Semibandwidth of Bipartite Graphs and Matrices

Richard A. Brualdi, Kevin F. McDougal

Department of Mathematics
University of Wisconsin
Madison, WI 53706
U.S.A.

Abstract. We define the semibandwidth of a bipartite graph (whose bipartition is specified), which is a bipartite analogue of the bandwidth of a graph, and develop some of its properties. The motivation for this concept comes from the question of transforming a matrix by row and column permutations to as close to triangular form as possible.

1. Introduction.

Let $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_n\}$ be disjoint sets with m and n elements respectively, and let $G = (X, Y; E)$ be an (m, n)-bipartite graph with vertex set $V = X \cup Y$ and edge set $E \subseteq \{\{x_i, y_j\}: 1 \le i \le m, 1 \le j \le n\}$. A *bipartite numbering* of G is a pair (σ, τ) where

$$\sigma: X \to \{1, \dots, m\} \quad \text{and} \quad \tau: Y \to \{1, \dots, n\}$$

are bijections. The *semibandwidths of the bipartite numbering* (σ, τ) of G are defined by

$$sb_X(G; \sigma, \tau) = \max\{\tau(y) - \sigma(x): x \in X,\ y \in Y,\ \{x, y\} \in E\}$$
$$sb_Y(G; \sigma, \tau) = \max\{\sigma(x) - \tau(y): x \in X,\ y \in Y,\ \{x, y\} \in E\}.$$

We observe that $-m \le sb_X(G; \sigma, \tau) \le n-1$ (if $E = \phi$, we define $sb_X(G; \sigma, \tau) = -m$), and $-n \le sb_Y(G; \sigma, \tau) \le m - 1$. The *semibandwidths* of G are the numbers

$$sb_X(G) = \min sb_X(G; \sigma, \tau)$$

and

$$sb_Y(G) = \min sb_Y(G; \sigma, \tau)$$

where the minima are taken over all bipartite numberings (σ, τ) of G. A bipartite graph, if not connected, does not have a unique bipartition. The semibandwidths depend on the choice of bipartition. Throughout, a bipartite graph always refers to a bipartite graph with a specified bipartition. The semibandwidths of a bipartite graph are bipartite analoques of the bandwidth of a graph [1, 2]. A bipartite numbering (σ, τ) for which $sb_X(G; \sigma, \tau) = sb_X(G)$ is said to *realize the semibandwidth* $sb_X(G)$.

Research partially supported by National Science Foundation Grant No. DMS-8521521.

It follows from the definition of semibandwidth that

$$sb_X(G) = sb_Y(G) = n - m. \tag{1.1}$$

Because of (1.1) we will confine our attention to the semibandwidth $sb_X(G)$ in the case that $m \leq n$. Suppose $m < n$ and let Z be a set disjoint from $X \cup Y$ with $|Z| = n - m$. If G' is the bipartite graph $(X \cup Z, Y; E)$, then $sb_X(G) = sb_{X \cup Z}(G') + (n - m)$. Thus in general discussion of the semibandwidth we may also assume that $m = n$.

Let $A = [a_{ij}]$ be the m by n *reduced adjacency* matrix of $G = (X, Y; E)$ determined by the orderings $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ of the vertices in X and Y. This matrix A is defined by: $a_{ij} = 1$ if and only if $\{x_i, y_j\}$ is an edge of G $(i = 1, \ldots, m;\ j = 1, \ldots, n)$. Let P and Q be the permutation matrices corresponding to the bijections σ and τ of the bipartite numbering (σ, τ). Let k be an integer with $-m \leq k \leq (n-1)$. The kth *diagonal* of A is the set of positions (i, j) with $j - i = k$. Then $sb_X(G; \sigma, \tau) \leq k$ if and only if there are permutation matrices P and Q such that all the nonzero entries of PAQ occur on or below its kth diagonal, that is PAQ has the form

$$\left[\overbrace{\ast\ \ }^{k+1}\!\!\diagdown\ 0\right]. \tag{1.2}$$

Hence $sb_X(G) \leq 1$ if and only if the rows and columns of A can be permuted to achieve a lower triangular matrix. The semibandwidth is one measurement of the closeness of A to a triangular matrix. If $-(m-1) \leq k < 0$, then (1.2) is to be interpreted as

$$-k\{\begin{bmatrix} \diagdown & 0 \\ \ast & \diagdown \end{bmatrix}. \tag{1.3}$$

In the next section we discuss some of the elementary properties of semibandwidth. In Section 3 we show how to determine the semibandwidth of a bipartite graph which is the pairwise disjoint union of complete bipartite graphs. In the final section we introduce a measure of how close a bipartite graph is to having semibandwidth at most equal to an integer k. In the case of a direct sum of two complete bipartite graphs, we show how to attain this measure.

We conclude this introduction by formulating the problem of the determination of the semibandwidth $sb_X(G)$ as an embedding problem. Let $H = (X, Y; F)$ be a bipartite graph. Let

$$\varepsilon_{ij} = \begin{cases} 1 & \text{if } j > i \\ -1 & \text{if } j \leq i \end{cases} \quad (i = 1, \ldots, m;\ j = 1, \ldots, n).$$

Let d_{ij} be the distance in H between x_i and y_j. We define the signed distance between x_i and y_j to be the number $\varepsilon_{ij}d_{ij}$. Now let k be an arbitrary integer. We define the kth *bipartite power of* H *with respect to* X to be the bipartite graph $H_X^{(k)} = (X, Y; F')$ in which $\{x_i, y_j\} \in F'$ if and only if $\varepsilon_{ij}d_{ij} \leq 2k - 1$ $(i = 1, \ldots, m;\ j = 1, \ldots, n)$.

Let $P_{n,n}$ be the path drawn in Figure 1.

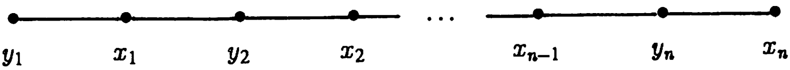

Figure 1

The powers $P_{3,3}^{(k)}$ are drawn in Figure 2.

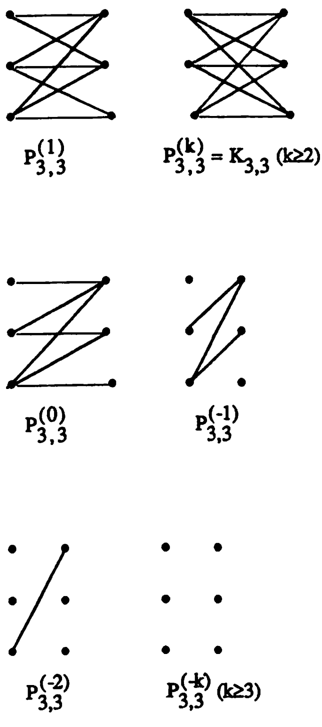

Figure 2

It follows that for an (n, n)-bipartite graph G the semibandwidth $sb_X(G)$ equals the smallest integer k with $-n \leq k \leq n - 1$ such that there is an embedding of G into the graph $P_{n,n}^{(k)}$.

2. Semibandwidth.

Our discussions are usually formulated in terms of m by n $(0,1)$-matrices A. We let $s'(A)$ be the smallest integer k such that A has the form (1.2) (or (1.3)), and we call

$$s(A) = \min\{s'(PAQ) : P, Q \text{ permutation matrices}\}$$

the *semibandwidth* of A. If P, Q are permutation matrices for which $s(A) = s'(PAQ)$, then PAQ is said to *realize the semibandwidth of* A. We have $-m \leq s(A) \leq n-1$ and by (1.1) $s(A) - s(A^T) = n - m$. We remark that if $s(A) \geq 0$, then $n - s(A)$ is the largest order of a square submatrix of A whose rows and columns can be permuted to give a triangular matrix. If $s(A) < 0$, then we may delete $m + s(A)$ zero rows of A and $n + s(A)$ zero columns and obtain a matrix whose rows and columns can be permuted to give a triangular matrix.

Unlike the bandwidth of a graph the semibandwidth of a bipartite graph does not in general equal the maximum of the semibandwidths of its connected components. For example, let

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then $s(A_1) = 0$ and $s(A_2) = 1$; however $s(A_1 \oplus A_2) = 0$ because we may permute the rows and columns of $A_1 \oplus A_2$ and obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Let p and q be the minimum row sum and maximum column sum of the m by n $(0,1)$-matrix A. It follows from definition that $s(A) \geq \max\{p-1, q-m\}$.

We now determine the semibandwidth of trees. Because a tree T is a connected bipartite graph there is a unique bipartition of its vertices into sets X and Y such that each edge of T is of the form $\{x, y\}$ for some x in X and y in Y.

Theorem 2.1. *Suppose that the m by n $(0,1)$-matrix A is the reduced adjacency matrix of a tree T. Then*

$$s(A) = n - \min\{m, n\}. \tag{2.1}$$

Proof: Let $T = (X, Y; E)$. If $m = 1$ or $n = 1$ then (2.1) clearly holds. We now assume that $m, n \geq 2$ and proceed by induction on $m + n$. Without loss of generality we assume that $m \geq n$. We then need to show that $s(A) = 0$. Because $m \geq n$, the set X contains a pendant vertex of T. We may permute the rows and columns of A and obtain

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ \star & & & \\ \vdots & & A' & \\ \star & & & \end{bmatrix}$$

where A' is the reduced adjacency matrix of a forest F with the trees $T_1, \dots, T_r$ as connected components. Let A_i be the m_i by n_i reduced adjacency matrix of $T_i (i = 1, \dots, r)$. We note that n_i may be 0 but this causes no difficulties in the argument that follows. By the inductive assumption $s(A_i) = n_i - \min\{m_i, n_i\}$. Let $d_i = m_i - n_i$ where we may assume that $d_1 \geq \dots \geq d_r$. We observe that $d_1 + \dots + d_r = m - n \geq 0$. There exist permutation matrices P and Q such that

$$PAQ = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline \star & B_1 & & 0 \\ \vdots & 0 & \ddots & \\ \star & & & B_r \end{array}\right]. \tag{2.2}$$

where B_i is an m_i by n_i matrix of one of the forms

$$\begin{bmatrix} \star & & 0 \\ & \ddots & \\ & & \star \\ & \star & \end{bmatrix} (m_i \geq n_i), \begin{bmatrix} & \star & & 0 \\ \star & & \ddots & \\ & & & \star \end{bmatrix} (m_i \leq n_i). \tag{2.3}$$

For $j = 1, \dots, r-1$ we have

$$0 \leq m - n = d_1 + \dots + d_r \leq d_1 + \dots + d_j + (r - j) d_{j+1}. \tag{2.4}$$

If $d_1 + \dots + d_j < 0$, then $d_j < 0$ and hence $d_{j+1} < 0$, and we contradict (2.4). Hence

$$d_1 + \dots + d_j \geq 0 \quad (j = 1, \dots, r). \tag{2.5}$$

It follows from (2.5) that both the $(1,1)$-entry and the (m_i, n_i)-entry of B_i are on or below the main diagonal of the matrix in (2.2) $(i = 1, \dots, r)$. If $m_i \geq n_i$ (respectively, $m_i < n_i$), we use the fact that the $(1,1)$-entry (respectively, (m_i, n_i)-entry) of B_i is on or below the main diagonal of (2.2) to conclude that the matrix in (2.2) has only 0's above its main diagonal. Hence $s(A) \leq 0$. But clearly $s(A) \geq 0$, and the induction is complete. ∎

Corollary 2.2. *Suppose that the m by n $(0,1)$-matrix A is the reduced adjacency matrix of a forest. Then*

$$s(A) \leq n - \min\{m, n\}.$$

We now consider the semibandwidth of a *regular bipartite graph of degree k.* Such graphs have a square reduced adjacency matrix A with exactly k 1's in every row and column. If $k = 1$, then clearly $s(A) = 0$. Now suppose that $k = 2$. Then there are permutation matrices P and Q such that PAQ is the direct sum of matrices of the form

$$\begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & 0 & \\ & & \ddots & \ddots & \\ & 0 & & \ddots & 1 \\ 1 & & & & 1 \end{bmatrix},$$

and it follows that $s(A) = 1$.

We now show that for $k \geq 3$, there exist matrices with exactly k 1's in each row and column having arbitrarily large semibandwidth. We recall that the *girth* g of a graph G equals the smallest length of a cycle of G. We first show that the girth of a regular bipartite graph determines a lower bound for the semibandwidth.

Lemma 2.3. *Let A be the reduced adjacency matrix of a regular bipartite graph G of degree k and girth g. If $k \geq 3$, then*

$$s(A) \geq \frac{(k-2)(g-2)}{2}. \tag{2.5}$$

Proof: Let

$$t = \lceil s/(k-2) \rceil \tag{2.6}$$

where $s = s(A)$. Let B realize the semibandwidth of A. Then the 1's of B in its first t rows are contained in the first p columns where $p = \min\{s + t, n\}$. Let B' be the leading t by p submatrix of B. Then B' is the reduced adjacency matrix of a bipartite graph with $t + p$ vertices and kt edges. Using (2.6) we obtain

$$kt \geq s + 2t \geq t + p.$$

Hence G' has at least as many edges as vertices and hence contains a cycle. Because G' is a (t,p)-bipartite graph, every cycle of G' has length of most $2t$. It follows that $g \leq 2t$. Using (2.6) we now obtain (2.5). ∎

We now show that there exist regular bipartite graphs of arbitrarily large girth.

Lemma 2.4. *Let $k \geq 2$ be an integer, and let c be an arbitrary positive even integer. Then there exists a regular bipartite graph of degree k with girth equal to c.*

Proof: It follows from Theorem 1 of Sauer [3] that there exists a regular graph H of degree k with girth c. Let $V = \{v_1, \ldots, v_n\}$ be the set of vertices of H. Let $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ be two disjoint sets of cardinality n. We define an (n, n)-bipartite graph $G = (X, Y; E)$ whose edges are obtained as follows: if $\{v_i, v_j\}$ is an edge of H, then $\{x_i, y_j\}$ and $\{x_j, y_i\}$ are edges of G $(1 \leq i, j \leq n)$. Thus the (symmetric) adjacency matrix of H is the reduced adjacency matrix of G. The bipartite graph G is regular of degree k. From a cycle of odd length t in H we obtain a cycle of length $2t$ in G. From a cycle of even length t in H we obtain two cycles of length t in G. On the other hand from a cycle of length ℓ in G we obtain a cycle of length at most ℓ in H. Because H has even girth c, we now conclude that G also has girth c. ∎

Theorem 2.5. *Let $k \geq 3$ be an integer. Then there exist $(0, 1)$-matrices with exactly k 1's in each row and column having arbitrarily large semibandwidth.*

Proof: The theorem follows by applying Lemma 2.3 and Lemma 2.4. ∎

3. Disjoint unions of complete bipartite graphs.

In this section we show how to realize the semibandwidth of a bipartite graph each of whose connected components is a complete bipartite graph K_{ab}. The reduced adjacency matrix of K_{ab} is the all 1's matrix J_{ab} of size a by b. We first consider the case of two complete bipartite graphs for which there is a simple expression for the semibandwidth.

Theorem 3.1. *Let a, b, m, and n be integers with $0 < a < m$, $0 < b < n$, and $m \leq n$. Let $A = J_{ab} \oplus J_{m-a,n-b}$. Then*

$$s(A) = -1 + \min\{\max\{n - a, b\}, \max\{n - m + a, n - b\}\}.$$

Proof: Let

$$B = \begin{bmatrix} & \overbrace{0 \quad \cdots \quad 0}^{s+1} \\ & \ddots \quad \vdots \\ & 0 \\ \star & \end{bmatrix} \tag{3.1}$$

realize the semibandwidth of A where $s = s(A)$. We may permute the first $s + 1$ columns of B and assume that the 1's in row 1 occur in the leftmost columns. We may then permute the rows of B so that the rows identical to row 1 are the uppermost rows and obtain a matrix which also realizes the semibandwidth of A. It follows that one of $J_{ab} \oplus J_{m-a,n-b}$, and $J_{m-a,n-b} \oplus J_{ab}$ realizes the semibandwidth of A. If it is $J_{ab} \oplus J_{m-a,n-b}$, then $s = \max\{b-1, n-a-1\}$. If it is $J_{m-a,n-b} \oplus J_{ab}$, then $s = \max\{n-b-1, n-(m-a)-1\}$. ∎

In order to determine how to realize the semibandwidth in the case of more than two complete bipartite graphs, we further discuss the realization of the semibandwidth of the matrix A above.

We call an all 1's matrix J_{pq} *horizontal, square,* or *vertical* according as $p < q$, $p = q$, or $p > q$. We show that in realizing the semibandwidth of A

(i) a vertical block precedes a square block precedes a horizontal block,
(ii) in the case of two vertical (respectively, horizontal) blocks the one with the smallest number of columns (respectively, rows) comes first (respectively, second), and
(iii) in the case of two square blocks both orderings realize the semibandwidth.

We choose our notation so that $s'(J_{ab} \oplus J_{m-a,n-b}) = s(A)$. Hence

$$\max\{b-1, n-a-1\} \leq \max\{n-b-1, n-(m-a)-1\}.$$

Case 1. $n-b-1 \leq n-(m-a)-1$.

We have $m-a \leq a, b, n-b$ and hence $J_{m-a,n-b}$ is either horizontal or square. If J_{ab} is vertical, there is nothing to verify. If J_{ab} and $J_{m-a,n-b}$ are both horizontal, then the inequality $m-a \leq a$ is in agreement with (ii). If J_{ab} and $J_{m-a,n-b}$ are both square, then $s'(J_{ab} \oplus J_{m-a,n-b}) = s'(J_{m-a,n-b} \oplus J_{ab})$ as required by (iii). We are left with the possibility that J_{ab} is horizontal and $J_{m-a,n-b}$ is square. Thus $m-a = n-b$ and $b > a$. We have

$$s'(J_{ab} \oplus J_{m-a,n-b}) = \max\{b-1, n-a-1\} = \max\{b-1, m-b-1\} \quad (3.2)$$

and

$$s'(J_{m-a,n-b} \oplus J_{ab}) = \max\{n-b-1, n-(m-a)-1\} = \max\{n-b-1, b-1\}. \quad (3.3)$$

Because $m-a \leq b$ and $m-a = n-b$, we have $2b \geq n$ and hence $b-1 \geq n-b-1 \geq m-b-1$, and hence (3.2) and (3.3) have the common value $b-1$, in conformity with (i).

Case 2. $n-(m-a)-1 < n-b-1$.

We have $b \leq a$, $n-b$ and $b < m-a$ and hence J_{ab} is either vertical or square. Because $m \leq n$, the matrix $J_{m-a,n-b}$ is either square or horizontal, and the conclusions easily follow in this case.

If $m > n$ we apply the above analysis to A^t.

If A is a direct sum of any number of all 1's matrices, then the following theorem shows how to realize the semibandwidth of A.

Theorem 3.2. *Let $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ be positive integers, and let $A = J_{a_1 b_1} \oplus \ldots \oplus J_{a_k b_k}$. Let*

$$V = \{i: 1 \leq i \leq k,\ a_i > b_i\}$$
$$S = \{i: 1 \leq i \leq k,\ a_i = b_i\}$$
$$H = \{i: 1 \leq i \leq k,\ a_i < b_i\}.$$

Let $i_1, \ldots, i_k$ be a permutation of $1, \ldots, k$ satisfying

$$i_p \in V \cup S,\ i_q \in H \text{ implies } p < q, \tag{3.4}$$

$$i_p \in V,\ i_q \in S \cup H \text{ implies } p < q, \tag{3.5}$$

$$i_p, i_q \in V,\ b_{i_p} < b_{i_q} \text{ imply } p < q, \tag{3.6}$$

$$i_p, i_q \in H,\ a_{i_p} > a_{i_q} \text{ imply } p < q. \tag{3.7}$$

Then

$$J_{a_{i1} b_{i1}} \oplus \ldots \oplus J_{a_{ik} b_{ik}} \tag{3.8}$$

realizes the semibandwidth of A.

Proof: It follows by induction using an argument like that in the proof of Theorem 3.1 that some matrix X of the form (3.6) realizes the semibandwidth of A. Suppose that in this realization there were two consecutive blocks which violated one of (3.4) to (3.7). It follows from the analysis preceding the statement of the theorem that we may interchange the order of these two blocks and obtain a matrix Y with $s'(Y) \leq s'(X)$. A finite number of interchanges leads to a realization of the semibandwidth of A satisfying (3.4) to (3.7). ∎

We conclude this section with an example illustrating Theorem 3.2. Let A be the direct sum of $J_{5,2}$, $J_{4,3}$, $J_{5,5}$, $J_{2,4}$ and $J_{1,3}$. Then

$$s(A) = s'(J_{5,2} \oplus J_{4,3} \oplus J_{5,5} \oplus J_{2,4} \oplus J_{1,3}).$$

4. Triangular numbers of a matrix.

Let A be an m by n $(0,1)$-matrix, and let k be an integer with $-(m-1) \leq k \leq n$. Let $t'_k(A)$ equal the number of 1's on or above the kth diagonal of A. We define the kth *triangular number* $t_k(A)$ by

$$t_k(A) = \min\{t'_k(PAQ)\}$$

where the minimum is taken over all permutation matrices P and Q of orders m and n, respectively. It follows that the semibandwidth of A satisfies

$$s(A) = -1 + \min\{k: t_k(A) = 0\}.$$

Let a and b be integers with $1 \le a, b < n$. The remainder of this section is concerned with the triangular numbers of the matrix $A = J_{ab} \oplus J_{n-a,n-b}$. Specifically we determine a matrix $P_k A Q_k$ where P_k and Q_k are permutation matrices such that

$$t_k(A) = t'_k(P_k A Q_k) \quad (-(n-1) \le k \le n).$$

Without loss of generality we assume that $b \ge a,\ n-a, n-b$. The following three cases arise: $b \ge 2a$; $b \ge 2(n-a)$; and $b < 2a, b < 2(n-a)$. These three cases can be described using Figure 3.

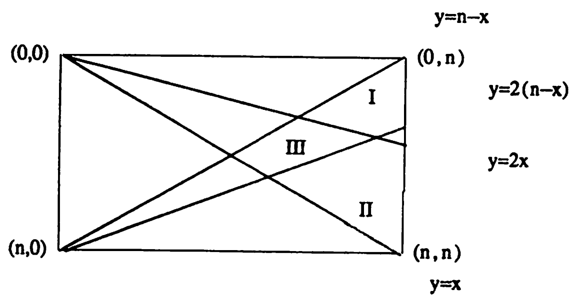

Figure 3. A partitioned n by n square

If we position an a by b rectangle in the above figure in such a way that its upper left vertex coincides with the point $(0,0)$ then its lower right vertex is at the point (a,b) in the rectangle which falls into one of the three regions I, II, or III (because $b \ge a$ and $b \ge n-a$). Thus the three cases described above correspond to the three regions I, II, and III.

Theorem 4.1. *Let a, b, and n be positive integers with $n > b \ge a,\ n-a, n-b$. Let $A = J_{n-a,n-b} \oplus J_{ab}$ and let k be an integer with $-(n-1) \le k \le n$. Then there is a matrix*

$$B_k = \begin{bmatrix} J_{n-a,r} & 0 & J_{n-a,s} \\ 0 & J_{ab} & 0 \end{bmatrix} \quad (r+s = n-b) \tag{4.1}$$

with $t_k(A) = t'_k(B_k)$ where

$r = 0$ *if one of the following holds:* (4.2)

I(i) $b \ge 2(n-a)$ *and* $k \le b - 2(n-a)$,

II(i) $b \ge 2a$ *and* $k \le n - 2(n-a)$,

III(i) $b < \min\{2(n-a), 2a\}$ *and* $k \le \left\lfloor \frac{b+1}{2} \right\rfloor - (n-a)$.

$s = 0$ *if one of the following holds:* (4.3)

I(ii) $b \geq 2(n-a)$ *and* $k \geq n-2(n-a)+1$,

II(ii) $b \geq 2a$ *and* $k \geq 2a-b-1$,

III(ii) $b < \min\{2(n-a), 2a\}$ *and* $k \geq a - \left\lfloor \frac{b-1}{2} \right\rfloor$.

r *and* s *are chosen so that* (4.4)

I(iii) *If* $b \geq 2(n-a)$ *and* $b-2(n-a) < k < n-2(n-a)$, *then the* k*th diagonal of* B_k *contains the* $(n-a-1)$ *th* 1 *from the right in the first row of* J_{ab} *(the* $(n-a)$ *th* 1 *works as well).*

II(iii) *If* $b \geq 2a$ *and* $n-2(n-a) < k < 2a-b-1$, *then the* k*th diagonal of* B_k *contains the* $(a+1)$ *th* 1 *from the left in the first row of* J_{ab} *(the* $(a+2)$ *th* 1 *works as well).*

III(iii) *If* $b < \min\{2(n-a), 2a\}$ *and* $\lfloor \frac{b+1}{2} \rfloor - (n-a) < k < a - \lfloor \frac{b-1}{2} \rfloor$, *then the* k*th diagonal of* B_k *contains the* $\lceil \frac{b-1}{2} \rceil$ *th* 1 *from the right in the first row of* J_{ab} *(the* $\lfloor \frac{b-1}{2} \rfloor$ *th* 1 *works as well).*

Before sketching the proof of Theorem 4.1 we give a different description of the conclusions of Theorem 4.1. We define the *critical* 1 of the matrix J_{ab} to be the 1 in the first row of J_{ab} which is

the $(n-a-1)$th from the right, if $b \geq 2(n-a)$;
the $(a+1)$th from the left, if $b \geq 2a$;
the $\lceil (b-1)/2 \rceil$th from the right, if $b < \min\{2(n-a), 2a\}$.

We note that since $b < n$, we cannot have both $b \geq 2(n-a)$ and $b \geq 2a$. If $r = 0$ in B_k, the critical 1 is above the kth diagonal of B_k. If $s = 0$, the critical 1 is below the kth diagonal of B_k. If $rs > 0$, the critical 1 is on the kth diagonal of B_k.

Sketch of the proof of Theorem 4.1: The theorem holds if $n = 2$, and the proof proceeds by induction on n. The difficulty in the proof is in showing that $t_k(A) = t'_k(B_k)$ for some matrix B_k of the form (4.1). Let C be any matrix obtained from A by row and column permutations. If $k \geq 1$, we delete row n and column 1 of C; if $k \leq 0$, we delete row 1 and column n. After row and column permutations the resulting matrix D is of the form $J_{n-1-c,n-1-d} \oplus J_{cd}$ for some integers c and d with $c = a$ or $a-1$ and $d = b$ or $b-1$. We let $\ell = k-1$ if $k \geq 1$ and $\ell = k+1$ if $k \leq 0$. It is not too difficult to show that it is enough to complete the induction in the case that $d \geq c$, $n-1-c$, $n-1-d$. It follows from the inductive assumption and our choice of row and column deleted from C that we may permute the rows and columns of C without decreasing the number of 0 's on or above the kth diagonal of C to obtain either

$$\left[\begin{array}{c|c} \beta & E_\ell \\ \hline & \alpha \end{array}\right] \quad (k \geq 1) \tag{4.5}$$

or

$$\left[\begin{array}{c|c} \alpha & \\ \hline E_\ell & \beta \end{array}\right] \quad (k \leq 0) \tag{4.6}$$

where

$$E_\ell = \begin{bmatrix} J_{n-1-c,p} & 0 & J_{n-1-c,q} \\ 0 & J_{cd} & 0 \end{bmatrix} \quad (p+q=n-1-d), \tag{4.7}$$

$$t_\ell(D) = t'_\ell(E_\ell), \tag{4.8}$$

and

$$\begin{array}{l} \text{one of (4.2), (4.3), and (4.4) holds} \\ \text{with } n, a, b, k, r, s \text{ replaced by } n-1, c, d, \ell, p, q. \end{array} \tag{4.9}$$

We only consider the case corresponding to I(iii). Thus we assume that

$$d \geq 2(n-1-c), d-2(n-1-c) < \ell < n-1-2(n-1-a) \tag{4.10}$$

and the ℓth diagonal of E_ℓ contains the $(n-1-c)$th 1 from the right in the first row of J_{cd}. In this case $\ell \geq 1$. We might have $k = 0$, but we only treat the case $k > 0$. Thus we have the situation of (4.5), namely

$$\begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c|c|c|c} & p & d & q \\ \beta_1 & J & 0 & J \\ \hline \beta_2 & 0 & J & 0 \\ \hline \star & \alpha_1 & \alpha_2 & \alpha_3 \end{array} \quad \begin{array}{c} \\ n-1-c \\ c \\ \\ \end{array} \quad . \tag{4.11}$$

Here J denotes an all 1 's matrix of an appropriate size. If $\beta_2 = 0$ and $\alpha_2 = J$, then (4.11) has the form (4.1). Suppose $\alpha_2 = 0$. Interchanging row $n-c$ and row n of (4.11) does not decrease the number of 0 's on or above the kth diagonal (because we lose q 0 's and gain $n-1-c$ 0 's, and $q \leq n-1-d \leq n-1-c$). If $\beta_2 = J$, we also interchange column 1 and column $p+1$ without any effect on the 0 's on or above the kth diagonal. We conclude that $t_k(A) \leq t'_k(B_k)$ for some matrix B_k of the form (4.1).

The arguments for the other eight cases are similar. At this point we know that $t_k(A) = t'_k(B_k)$ for some matrix B_k of the form (4.1). It is now an elementary exercise to show that $t'_k(B_k)$ is maximized by choosing r and s according to the rules given in (4.2), (4.3), and (4.4). ∎

References

1. P.Z. Chinn, J. Chvátalová, A.K. Dewdney, and N.E. Gibbs, *The bandwidth problem for graphs and matrices — a survey*, J. Graph Theory **6** (1982), 223-254.
2. F.R.K. Chung, *Labelings of graphs*, in "Selected Topics in Graph Theory III", L. Beineke and R. Wilson, eds. (to appear).
3. N. Sauer, *On the existence of regular n-graphs with given girth*, J. Combin. Theory **9** (1970), 144-147.