degree 2. But then the only candidates are C_6 and $K_{2,3}$; both have six edges incident with degree 2 vertices, and hence force $t=6$.

Now if $s=9$, we may have three portions of size 4 and one of size 6, or three portions of size 6. In the first case, we have $t \geq 6$ since each portion of size 4 contributes 2 doubly repeated edges. In the second case, two portions must be $K_{3,3}$ $(=P_2)$, and hence $R(\mathcal{F})$ is $K_{3,3}$ since it has only nine edges. But then $t=0$ or $t \geq 4$.

If $s=10$, first observe that no portion is P_5 or P_6 since both require more than ten edges. Moreover, since the unique ten-edge graph containing $K_{3,3}$ has a degree 1 vertex, no portion is P_1. Now consider the possible portion sizes; they are $\{8,6,6\}$, $\{8,4,4,4\}$, $\{6,6,4,4\}$ and $\{4,4,4,4,4\}$. The last two require $t \geq 8$ since each possible portion has at least two double edges. Similarly, $\{8,4,4,4\}$ requires $t \geq 8$ since the portion of size 8 has at least two double edges. This leaves only $\{8,6,6\}$. Now consider the degrees in a class of the bipartition of $R(\mathcal{F})$. They must be $\{4,2,2,2\}$, $\{3,3,2,2\}$ or $\{2,2,2,2,2\}$. The last requires $t \geq 10$, and the first forces the presence of at least 8 edges incident with degree 2 vertices, and hence $t \geq 8$. Three nonisomorphic candidates for $R(\mathcal{F})$ remain. One contains eight edges incident with degree 2 vertices, forcing $t \geq 8$. A second contains an induced path on four vertices whose interior vertices have degree 2. Replacing the three edges by a single edge, and deleting the interior vertex, yields in the obvious way a factorization of a graph on 8 edges having two fewer double edges. But this requires $t \geq 10$. The final candidate has six edges incident with degree 2 vertices, and hence $t \geq 6$; moreover, t can be less than eight only if at most one edge not incident with a degree 2 vertex appears as a double edge. One can verify that this cannot be achieved in this last candidate, which we display here:

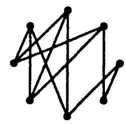


If $s = 11$, first suppose that there is a portion isomorphic to P_2. Then $R(\mathcal{F})$ is



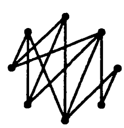
and the portion sizes are $\{6,6,6,4\}$ if $t \leq 7$. If the degree 2 vertex is in the portion of size 4, let F_1 and F_2 be the portions not incident with the vertex of degree 2. Then $\{F_1, F_2, 3K_{3,3} - F_1 - F_2\}$ forms a 3-factorization of type $(t-2, s-2)$ with $t-2 \geq 1$, and hence $t = 6$ or $t \geq 8$. On the other hand, if the degree 2 vertex is not in the portion of size 4, $t \geq 6$ is immediate; exhaustive checking eliminates $t = 7$.

If no portion of P_2 is present, a portion of P_6 must be present, with portion sizes $\{8,6,4,4\}$. Since $s = 11$, $R(\mathcal{F})$ is



and hence if $t \leq 7$, the portion of size 6 is P_3. Now whenever $R(\mathcal{F})$ contains an induced path of length 3, we can replace this path by a single edge, making the obvious modifications to two of the factors, to obtain a 3-factorization of type $(t-3, s-3)$ with $t-3 \geq 1$. Hence $t \geq 9$ in this case.

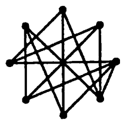
If $s = 12$, the possible portion sizes are $\{8,8,8\}$, $\{8,8,4,4\}$, $\{8,6,6,4\}$ and $\{6,6,6,6\}$ for $t \leq 5$. If some portion is P_5, $R(\mathcal{F})$ is P_5 (the 3-cube). The remaining factors partition $2P_5$, and hence by Fu's theorem [2], $t \notin \{1,2,3,5\}$. This eliminates $\{8,8,4,4\}$. If some portion is P_6, $R(\mathcal{F})$ is



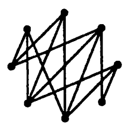
whence $t \geq 3$ is immediate. Moreover, since this graph has minimum degree 2, $\{8,8,8\}$ is eliminated since all portions would have at least two doubly repeated edges. However, we must still consider $\{8,6,6,4\}$. If the degree two vertices appear in a portion of size 6, that portion must be P_4 (since a portion P_6 must remain). This leads to $t \geq 6$. Otherwise the degree 2 vertices appear in the portion of size 4. By eliminating the portion of size 4, and modifying the portion of size 8, we obtain a 3-factorization of type $(t-3, s-3)$ and hence $t \neq 5$. Moreover, if any of portions P_7–P_{10} is present, so must portion P_2, and hence the graph is as depicted above. This eliminates $\{8,6,6,4\}$.

What remains for $s=12$ is the set $\{6,6,6,6\}$ of portion sizes. Now $t \neq 1$, as no portion of size 6 has exactly one doubly repeated edge. The unique candidate for $R(\mathcal{F})$ is $K_{3,4}$, and it is an easy exercise to check that $t \notin \{1,2,3,5\}$ here.

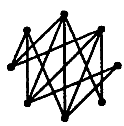
Next we turn to $s=13$. If there is a component of size 10, $R(\mathcal{F})$ either has maximum degree 3 (whence $t \notin \{1,2,3,5\}$) or it has at least five vertices of degree 2. In the latter case, at least six edges are incident with such vertices and hence $t \geq 6$. The remaining sets of portion sizes which are candidates for $t<6$ are $\{8,8,6,4\}$ and $\{8,6,6,6\}$. If P_5 is a portion, $R(\mathcal{F})$ is



and hence neither P_2 nor P_6 are portions. This eliminates $\{8,8,6,4\}$, and also eliminates P_5 as a possible portion. Hence for $t \geq 5$, P_2 must be a portion. If P_6 is a portion, $R(\mathcal{F})$ is



and the portions must be $\{P_2,P_3,P_3,P_6\}$. However, after removing portions of P_2 and P_6, no portion of P_3 remains. Hence P_6 is not a portion, and there are *two* portions of P_2. These are necessarily on the same vertex set since $s=13$ (otherwise, $K_{3,4}$ is contained or more than 13 edges result). But then all remaining vertices have degree 2 and appear in both remaining portions. From this, $t \geq 4$ and $R(\mathcal{F})$ is



Once two portions of P_2 are taken, the remainder partitions into $\{P_3,P_7\}$ giving $t=4$. Hence $t=4$ or $t \geq 6$.

Next we consider $s=14$. We must establish that $t \neq 1$. If there is a portion of size 10, there are at least two edges incident with a degree 2 vertex. If there is a portion of size 4, $t \geq 2$ is immediate. Hence the portion sizes must be $\{8,8,6,6\}$, with portions $\{P_2,P_2,P_5,P_6\}$. However, the double edge of the P_6 is forced to be incident with a degree three vertex in $R(\mathcal{F})$, but then $t \geq 2$.

Finally, we consider $s=17$, and must prove that $t \neq 1$. There are no portions of size 12. If there were a portion of size 10 with one double edge, we would require $s \geq 18$ since both incident vertices require degree at least four. Hence $t=1$ implies that P_6 is a portion. Now if there is a portion of size 10 at all, either a degree two vertex is present or the double edge is incident to a degree 3 vertex; both are forbidden. Hence the portion sizes are $\{8,8,6,6,6\}$ and the portions are $\{P_2,P_2,P_2,P_5,P_6\}$, and $R(\mathcal{F})$ has nine vertices. The only possibility is $K_{4,5}$ minus three disjoint edges; here $t=0$ or $t \geq 4$. This completes the proof. □

Naturally, this type of proof is better suited to computational verification than to hand calculations. We have verified by computer that all candidates for $R(\mathcal{F})$ were considered.

3. 3-latin rectangles and squares

In proving sufficiency in the Main Theorem, we develop a number of constructions for 3-factorizations. A useful device in these constructions is the use of an equivalent combinatorial structure. A *3-latin rectangle* is a $d \times n$ array on symbols $\{1, \cdots, n\}$, satisfying

(1) each entry contains three (not necessarily distinct) symbols,

(2) each symbol appears three times in each row, and

(3) each symbol appears zero or three times in each column.

A *3-latin square* of order n is an $n \times n$ 3-latin rectangle. Now let G be a d-regular subgraph of $K_{n,n}$ and let $\{F_1, \cdots, F_d\}$ be a 3-factorization of $3G$. Suppose that the bipartition of G is $\{x_1, \cdots, x_n\} \cup \{y_1, \cdots, y_n\}$. Form a 3-latin rectangle as follows: for each occurrence of edge $\{x_j, y_k\}$ in factor F_i, place symbol k in entry (i,j). This process is reversible to recover the 3-factorization. The type is easily obtained from the 3-latin square: s is just nd minus the number of entries of the form $\{x,x,x\}$, while t is the number of entries of the form $\{x,x,y\}$.

We give some examples here:

(0,9)	123	123	123	(4,9)	113	223	123	(6,6)	111	223	233
	123	123	123		223	113	123		233	111	223
	123	123	123		123	123	123		223	233	111
(6,8)	111	223	233	(6,9)	123	112	233	(7,8)	111	223	233
	223	113	123		233	123	112		223	123	113
	233	123	112		112	233	123		233	113	122
(7,9)	123	112	233	(9,9)	112	223	133	(0,0)	111	222	333
	122	133	123		133	112	223		333	111	222
	133	223	112		223	133	112		222	333	111

This effectively translates our problem into one concerning 3-latin squares, and hence suggests that standard techniques for constructing latin squares be employed here. In this vein, we observe that every latin rectangle can be completed to a latin square. A similar argument can be used here, to obtain what we call the *basic construction*.

Lemma 3.1: Let (t,s) be the type of a $d \times n$ 3-latin rectangle, and let x,y be integers satisfying $0 \le x \le y \le n-d$ and $(x,y) \ne (0,1), (0,2), (1,1), (1,2)$ or $(1,3)$. Then $(xn+t, yn+s)$ is the type of an $n \times n$ 3-latin square.

Proof:

Given a $d \times n$ 3-latin rectangle, form a bipartite graph whose two parts are the n columns and the n symbols. Join a symbol and a column if and only if that symbol does not appear in the specified column. Now form a 1-factorization $\{F_1, \cdots, F_{n-d}\}$ of the bipartite graph. Take three copies of each 1-factor, to obtain $3n-3d$ 1-factors in total. Group these in threes, so that $n-d-y$ groups contain three identical 1-factors, and x groups contain precisely two the same. For the ith resulting 3-factor, fill row $d+i$ by placing for each column (vertex) the three symbols corresponding to the symbol vertices adjacent to it. The only issue is to accomplish the grouping appropriately, and it is easy to see that this can be done with the exceptions listed above (see [1]). □

The next main tool is to combine small 3-latin rectangles to form larger ones. We first consider adding columns to 3-latin rectangles (since adding rows may be accomplished by the basic construction). Perhaps the simplest method is to simply "catenate" two rectangles:

Lemma 3.2: If (t,s) is the type of a $d \times n$ 3-latin rectangle and (t',s') is the type of a $d \times n'$ 3-latin rectangle, then $(t+t', s+s')$ is the type of a $d \times (n+n')$ 3-latin rectangle.

Proof:

Write the first rectangle on symbols $\{1, \cdots, n\}$ and the second on symbols $\{n+1, \cdots, n+n'\}$; now simply concatenate the rows. □

We can add fewer columns by "splicing" two rectangles together. Define the *profile* of a column to be the multiset of pairs $\{(r,e)$: symbol e appears in row r in the column$\}$, and the profile of a symbol to be the multiset of pairs $\{(r,c)$: the symbol appears in entry $(r,c)\}$ (multiple occurrences are included multiple times). The *type* of a profile is, as one would expect, the pair (b,a) where b is the number of doubly repeated entries, and $d-a$ is the number of triply repeated entries. Then we have

Lemma 3.3: Let L be a $d \times n$ 3-latin rectangle of type (t,s), and let L' be a $d \times n'$ 3-latin rectangle of type (t',s'). Suppose there is a column of L and a symbol of L' having the same profile, and that the type of this profile is (b,a). Then there is a $d \times (n+n'-1)$ 3-latin rectangle of type $(t+t'-b, s+s'-a)$.

Proof:

We outline the strategy. Let c be the column of L and e be the symbol of L' having the same profile. Form L' on symbols disjoint from those in L and then catenate the two rectangles. Now strike out all occurrences of symbol s. The equality of the profiles guarantee that, if s is removed from a column of L', there is a symbol in column c appearing precisely in the rows which symbol s occupied. Hence for each such column vacated by symbol s, we can replace it by one of the symbols in column c. In the process, column c is emptied, and can then be removed. □

Since there is a natural duality between columns and symbols in a 3-latin rectangle, the same 3-latin rectangle can be used to provide both rectangles used in Lemma 3.3; this is particularly useful if one wants to avoid cataloguing solutions by column profile.

An illustration of Lemma 3.3 may be helpful. Take the 3×3 3-latin rectangle of type (7,9) given earlier. The last column has profile type (2,3). Interchanging the roles of columns and symbols, we obtain the 3-latin square with rows 122 123 133, 123 113 223, 133 223 112. The profile of symbol 3 is the same as the original profile of column 3. Applying Lemma 3.3, we obtain

123 112 455 245 334
122 133 145 244 355
133 223 114 255 445

The type is (7+7-2,9+9-3) = (12,15) as expected.

Both Lemmas 3.2 and 3.3 have the drawback that they add at least $d-1$ columns. It is possible, however, to add a single column. A *near 3-transversal* is a set $c_1, \cdots, c_{d-1}$ of distinct columns, and a set $e_1, \cdots, e_{d-1}$ of distinct symbols, so that e_i appears in column c_i, and for each row, the total number of occurrences of e_i in c_i, for $1 \leq i < d$, is at most three. The *type* of the near 3-transversal is obtained as follows. Let x be the number of double occurrences of e_i in the same row of c_i, and let y be the number of triple occurrences. Now since only $d-1$ pairs have been chosen, the row sums add up to $3d-3$, not to $3d$. Three possibilities arise. If any row sum is zero, we add one to y. If any row sum is 1, we add 1 to x. Otherwise, three row sums are 2 and the rest are three; in this case, we leave x and y unchanged. Now the type is $(x, d-y)$.

Lemma 3.4: Let L be a $d \times n$ 3-latin rectangle of type (t,s) having a near 3-transversal of type (b,a). Then there is a $d \times (n+1)$ 3-latin rectangle of type $(t+b, s+a)$.

Proof:

We append an $n+1$'st column; whenever symbol e_i appears in column c_i, it is moved to the new column (without changing its row), and is replaced by the symbol $n+1$. The last column then has up to three entries not fully occupied; the symbol $n+1$ is placed in each enough times so that each contains three symbols. □

Again, an example may prove helpful. Consider again the 3-latin rectangle of type (7,9). A near 3-transversal $\{(c_i, e_i)\}$ is $\{(2,3),(3,1)\}$. Its type is (2,2). Applying Lemma 3.4, we obtain

123 112 233 444
122 144 234 133
133 224 244 113

which has type (9,11) as expected.

These tools enable us to construct a large number of small examples. Our strategy for settling sufficiency is to determine cases for $d=3$ and $d=4$. Applying Lemmas 3.2 and 3.3 then determines the spectrum for $\mathcal{Y}(n,4)$ for all $n \geq 12$. Using Lemma 3.1 to factorize the $(n-4)$-regular complement, we use this to determine $\mathcal{Y}(n,n)$ for all $n \geq 16$. Finally, some ad hoc constructions are used to establish sufficiency for $10 \leq n \leq 15$.

4. The case $d=3$

In this section, we exhibit a number of small $3 \times n$ 3-latin rectangles. In the previous section, we gave nine examples; these are, in fact, all the cases for $n=3$. For

$n = 4$, in addition to those produced by Lemma 3.1, we obtain the following.

(4,4)	112 122 333 444	(6,7)	122 112 444 333
	122 112 444 333		444 233 223 111
	333 444 111 222		112 123 233 444
(6,9)	111 234 224 334	(6,11)	111 224 334 234
	222 334 114 134		223 114 134 234
	333 224 124 114		233 124 114 234
(6,12)	344 224 123 113	(7,7)	112 122 444 333
	223 114 123 344		444 233 223 111
	234 124 123 134		122 113 233 444
(7,12)	234 112 334 124	(8,9)	111 223 244 334
	224 133 134 124		334 111 224 234
	334 223 114 124		344 233 111 224
(8,10)	111 224 244 333	(8,11)	111 223 334 244
	223 334 112 144		223 344 134 112
	233 234 124 114		233 234 114 124
(9,10)	111 223 233 444	(9,11)	111 344 233 224
	344 234 113 122		224 134 113 234
	334 344 122 112		244 113 122 334
(9,12)	124 112 334 234	(10,10)	223 233 111 444
	122 133 134 244		112 344 224 133
	144 223 114 233		133 224 244 113
(10,11)	111 223 244 334	(10,12)	134 112 233 244
	344 123 114 223		144 133 234 122
	334 113 122 244		133 223 244 114

This large set of examples, together with those given earlier and those from Lemma 3.1, can now be extended to produce many values for larger n. Since every $3 \times n$ 3-latin rectangle can be extended to a $4 \times n$ 3-latin rectangle of the same type, we employ the resulting collection as a starting point for a solution for $d = 4$.

5. The case $d = 4$

We first exhibit 4×4 3-latin rectangles, completing in the process the precise determination of $\mathcal{Y}(4,4)$. We omit those arising from 3×4 rectangles, and those from Lemma 3.1.

(0,15)	111 234 234 234	(1,15)	111 234 234 234
	234 123 124 124		234 123 124 134
	234 134 123 124		234 124 123 134
	234 124 134 123		234 134 134 122
(1,16)	123 234 134 124	(2,14)	123 124 134 234
	123 124 234 134		124 123 134 234
	144 123 123 234		124 123 134 234
	234 134 124 123		334 344 222 111
(2,15)	111 234 234 234	(2,16)	123 234 134 124
	234 123 234 114		124 123 244 133
	234 134 112 234		134 124 123 234
	234 124 134 123		234 134 123 124

(3,15)
111 234 234 234
234 134 134 122
234 124 124 133
234 123 123 144

(3,16)
223 134 134 124
114 223 234 134
134 124 123 234
234 134 124 123

(4,14)
124 124 134 233
124 123 134 234
123 123 134 244
334 344 222 111

(4,15)
111 234 234 234
244 123 123 113
234 134 112 234
233 124 134 124

(5,14)
122 124 134 334
124 123 134 234
134 123 134 224
334 344 222 111

(5,15)
111 234 234 234
234 123 234 114
233 144 112 234
244 123 134 123

(5,16)
123 234 134 124
124 123 244 133
123 124 123 344
344 134 123 122

(6,14)
223 114 134 234
114 223 134 234
124 123 134 234
334 344 222 111

(6,15)
111 234 234 234
344 123 234 112
233 144 112 234
224 123 134 134

(6,16)
123 234 113 244
124 123 244 133
123 124 234 134
344 134 123 122

(6,13)
111 234 234 234
222 111 344 334
334 234 112 124
344 234 123 112

(7,13)
111 234 233 244
222 113 444 133
334 234 112 124
344 124 123 123

(7,14)
124 124 134 233
234 112 134 234
112 233 134 244
334 344 222 111

(7,15)
111 223 344 234
244 123 123 113
234 134 112 234
233 144 123 124

(7,16)
123 234 134 124
124 233 244 113
123 124 123 344
344 114 123 223

(8,13)
111 234 233 244
222 113 444 133
334 244 112 123
344 123 123 124

(8,14)
114 224 134 233
224 113 134 234
123 123 134 244
334 344 222 111

(8,15)
111 234 223 344
234 123 234 114
233 144 114 223
244 123 134 123

(9,13)
111 234 233 244
222 133 444 113
334 244 112 123
344 112 123 234

(9,14)
124 122 134 334
234 114 134 223
112 233 134 244
334 344 222 111

(9,15)
111 223 344 234
244 123 123 113
234 344 112 123
233 114 123 244

(9,16)
123 234 114 234
124 233 244 113
123 124 123 344
344 114 233 122

(10,13)
111 234 233 244
222 113 444 133
334 234 122 114
344 124 113 223

(10,14)
124 122 134 334
234 114 133 224
112 233 144 234
334 344 222 111

(10,15)	111 234 223 344 234 123 244 113 233 144 114 223 244 123 133 124
(10,16)	123 234 114 234 124 233 244 113 233 124 123 144 144 114 233 223
(11,12)	111 344 233 224 222 113 444 133 334 223 111 244 344 124 223 113
(11,13)	111 334 223 244 222 113 444 133 334 244 112 123 344 122 133 124
(11,14)	123 122 134 344 244 114 133 223 112 233 144 234 334 344 222 111
(11,15)	111 223 344 234 244 133 234 112 234 244 112 133 233 114 123 244
(11,16)	123 224 114 334 124 233 244 113 123 134 123 244 344 114 233 122
(12,13)	111 234 233 244 222 144 113 334 334 112 144 223 344 233 224 111
(12,14)	134 222 114 334 222 334 144 113 113 134 223 244 344 114 233 122
(12,15)	111 234 223 344 234 113 244 123 233 144 114 223 244 223 133 114
(13,13)	111 222 334 344 222 114 344 133 334 344 122 112 344 133 112 224
(13,14)	111 222 344 334 223 334 112 144 244 113 334 122 334 144 122 123
(13,15)	111 223 344 234 344 133 224 112 234 244 112 133 223 114 133 244
(13,16)	123 224 114 334 224 133 244 113 113 234 123 244 344 114 233 122
(14,14)	111 223 334 244 222 114 344 133 334 144 112 223 344 233 122 114
(14,15)	111 334 223 244 234 112 244 133 233 144 114 223 244 223 133 114
(14,16)	123 224 114 334 224 233 144 113 113 134 223 244 344 114 233 122
(15,15)	444 112 133 223 133 344 224 112 223 334 112 144 112 122 344 334
(15,16)	244 223 133 114 133 112 344 334 223 144 112 224 114 334 224 123

This list is complete for $n=4$. For $n=5$, we do not attempt to produce an exhaustive list, but rather supply those examples which are most useful in the recursion. Two general methods are useful here. The first employs a construction of [1] directly, called Construction E in that paper. We restate the construction here, correcting in the process some errors in the original presentation:

Construction E: Let $\mathcal{F}$ be a 3-factorization of type (t,s) of a r-regular subgraph of K_{2n}, and let ∞ be a vertex of $\mathcal{F}$ incident with d doubly repeated and c triply repeated edges. Then $(2t-d, 2s-r+c)$ is the type of a 3-factorization of an r-regular subgraph of $3K_{2n-1,2n-1}$. □

Given the known solutions for 3-factorizations of $3K_6$ [1], we obtain by Construction E many 3-factorizations of subgraphs of $K_{5,5}$. In this way, we obtain that $\{(1,20),(2,20),(3,18),(3,20),(4,20)\} \subseteq \mathcal{Y}(5,4)$.

A second general method is to prescribe a large portion of the 3-latin rectangle, and examine all possible completions. Consider, for example, the following partial 5×4 3-latin rectangle:

```
555
123 4 5 5 5
123 4 5 5 5
123 4 5 5 5
```

Notice that the edges that remain form the edges of the 3-cube with each edge triplicated; any completion of the rectangle corresponds to a 3-factor and 3 2-factors of the triplicated 3-cube. Hence we can produce many different types by choosing the factors of the triplicated 3-cube appropriately. In this way, one can construct $\{(0,19),(1,19),(2,18),(3,19),(4,18),(4,19),(5,18),(5,19)\} \cup \mathcal{Y}(5,4)$.

Next we exhibit some ad hoc constructions.

(0,18)	111	345	245	235	234	(2,17)	235	145	145	234	123
	345	345	124	125	123		245	135	145	234	123
	345	222	145	135	134		245	135	145	234	123
	345	345	125	123	124		334	344	222	111	555
(4,17)	245	145	145	233	123	(5,17)	122	245	145	335	123
	245	135	145	234	123		124	235	145	235	123
	235	135	145	244	123		134	235	145	225	123
	334	344	222	111	555		334	344	222	111	555

For $n=6$, we can again employ the factorization of the triplicated 3-cube to complete the following partial 4×6 3-latin square:

```
111 222
234 156 1 1 2 2
234 156 1 1 2 2
234 156 1 1 2 2
```

We also employ an ad hoc example:

(1,23)	555	126	124	134	236	346
	236	224	145	135	356	146
	236	146	245	145	235	136
	236	146	125	345	256	134

In the remaining examples, we exhibit only those entries which are not 3-times repeated; the completion of these to latin rectangles of suitable (larger) sizes is easy (and omitted).

(0,14)	123	124	134	234	
	234	124	134	123	
	134		134	134	
	124	124		124	

(0,17)	135		235	123	125
	134	124	234	123	
	345	124	345	123	125
	145	124	245		125

(3,12)	123	123	123	
	123	123	123	
	123	123	124	344

(3,14)	123	123	123	
		234	234	234
	123	144	134	223

(3,17)	135		235	123	125
	134	124	234	123	
	135	244	345	123	125
	445	112	245		125

(4,17)	135		235	123	125
	113	244	234	123	
	345	124	345	123	125
	445	112	245		125

(1,18)	123	125	135		235
		245	345	235	234
	123	145	145	235	234
	123	124	134	235	455

Now we apply Lemmas 3.2 and 3.3 to this large collection of 3-latin rectangles with $d=4$. We employed a computer program to determine the values for n, s and t for which a factorization of a 4-regular subgraph of $3K_{n,n}$ of type (t,s) results; this was carried out for all $n \leq 15$. We found that for $12 \leq n \leq 15$, $\mathcal{Y}(n,4)=\mathcal{B}(n,4)$. For $n \leq 11$, the lemmas fail to construct certain required types from the small rectangles given here. For $n=11$, only type (1,31) is missed. For $n=10$, types (0,27), (1,27), (2,27) and (1,31) are missed. For $n=9$, the following values are missed: (0,22), (1,22), (2,22), (3,22), (5,22), (0,24), (1,24), (2,24), (1,27), (2,27), (3,27) and (5,27).

Numerous exceptions occur for $5 \leq n \leq 8$. Nevertheless, we can now apply Lemma 3.2 to establish

Lemma 5.1: For $n \geq 12$, $\mathcal{Y}(n,4)=\mathcal{B}(n,4)$.

Proof:

For $12 \leq n \leq 15$, this follows from Lemmas 3.2, 3.3 and the small rectangles given. For $n \geq 16$, choose n_1 and n_2 so that $n = n_1+n_2$, with $n_1 \geq 12$, $n_2 \geq 4$, and n_1 and n_2 as equal as possible. Apply Lemma 3.2 to combine $\mathcal{Y}(n_1,4)$ and $\mathcal{Y}(n_2,4)$. □

6. Proof of the Main Theorem

We extend a $4 \times n$ rectangle to an $n \times n$ square. The appended rows form an $(n-d) \times n$ rectangle, which we construct in general using Lemma 3.1. Our strategy is to introduce first the general method which works for $n \geq 16$; subsequently, using this general method together with some special constructions for each n, we complete the determination for $10 \leq n \leq 15$.

We assume throughout the proof that $n \geq 10$. Consider $(t,s) \in \mathcal{B}(n,n)$. If $s \leq 4n$, $(t,s) \in \mathcal{Y}(n,n)$ with the possible exceptions for $n=10$ and 11 stated earlier. Hence we may assume that $s>4n$.

If $t \geq 3n$, choose f, g, h and k so that $t=fn+g$, $s=hn+k$, $2 \leq f \leq n-4$ and $n \leq g \leq k \leq 4n$. Then $(g,k) \in \mathcal{Y}(n,4)$ and Lemma 3.1 provides the required $(n-4) \times n$ rectangle of type (fn,hn). Hence we may assume that $t<3n$.

If $t \geq 2n$, write $t=2n+g$, $s=hn+k$ where $2 \leq h \leq n-4$, $0 \leq g<n$ and $2n<k \leq 4n$. Then $(g,k) \in \mathcal{Y}(n,4)$ with the noted possible exceptions on $n=10$ and 11. Lemma 3.1 completes the construction.

At this point, we have $t<2n$ and $s>4n$. If $s\geq 5n$, write $t=g$, $s=hn+k$ where $3\leq h\leq n-4$, $0\leq g<2n$ and $2n\leq k\leq 4n$ and proceed as above.

For $n\geq 12$, what remains is just those types (t,s) with $0\leq t<2n$ and $4n<s<5n$; some additional omissions occur for $n=10$ and 11 as a result of the possible exceptions in $\mathcal{Y}(n,4)$. To handle these remaining values, we treat cases for n.

If $n\geq 16$, Lemma 5.1 ensures that all values in $\mathcal{Y}(n,4)$ can be constructed by an application of Lemma 3.2 in which n_1 and n_2 are chosen as equal as possible subject to $n_1\geq 12$ and $n_2\geq 4$. Hence we may assume that the 4 3-factors chosen partition $3G$, where G is a 4-regular spanning subgraph of $K_{n_1,n_1}\cup K_{n_2,n_2}$. One can 3-factorize $3(K_{n,n}-G)$ with type $(0,3n_1)$. Now writing $t=g$, $s=3n_1+k$ for the remaining values (t,s) guarantees that $k\leq 4n$. Moreover, $k>n+3n_2$. We must verify that $g\leq k$. Since $g<2n$ and $k>n+3n_2$, this holds provided $3n_2\geq n$ and hence holds for all $n\geq 16$.

Finally, we must treat small values of n. For $n=15$, the method for $n\geq 16$ leaves one possible omission, (1,64). Observe then that $K_{15,15}$ can be partitioned into three subgraphs isomorphic to $K_{5,5}\cup K_{5,5}\cup K_{5,5}$. Hence adding any nine types in $\mathcal{Y}(5,5)$ gives a type in $\mathcal{Y}(15,15)$. Since $\{(0,0),(0,20),(0,25),(1,19)\}\subseteq\mathcal{Y}(5,5)$, this constructs the required value.

For $n=14$, the general method leaves the possible exceptions of $\{(0,57),(1,57),(2,57),(1,61)\}$. $K_{14,14}$ can be partitioned into two factors, each isomorphic to $K_{7,7}\cup K_{7,7}$. Since $\mathcal{Y}(7,4)$ contains (0,0), (0,12), (0,21), (0,24), (1,24), (1,28) and (2,24), the required values are all constructed.

For $n=13$, we proceed similarly, applying Lemma 3.2 with $n_1=9$ and $n_2=4$. Since we are only concerned with types with $53\leq s\leq 64$, the only exceptions to handle are (1,54), (2,54), (3,54) and (5,54). The remaining 9-regular graph from this construction has a 3-factorization of type (0,27), and hence since $(0,12)\in\mathcal{Y}(9,4)$ and $\{(1,15),(2,15),(3,15),(5,15)\}\subseteq\mathcal{Y}(4,4)$, these omissions are constructed.

For $n=12$, employing the general method is complicated by the large number of omissions in $\mathcal{Y}(8,4)$. Hence we handle the types (t,s) with $49\leq s\leq 59$ and $0\leq t\leq 23$ in a different way. $K_{12,12}$ has a partition into three copies of $K_{4,4}\cup K_{4,4}\cup K_{4,4}$; hence we can add any nine types in $\mathcal{Y}(4,4)$ to obtain a type in $\mathcal{Y}(12,12)$. Now $\mathcal{Y}(4,4)$ contains (0,0), (0,12), (0,15), (0,16), (15,15), (16,16) and all types $\{(i,15),(i,16)\}$ with $0\leq i\leq 15$. Summing these types, we obtain all values with $s\in\{51,52,53,54,57,58,59\}$. Since $(4,4)\in\mathcal{Y}(4,4)$, we obtain in addition all types with $t\geq 4$ and $s\in\{49,50,55,56\}$. This leaves only $t\in\{0,1,2,3\}$ and $s\in\{49,50,55,56\}$. Next apply Lemma 3.2 with $n_1=7$ and $n_2=5$. The complement has a factorization of type (0,21); since $(0,9)\in\mathcal{Y}(7,4)$ and $\{(i,19),(i,20)\}\subseteq\mathcal{Y}(5,4)$ for $i\in\{0,1,2,3\}$, this constructs the remaining examples with $t\in\{49,50\}$. Finally, applying Lemma 3.2 with $n_1=8$ and $n_2=4$, we factor the complement with type (0,24); since $(0,16)\in\mathcal{Y}(8,4)$ and $\{(i,15),(i,16)\}\subseteq\mathcal{Y}(4,4)$ for $i\in\{0,1,2,3\}$, this settles the cases with $s\in\{55,56\}$ and completes the determination for $n=12$.

For $n=11$, we must handle all types (t,s) with $0\leq t\leq 21$ and $45\leq s\leq 54$, *in addition to* those arising from the possible omission of (1,31) in $\mathcal{Y}(11,4)$. We handle the latter cases first. Suppose in the earlier constructions that $g=1$ and $k=31$. Since (1,20) and (1,42) are both in $\mathcal{Y}(11,4)$, we could instead choose $k=20$ unless $h=n-4=7$; in this single case, we choose $k=42$ and decrement h by one. This leaves the exception of (1,31) itself. For this value, apply Lemma 3.2 with $n_1=6$ and $n_2=5$;

the bipartite complement is precisely the graph obtained by applying Lemma 3.3 with $n_1=n_2=6$. In this latter graph, we have type (0,9). Then taking $(0,0) \in \mathcal{Y}(5,5)$ and $(1,22) \in \mathcal{Y}(6,5)$ constructs (1,31) in $\mathcal{Y}(11,11)$. Now we must deal with the large block of omissions.

Apply Lemma 3.2 with $n_1=7$ and $n_2=4$. The 7-regular complement has a 3-factor which consists of two copies of a 3-cube together with $K_{3,3}$. Observing that (0,0), (0,9), (0,12), (0,21), (9,9), (9,21) and (12,12) all appear as types of factorizations of the complement, and that (0,21), (0,26), (0,27) and (0,28) all appear in $\mathcal{Y}(7,4)$, we add to types $\{(i,15),(i,16):0 \le i \le 15\} \subseteq \mathcal{Y}(4,4)$ to settle all remaining cases, except those with $s=47$. In this last case, we take (0,9) in the complement, $(0,12) \in \mathcal{Y}(4,4)$ and $(i,26) \in \mathcal{Y}(7,4)$ for $0 \le i \le 26$ to produce all types required with $s=47$.

For $n=10$, we treat the exceptions in the general constructions as above. To handle the omission of (0,27), we employ (0,17) and (0,37) which are present. For the omission of (2,27), we employ (2,17) and (2,37). However, for (1,27) we cannot employ (1,17), which is forbidden by the necessary conditions; this results in the further omission of (1,57) and (21,47). Similarly, for the omission (1,31), we cannot employ (1,41) and hence we obtain the further omissions of (1,91) and (21,91). Hence we have the four original omissions, four additional ones, and all types (t,s) with $0 \le t \le 19$ and $41 \le s \le 49$ to handle. Since $K_{10,10}$ can be partitioned into four copies of $K_{5,5}$, we require some knowledge of $\mathcal{Y}(5,5)$. In [1], a complete solution is given for the complete graph $3K_6$; applying Construction E from [1] (as corrected earlier here), we find (at least) that all types (t,s) with $0 \le t \le 25$ and $\max(t,20) \le s \le 25$ appear in $\mathcal{Y}(5,5)$. Hence adding any four types in $\mathcal{Y}(5,5)$ produces all of the required values except the four original exceptions (remark that for (1,57) we are also employing that $(0,15) \in \mathcal{Y}(5,5)$).

Let us then handle the last four omissions. Let S_6 be the 4-regular spanning subgraph of $K_{6,6}$ whose bipartite complement forms two 6-cycles. Since S_6 is 1-factorable, $3S_6$ has a 3-factorization of type (0,0). Now apply Lemma 3.2 with $n_1=6$ and $n_2=4$, so that the subgraph of $K_{6,6}$ covered is S_6. Then the 6-regular complement left by Lemma 3.2 can be partitioned into two 3-factors, each consisting of two copies of $K_{3,3}$ and one 3-cube. Hence the complement can be factored with type (0,12). Then since (0,15), (1,15) and (2,15) appear in $\mathcal{Y}(4,4)$, the types (0,27), (1,27) and (2,27) in $\mathcal{Y}(10,10)$ are handled. If instead one chooses the 4-regular subgraph of $K_{6,6}$ to have bipartite complement consisting of three 4-cycles, one can factor the resulting 6-regular complement with type (0,15). Then since (1,16) is in $\mathcal{Y}(4,4)$, (1,31) is in $\mathcal{Y}(10,10)$.

We have established that for all $n \ge 10$, $\mathcal{B}(n,n) \subseteq \mathcal{Y}(n,n)$. Together with Lemma 2.1, this proves the Main Theorem.

7. Intermediate Cases

We have obtained a complete solution thus far for $n=3$ and 4, and for $n \ge 10$. In this section, we outline what remains for the intermediate cases.

For $n=9$, applying Lemma 3.1 to the solutions in $\mathcal{Y}(9,4)$ as in the proof of the Main Theorem leaves all values (t,s) for $0 \le t \le 17$ and $37 \le s \le 44$, in addition to those resulting from the possible exceptions in $\mathcal{Y}(9,4)$, namely (0,22), (1,22), (2,22), (3,22), (5,22), (0,24), (1,24), (2,24), (1,27), (2,27), (3,27), (5,27), (18,40), (19,40), (20,40), (21,40), (23,40), (0,49), (1,49), (2,49), (3,49) and (5,49). Earlier we observed that Construction E of [1] can be applied to produce many types in $\mathcal{Y}(5,5)$; applying Lemma 3.3 to the resulting rectangles produces values in $\mathcal{Y}(9,5)$. In this way, we immediately handle all possible exceptions with $37 \leq s \leq 44$. Moreover, the complement remaining contains $K_{4,4}$, and hence we obtain the types with $s=49$, and the remaining values with $s=27$, along with (0,24). Hence the only types whose membership in $\mathcal{Y}(9,9)$ is in doubt are: (0,22), (1,22), (2,22), (3,22), (5,22), (1,24) and (2,24).

For $n=8$, there is a large number of omissions in $\mathcal{Y}(8,4)$. We employed three constructions here, in addition to those used already for $\mathcal{Y}(8,4)$. First of all, Lemma 3.1 applies to extend those values in $\mathcal{Y}(8,4)$. Second, we observe that $K_{8,8}$ has a partition into four copies of $K_{4,4}$ and hence we can add any four types in $\mathcal{Y}(4,4)$ to obtain a type in $\mathcal{Y}(8,8)$. Third, remark that Lemma 3.3 with $n_1=5$ and $n_2=4$ partitions a 4-regular subgraph of $K_{8,8}$ which is isomorphic to its bipartite complement, and hence we can add any two types resulting from this application of Lemma 3.3 to obtain a type in $\mathcal{Y}(8,8)$.

Many exceptions remain. Hence we resort to constructions of factorizations of $3K_{2n,2n}$ from 3-factorizations of $3K_{2n}$. Our basic device is Construction C of [1]: whenever there is a 3-factorization of $3K_{2n}$ of type (t,s), there is a 3-factorization of $3K_{2n,2n}$ of type $(2t,2s)$. This can be done quite simply. Given a 3-factorization of $3K_{2n}$ on symbols $1,\cdots,2n$, form a 3-factorization of $3K_{2n,2n}$ on symbols $\{x_i,y_i: i=1,\cdots,2n\}$. In the kth 3-factor, connect x_i and y_j the same number of times that i and j are connected in the kth factor of $3K_{2n}$. Finally, add the triplicated 1-factor $\{\{x_i,y_i\}: i=1,\cdots,2n\}$.

By exchanging edges from the triplicated 1-factor, we can achieve many different values. For example, suppose that the kth factor of $3K_{2n}$ contains $\{i,j\}$ as a singly repeated edge. Then in the 3-factorization of $3K_{2n,2n}$, we "trade" edges $\{x_i,y_j\}$ and $\{x_j,y_i\}$ of the kth factor with edges $\{x_i,y_i\}$ and $\{x_j,y_j\}$ of the $2n$th factor to obtain a 3-factorization of type $(2t+2,2s+2)$. In fact, since this can be done on disjoint singly repeated edges, we also obtain types $(2t+4,2s+4)$ and $(2t+6,2s+6)$ in this way. In a similar vein, if $\{i,j\}$ is singly repeated in a factor and $\{i,k\}$ is singly repeated in a different factor, trading twice as above gives type $(2t+2,2s+3)$. If instead $\{i,k\}$ is singly repeated in the same factor, we obtain type $(2t+3,2s+3)$. If $\{i,j\}$, $\{i,k\}$ and $\{j,k\}$ are all singly repeated in different factors, trading three times as above gives type $(2t,2s+3)$. If $\{i,j\}$, $\{i,\ell\}$, $\{j,k\}$ and $\{k,\ell\}$ appear singly in factors, so that incident edges are in different factors, four such trades gives type $(2t,2s+4)$.

After applying this generalized construction to the known 3-factorizations of $3K_8$ in [1], the following exceptions remain in $\mathcal{Y}(8,8)$:

t	Omissions for s
0	18-23, 25, 26
1	19-26, 33-38, 49
2	17-25
3	18-26, 34, 50
4	18, 21-23, 25
5	17, 18, 21-25
6	17
7	17
8	17
9	17

Now turning to $n=7$, we observe first that applying Lemma 3.3 with $n_1=n_2=4$ leaves uncovered a 3-regular graph which is $K_{3,3}$ and a 3-cube. Hence we can add a type in $\mathcal{Y}(3,3)$, a type in $\mathcal{Y}(4,3)$ and a type resulting from Lemma 3.3 to obtain a type in $\mathcal{Y}(7,7)$. We can also apply Construction E of [1] to the 3-factorizations of $3K_8$ given in [1]; after doing so, we leave only the following possible exceptions:

t	Omissions for s
0	15-20, 22, 23, 25, 32, 43, 44
1	15, 16, 18-23, 25, 32, 34, 41, 43, 44, 46
2	14-22, 41, 43
3	12, 14-23, 34, 43, 44
4	14, 15, 17, 18, 20, 22
5	15, 16, 18-22, 43
6	13, 14, 17
7	13, 14, 15, 17, 18, 22
8	14
9	13, 14, 16, 17
10	14
11	14

For $n = 6$, we can add four types in $\mathcal{Y}(3,3)$ to obtain a type in $\mathcal{Y}(6,6)$. We can also apply Construction C of [1] to the 3-factorizations of $3K_6$; in fact, we use the augmented Construction outlined above for $n=8$. Combining all of these techniques, we leave a large number of possible exceptions, as follows:

t	Omissions for s
0	12, 14-16, 19, 21, 23, 25, 27, 29, 31-35
1	15, 16, 18-21, 23-36
2	14-19, 23, 25, 27, 29, 31, 33-36
3	14-16, 18-21, 23, 24, 34-36
4	12-16, 19, 20, 23, 25, 35
5	14-19, 24, 34, 36
6	11-14, 16, 19, 23
7	12-16, 19, 20, 23, 24, 34
8	9, 11, 12, 14, 16, 19, 23, 25
9	11-16, 34
10	19
11	19, 20, 22, 34
12	19
13	20
17	17, 18, 20
29	29, 30
31	31
33	34
35	35, 36

Finally, earlier results, together with the application of Construction E of [1], leaves the following possible exceptions for $n=5$:

t	Omissions for s
0	9, 12, 14, 17
1	15, 16, 18
2	14-16
3	12, 14-17
4	9, 12-14, 16
5	14-16
6	8, 9, 11-13, 16
7	8, 9, 12-17
8	8, 11, 16
9	11, 13

We expect that most of the omissions left for $5 \leq n \leq 9$ could in fact be constructed.

We should also note that the results here can in fact be used to eliminate some of the possible exceptions left in the case of complete graphs in [1].

8. Concluding Remarks

Of most interest in the results here are the differences between complete graphs and complete bipartite graphs. For complete graphs, we find only forty-three omitted types while here we find sixty-six. Nevertheless, the same conclusion holds: if t or s is large enough, there are no exceptions.

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